# TOPICS IN ALGEBRAIC TOPOLOGY, TUTORIAL 9.12.2020

ANDREA BIANCHI

Here are some problems for the tutorial. We will try to cover as much as possible of them. Some problems are exercises, some are more difficult, some are open questions (at least for me). This is how life works...

## 1. Categories with spaces of objects

Recall that a (small) category $\mathcal{C}$ enriched in spaces (CGWH, or Kan complexes) is the datum of a *set* of objects $obj(\mathcal{C})$ plus, for each pair of objects $x, y$, the datum of a *space* $\mathcal{C}(x, y)$, plus an identity $I_x \in \mathcal{C}(x, x)$ for every object $x$, plus composition maps $\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$, such that associativity and unitality hold.

Infinity categories are essentially modeled on this notion: in particular if $\mathcal{C}$ is enriched in Kan complexes, then the coherent nerve of $\mathcal{C}$ is an $\infty$-category.

But what happens if we take a *space* of objects? Consider the following definition: $\mathcal{C}$ is given by a couple of spaces $O$, $M$ (of obects and of *all* morphisms in the category), plus maps of spaces $s, t\colon M \to O$ (every morphism is assigned its source and target), plus a map $I\colon O \to M$ (assigning to each object its identity morphism), plus a composition map $- \circ -$ from the following pullback $M \times_O M$ to $M$.

$$\begin{tikzcd} M \times_O M \arrow[r] \arrow[d] & M \arrow[d, "s"] \\ M \arrow[r, "t"] & O. \end{tikzcd}$$

Is the previous a genuine pullback or a homotopy pullback? If we work $\infty$-categorically there is no difference, but if we are old-fashoned and work with CGWH spaces or Kan complexes, then there is. However if we assume (as we do henceforth) that **the map $(s, t)\colon M \to O \times O$ is a Serre/Kan fibration**, then the two notions agree up to weak equivalence.

(1) In the old-fashoned way of thinking, we would require strict associativity and strict unitality for $- \circ -$ and $I$. In the $\infty$-categorical way of thinking, what would be the natural analogue of the previous definition? Suggestion; consider the $\infty$-category $Idem$ from [HTT, Definition 4.4.5.2]; then an $\infty$-category $\mathcal{C}$ whose objects form a space should be an inner fibration $\mathcal{C} \to Idem$ such that the fibre over the (unique) object of $Idem$ is a Kan complex/anima/$\infty$-groupoid. Check whether starting from a hard category $(O, M, I, - \circ -)$ as above and taking coherent nerve (of the singular set) yields this notion or something different.

(2) For $\mathcal{C} = (O, M, I, - \circ -)$ as above, we say that two objects $x, y$ are *path-connected* if they lie in the same path-component of $\mathcal{C}$; and they are *equivalent* if there are morphisms $f\colon x \to y$ and $g\colon y \to x$ such that

- $g \circ f$ is in the same path component of $I_x$ inside the space $M(x, x)$ (the fibre over $(x, x)$ of the map $(s, t)\colon M \to O \times O$);
- $f \circ g$ is in the same path component of $I_y$ inside the space $M(y, y)$.

Prove that, under the hypothesis $(s, t)\colon M \to O \times O$ is a Serre/Kan fibration, path-connected objects are equivalent.

(3) For $\mathcal{C} = (O, M, I, - \circ -)$ as above, we can define a simplicial space $B_\bullet\mathcal{C}$ by setting $B_p\mathcal{C} = M \times_O M \times_O \cdots \times_O M$ (again, fibre products are taken in CGWH or Kan complexes, but since $(s, t)$ is a fibration...). Its geometric realisation (as a CGWH[1], or as a Kan complex) is called $B\mathcal{C}$. Now from

[1]Maybe one should here assume that $I\colon O \to M$ is a cofibration of CGWH, or take the fat geometric realisation

point (2) we could expect that if we take a full subcategory $\mathcal{C}_0\mathcal{C}$ with a *discrete* set of objects, say one for each component of $O$, then the inclusion of $\mathcal{C}_0$ in $\mathcal{C}$ is fully faithful and hits up to equivalence all objects; so we would guess that $B\mathcal{C}_0$ (which is the classical nerve of a category whose objects have no topology) is weakly equivalent to $B\mathcal{C}$. So the topology on objects, in a certain way, doesn't count much. Prove it!

## 2. Grothendieck topologies on $\infty$-categories of embeddings

Recall that for fixed $n \geq 1$, we denoted $\mathrm{Man}$ the category of smooth, oriented $n$-dimensional manifolds with empty boundary, and for $M, N \in \mathrm{Man}$ we denote by $Emb(M, N)$ the space of morphisms from $M$ to $N$ in $Man$, i.e. the space (CGWH, and after taking singular set it is a Kan complex) of embeddings of $M$ into $N$[2].

We passed from $\mathrm{Man}$ to its coherent nerve, thus regarding $\mathrm{Man}$ as an $\infty$-category. We considered presheaves over $\mathrm{Man}$ and over its full subcategories $\mathrm{Disc}_k$: for $k \geq 0$ the category $\mathrm{Disc}_k$ is spanned by objects diffeomorphic to $\coprod_h \mathbb{R}^n$ for some $0 \leq h \leq k$.

And for a presheaf $F \in \mathcal{P}_{\mathrm{Man}}$ we defined $\mathcal{T}_k F \in \mathcal{P}_{\mathrm{Man}}$ as the right Kan extension from $\mathrm{Disc}_k$ to $\mathrm{Man}$ of the restriction of $F$ from $\mathrm{Man}$ to $\mathrm{Disc}_k$. This worked smoothly in the context of $\infty$-categories.

Now there is another classical characterisation of $\mathcal{T}_k \colon \mathcal{P}_{\mathrm{Man}} \to \mathcal{P}_{\mathrm{Man}}$ due to Weiss. Consider again $\mathrm{Man}$ as a category enriched over CGWH or over Kan complexes, with strictly associative composition. Weiss considers for every $k \geq 0$ (maybe it is better to think of $k \geq 1$, though the definition makes sense also for $k = 0$) a Grothendieck topology $\mathcal{I}_k$ on $\mathrm{Man}$. The question of this exercise will be: **can we give this characterisation also in the language of $\infty$-categories?**

First, recall the definition of Grothendieck topology on an ordinary (non-enriched, strict) category $\mathcal{C}$. A *sieve* $S$ on an object $x \in \mathcal{C}$ is a subpresheaf of the presheaf (in sets) represented by $x$: we have $S \subseteq \mathcal{C}(-, x)$, i.e. $S(y) \subseteq \mathcal{C}(x, y)$ for all $y \in \mathcal{C}$, and $S$ is also a presheaf. A Grothendieck topology on $\mathcal{C}$ is a family of sieves over objects of $S$, or equivalent, a determination of *which* sieves should be considered as "covering" or "distinguished" or "admissible", and which not. There are some axioms to be satisfied, see [HTT,Definition 6.2.2.1] (the notion for $\infty$-categories specialises to the classical one for ordinary categories).

Classically there is a set-theoretic condition $S(y) \subseteq \mathcal{C}(y, x)$. If $\mathcal{C}(x, y)$ happens to be a space (CGWH, or Kan complex), then one could ask a condition that takes the topology into account. In [HTT,Definition 6.2.2.1] this condition is that $S(y)$ should be a union of path-components of $\mathcal{C}(x, y)$: after all, in an infinity category $\mathcal{C}$ there is no sensible notion that discriminates homotopic morphisms from $x$ to $y$! Weiss definition of $\mathcal{I}_k$ however uses sieves with no evident extra topological condition.

So what is a covering sieve for the topology $\mathcal{I}_k$? Let $M \in \mathrm{Man}$; then a sieve $S \subseteq \mathrm{Emb}(-, N)$ is covering for $\mathcal{I}_k$ if for every subset $J \subset M$ of cardinality $\leq k$ there exists an object $U \in M$ and an $S$-admissible embedding $f \in S(U) \subseteq \mathrm{Emb}(U, M)$ such that the image of $f$ contains $J$.[3].

The result of this definition is astonishing: the functor $\mathcal{T}_k \colon \mathcal{P}_{\mathrm{Man}} \to \mathcal{P}_{\mathrm{Man}}$ that we defined before using the subcategory $\mathrm{Disc}_k$, is indeed the[4] sheafification with respect to the topology $\mathcal{I}_k$. So the following diagram commutes

$$\begin{tikzcd} \mathcal{P}_{\mathrm{Man}} \arrow[r, "U_k"] \arrow[d, "sheafify"'] \arrow[dr, "\mathcal{T}_k"] & \mathcal{P}_{Disc_k} \arrow[d, "Ran\iota_k"] \\ Shv_{\mathrm{Man},\mathcal{I}_k} \arrow[r, "\subseteq"] & \mathcal{P}_{\mathrm{Man}}. \end{tikzcd}$$

Yet it seems to me that a sieve for Weiss is not following Lurie's definition. For example, take a manifold $M$ and let $U_1, \dots, U_r$ be open sets covering $M$ in the most classical sense (every point of $M$ is contained in some $U_i$). Consider the set-theoretic sub-presheaf $S \subset \mathrm{Emb}(-, M)$ that associates with any manifold $U$ the set of embeddings $U \hookrightarrow M$ with image contained in some $U_i$. Then Weiss will say

[2]Actually we defined $\mathrm{Man}^Z$ to have objects manifolds with an identification of the boundary with $Z$, and morphisms being those embeddings which restrict to "the identity of $Z$" on the boundary. Here for simplicity we restrict to the case $Z = \emptyset$

[3]Actually Weiss defines the Grothendieck topology $\mathcal{I}_k$ in terms of *coverings* and not of *covering sieves*, but I think it is evident that this is the reformulation

[4]the=a model of

that $S$ is a sieve, and $S$ is even a covering sieve for $\mathcal{I}_1$; but Lurie will say that $S$ is not a sieve at all, because there exist in general isotopic embeddings $f_1, f_2 \colon U \hookrightarrow M$ with $f_1$ landing in some $U_i$, but $f_2$ having image contained in no $U_i$. For higher $k$ the situation is similar: given a covering $U_1, \dots, U_r$ of $M$ which is redundant enough so that every subset $J \subset M$ of cardinality $\leq k$ is in some $U_i$, define the associated Weiss-sieve $S$ by selecting for $U \in \mathrm{Man}$ the embeddings $U \hookrightarrow M$ with image in some $U_i$; again $S$ is not a Lurie-sieve.

**Can we generalise the notion of $\infty$-sieve in such a way that we are able to model Weiss' constructions and results?**

Suggestion: we may try to define a sieve on an object $x$ in an infinity category $\mathcal{C}$ just as a presheaf $S \in \mathcal{P}_{\mathcal{C}}$ (presheaf means with values in spaces/animae/animated sets, not sets), plus a map of presheaves $S \to \mathcal{C}(-, x)$, that we think of as an *inclusion* (but it doesn't have to be injective in any sense, and probably it makes more sense to think of $S(y) \to \mathcal{C}(y, x)$ as being a Serre/Kan fibration). Can we adapt [HTT, Definition 6.2.2.1] to define a sensible notion of $\infty$-Grothendieck topology? Can we describe $\mathcal{I}_k$ in this way?

*Email address*: `anbi@math.ku.dk`

# TOPICS IN ALGEBRAIC TOPOLOGY, TUTORIAL 6.1.2021

ANDREA BIANCHI

Here are some problems for the tutorial. We will try to cover as much as possible of them. Some problems are exercises, some are more difficult, some are open questions (at least for me). This is how life works...

## 1. Perverse sheaves over a space with two strata

Let us look again at Thais' general example, and try to elaborate on it.

Let $X$ be a topological space, written as union of a closed subspace $Y$ and an open complement $U$. As usual we denote $i\colon Y \to X$ and $j\colon U \to X$ the inclusions. We denote by $S$ the stratification of $X$ by $Y$ and $U$. Let $\mathbb{K}$ be a field[1]. We consider the bounded below derived category $D^-(X;\mathbb{K})$ of bounded below complexes of sheaves of $\mathbb{K}$-modules over $X$. For old-fashoned people, $D^-(X;\mathbb{K})$ is a triangulated 1-category with a Postnikov t-structure, whose heart $D^-(A)^\heartsuit$ is equivalent to $Sh(X;\mathbb{K})$, the abelian category of sheaves of $\mathbb{K}$-modules over $X$. For $\infty$-people, $D^-(X;A)$ is a stable $\infty$-category, it has a Postnikov t-structure as such and its heart is homotopy discrete and equivalent to (the nerve of) the 1-category $Sh(X;\mathbb{K})$.

Recall that for an object $\mathcal{F}_\bullet \in D^-(X;\mathbb{K})$ we have associated objects $\underline{H}_i(\mathcal{F}_\bullet)$ in $Sh(X;\mathbb{K})$. The functor $\underline{H}_i\colon D^-(X;\mathbb{K}) \to D^-(X;\mathbb{K})^\heartsuit \cong Sh(X;\mathbb{K})$ is concretely described by taking the $i$-th homology on each stalk of $\mathcal{F}_\bullet$; on the other hand it coincides with the double truncation $\tau_{\le i} \circ \tau_{\ge i}$ associated with the Postnikov t-structure on $D^-(X;\mathbb{K})$.

Recall that we use homological notation: for example we have a distinguished triangle (aka fibre sequence) in $D^-(X;A)$ of the form $\tau_{\ge 0}\mathcal{F}_\bullet \to \mathcal{F}_\bullet \to \tau_{\le -1}\mathcal{F}_\bullet$. Similar remarks as above hold for $Y$ and $U$.

Consider the following categories of sheaves on $X$ (and similarly on $U$ and $Y$), and derived categories thereof.

- $Sh(X;\mathbb{K})$, i.e. *all* sheaves;
- $Sh_{lc}(X;\mathcal{C})$, i.e. *locally constant* sheaves on $X$; these are all sheaves $\mathcal{F}$ satisfying the following property: for $V_1 \subset V_2 \subset X$ small enough (with respect to some open cover of $X$) and connected, the restriction map $\mathcal{F}(V_2) \to \mathcal{F}(V_1)$ is an isomorphism;
- $Sh_c(X,S;\mathbb{K})$, i.e. *$S$-constructible* sheaves; a sheaf $\mathcal{F}$ here is required to satisfy that $i^*\mathcal{F}$ and $j^*\mathcal{F}$ are locally constant *with finitely generated stalks*,[2] on $Y$ and $U$ respectively.
- $D^-(X;\mathbb{K})$, i.e. the entire bounded below derived category;
- $D^-_c(X;\mathbb{K})$, the constructible derived category; a complex of sheaves $\mathcal{F}_\bullet$ here satisfies that $\underline{H}_i(\mathcal{F}_\bullet)$ is constructible (but the $\mathcal{F}_i$'s making the complex need not be constructible).

More precisely, $D^-_c(X;\mathbb{K})$ is the full subcategory of the $\infty$-category $D^-(X;\mathbb{K})$ spanned by the mentioned objects. In the following we focus on $D^-(X)$, but similarly we could work with $D^-_c(X;\mathbb{K})$; we make some remarks about it later.

Next, we used the recollement to produce a t-structure on $Sh(X;\mathbb{K})$ starting from t-structures on $Sh(U;\mathbb{K})$ and $Sh(Y;\mathbb{K})$.

**Question 1.1.** *Prove (or disprove): the recollement of the Postnikov t-structures $(Sh(Y;\mathbb{K})_{\ge 0}, Sh(Y;\mathbb{K})_{\le 0})$ and $(Sh(U;\mathbb{K})_{\ge 0}, Sh(U;\mathbb{K})_{\le 0})$ gives precisely the Postnikov t-structure $(Sh(X;\mathbb{K})_{\ge 0}, Sh(X;\mathbb{K})_{\le 0})$.*

[1] This is a simplifying assumption; a commutative noetherian ring of finite global dimension should work as well

[2] This assumption is responsible for Verdier duality on $D_b^-(X;\mathbb{K})$ to be a perfect duality

**Question 1.2.** *Prove (or disprove): the heart of the Postnikov t-structure* $(Sh(X;\mathbb{K})_{\geq 0}, Sh(X;\mathbb{K})_{\leq 0})$ *is equivalent to* $Sh(X;\mathbb{K})$.

More generally, we considered *any* function $p\colon \{Y,U\} \to \mathbb{Z}$ and considered on $Sh(X;\mathcal{C})$ the $p$-perverse t-structure $({}^pSh(X;\mathbb{K})_{\geq 0},\ {}^pSh(X;\mathbb{K})_{\leq 0})$ whose *decollement*[3] is given by the t-structures

$$(Sh(Y;\mathbb{K})_{\geq p(Y)}, Sh(Y;\mathbb{K})_{\leq p(Y)}) \text{ and } (Sh(U;\mathbb{K})_{\geq p(U)}, Sh(U;\mathbb{K})_{\leq p(U)}).$$

We defined *perverse sheaves* $Perv(X,p;\mathbb{K})$ as the heart of the $p$-perverse t-structure. Let us assume for simplicity that $p(Y)=0$ (we are free to translate all t-structures involved in a recollement by the same integer).

**Question 1.3.** *For which values of* $p(U)$ *can we prove the following?*

- *The functor* $i_*\colon D^-(Y;\mathbb{K}) \to D^-(X;\mathbb{K})$ *sends the full subcategory* $Sh(Y;\mathbb{K}) \subset D^-(Y,\mathbb{K})$ *inside* $Perv(X,p;\mathbb{K})$*;*
- *The abelian category* $Perv(X,p;\mathbb{K})$ *splits as direct sum of two categories, one of which is* $i_*Sh(Y;\mathbb{K})$.

Observe that one can use recollement also with the categories $D^-_c(X,S;\mathbb{K})$, $D^-_{lc}(Y;\mathbb{K})$ and $D_{lc}(U;\mathbb{K})$; note that $D^-_{lc}(Y;\mathbb{K}) = D^-_c(Y,S|_Y;\mathbb{K})$ and similarly for $U$.

**Reason for the last question**: in the very, very special case $X=\mathbb{C}$, $Y=\{0\}$ and $U=\mathbb{C}^*$, then the following happens, according to the value of $p(Y)=0$ (I really hope to get indices correctly):

- for $p(Y)=0$ we have $Perv(\mathbb{C},p;\mathbb{K}) \simeq Sh^-_c(\mathbb{C},S;\mathbb{K})$, the latter being the usual abelian category of constructible sheaves in $\mathbb{K}$-vector spaces; $S$ is the filtration on $\mathbb{C}$ by $\{0\}$ and $\mathbb{C}^*$;
- for $p(Y)=2$ we get precisely the Verdier duals of the complexes of sheaves in the previous example, i.e. Verdier duals of constructible sheaves in $\mathbb{K}$-vector spaces;
- for $p(Y)=1$ we get the *middle perversity*, and Verdier duality restricts to a self-duality
$$\mathcal{D}\colon Perv(\mathbb{C},p;\mathbb{K}) \to Perv(\mathbb{C},p;\mathbb{K});$$
- in all other cases $Perv(\mathbb{C},p;\mathbb{K})$ splits as a direct sum of the subcategories spanned, respectively, by skyscrapers on $\{0\}$ and by those perverse sheaves of the form $Im(j_!\mathcal{F}_\bullet \to j_*\mathcal{F}_\bullet)$.

See Geordie Williamson, "An illustrated guide to perverse sheaves" for further details (but pay attention, he uses cohomological notation!).

## 2. Mixing t-structures and sheaves

Let $\mathcal{C}$ be a stable $\infty$-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, and let $X$ be a topological space.

**Question 2.1.** *Is there a natural t-structure on the stable $\infty$-category $Sh(X;\mathcal{C})$ coming from the one on $\mathcal{C}$?*

Using homological notation, one is tempted to define $Sh(X;\mathcal{C})_{\geq 0}$ as the full subcategory on those sheaves $\mathcal{F}$ taking values in $\mathcal{C}_{\geq 0}$, and then $\mathcal{C}_{\leq 0}$ would be automatically the full subcategory on those sheaves $\mathcal{F}'$ such that $Map_{Sh}(\mathcal{F},\mathcal{F}') \simeq *$ for all $\mathcal{F} \in Sh(X;\mathcal{C})_{\geq 0}$. Does it work?

If this work, one can hopefully stratify a space $X$, give a perversity function $p$ on strata, and define perverse sheaves as the heart of $Sh(X;\mathcal{C})$, endowed with the perverse $t$-structure coming from recollement of the $p$-shifted t-structures on strata. Can this be good for anything?

Another, related question, is the following.

**Question 2.2.** *Suppose the previous question has positive answer, at least under some additional assumptions. Let $X$ and $Y$ be spaces. How do the t-structures on $Sh(X\times Y;\mathcal{C})$ and $Sh(X;Sh(Y;\mathcal{C}))$ that we obtain interrelate with each other?*

[3]Aka *cassement*

## 3. Quiver representations

Recall that perverse sheaves have often a reinterpretation as much more down-on-to-earth abelian categories of diagrams of vector spaces satisfying some constraints. The typical example is, once again, given by $Perv(\mathbb{C}, p; \mathbb{K})$, where $\mathbb{C}$ is stratified by $\{0\}$ and $\mathbb{C}^*$, $p$ is the middle perversity (i.e. $p(0) = 0$ and $p(\mathbb{C}^*) = 1$), and $\mathbb{K}$ is a field.

Then the datum of a perverse sheaf is equivalent to a diagram of vector spaces $\Phi$ and $\Psi$ interrelated by maps $u, v$ as follows

$$\Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi,$$

with the additional condition that both $1 - uv$ and $1 - vu$ are invertible[4]. More precisely, the category $Perv(\mathbb{C}, p; \mathbb{K})$ is equivalent to the category of diagrams as above and natural transformations.

Another incarnation of $Perv(\mathbb{C}, p; \mathbb{K})$, different from the $(\Phi, \Psi)$ description above, is the Dirac description: this time we consider the category of diagrams (and natural transformations) of the form

$$E_- \underset{\delta_-}{\overset{\gamma_-}{\leftrightarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+$$

with the additional condition that $\gamma_\pm \delta\pm$ is invertible for all choices of signs, and is the identity whenever it has same source and target.

See Mikhail Kapranov, Vadim Schechtman, "Perverse sheaves and graphs on surfaces" and "Shuffle algebras and perverse sheaves" for more details, and for other incarnations.

**Question 3.1.** *Assuming that we can make sense of perverse sheaves with values in our favourite $\infty$-category, e.g. spectra; what alternative description as quiver representations do such perverse sheaves have, if any?*

*Email address*: `anbi@math.ku.dk`

[4]But by the formal identity $(1 - uv)^{-1} = 1 + u(1 - vu)^{-1}v$ it suffices one invertibility

## TOPICS IN ALGEBRAIC TOPOLOGY, TUTORIAL 13.1.2021

ANDREA BIANCHI

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### 1. Complex oriented cohomology theories

Let $E$ be a spectrum, and let $E^*$ be the associated cohomology theory (for spaces, and for spectra as well). In particular, for a space $X$ we have isomorphisms of sets

$$E^i(X) = \pi_0(\mathrm{Map}(\Sigma^\infty_+ X, E[i])) = \pi_0(\mathrm{map}(X, E(i))).$$

Here $E[i]$ is the shifted spectrum, $E(i)$ is the $i$-th space of the spectrum $E$, $X$ is a space, $\mathrm{Map}$ is the mapping spectrum and $\mathrm{map}$ is the mapping space. For another spectrum $F$ we have $E^*(F) = \pi_*(\mathrm{Map}(F, E))$, since only the second formula is available.

A cohomology theory coming from an $\mathbb{E}^\infty$ ring spectrum [1] $E$ is called *multiplicative*; in this case $E^*(F)$ is an (associative, unital) graded-commutative ring.

**Example 1.1.** Here are some examples of $\mathbb{E}_\infty$ ring spectra:

- the sphere spectrum $S = \Sigma^\infty_+ S^0$;
- the Eilenberg-MacLane spectrum $HR$, for all commutative ring $R$;
- the complex K-theory spectrum $KU$;
- the complex cobordism spectrum $MU$.

**Question 1.2.** *Prove or disprove: every graded-commutative ring $R$ arises as $E^*(*)$ for some $\mathbb{E}^\infty$ ring spectrum $E$.*

**Question 1.3.** *Prove or disprove: every spectrum $E$ with $\pi_0(E) \neq 0$ can be endowed with a structure of $\mathbb{E}_\infty$ ring spectrum.*

Recall: a multiplicative cohomology theory $E^*$ is *complex orientable* if the map $E^2(\mathbb{C}P^\infty) \to E^2(S^2)$ coming from the inclusion $S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^\infty$ is surjective. Some people will only ask that this map hits the element $\eta \in E^2(S^2) \cong E^0(*)$ corresponding to the unit $1 \in E^0(*)$ of the ring $E^*(*)$: check that for $E$ being a *multiplicative* ring spectrum this is an equivalent requirement, using that $E^2(S^2)$ is a rank 1 free module over $E^*(*)$. Of the examples above, $HR$, $KU$ and $MU$ are well-known to be complex orientable.

**Question 1.4.** *Prove that $S$ is not complex orientable*[2].

Recall: a *complex orientation* on a multiplicative cohomology theory $E^*$ is an explicit choice of $\theta \in E^2(\mathbb{C}P^\infty$ hitting $\eta$. So a complex orientable cohomology theory $E$ may admit several complex orientations.

**Question 1.5.** *Prove or disprove: $HR$ admits exactly one complex orientation. In general, what properties should $E$ satisfy, besides being complex orientable, so that $E^*$ admits exactly one complex orientation?*

[1] I guess all examples we are going to consider today are indeed $\mathbb{E}_\infty$ ring spectra, but in principle one can consider also just $\mathbb{E}_1$-ring spectra

[2] Oh, how easy would life be, if only $S$ were complex orientable...

Recall that the ring $L = \pi_*(MU)$, also known as *Lazard ring*, has the following universal property: it is a commutative ring with a canonical formal group law $f_L \in L[[x,y]]$, and every formal group law $f \in R[[x,y]]$ on every commutative ring $R$ can be obtained by change-of-coefficients from $\theta_L$ along exactly one ring homomorphism $L \to R$. The fact that the ring $L$ with the aforementioned universal property is isomorphic to $\pi_*(MU)$ is the content of Quillen's theorem.

The ring $L$ is abstractly isomorphic to $\mathbb{Z}[t_1, t_2, \dots ...]$, a polynomial ring in countably many variables: this is the content of Lazard's theorem. The description of $L$ coming from the universal property is not so easy: $L$ is generated over $\mathbb{Z}$ by variables $c_{i,j}$ for $i, j \geq 0$; the canonical formal group law $f_L$ has the tautological form $f_L(x,y) = \sum_{i,j\geq 0} c_{i,j} x^i y^j$; imposing that this is a formal group law yields many polynomial relations that the generators $c_{i,j}$ must satisfy, and $L$ is the quotient of the free polynomial algebra $\mathbb{Z}[c_{i,j}]_{i,j\geq 0}$ by all these relations.

To specify a meaningful grading on $L$, let us define what a formal group law in the graded setting should be. It should be a power series $f(x,y) = \sum_{i,j\geq 0} r_{i,j} x^i y^j \in R[[x,y]]$ satisfying the usual properties, and we would like $x$, $y$ and $f(x,y)$ to be homogeneous of the same degree $d \in \mathbb{Z}$. Moreover we want $x$ and $y$ to commute without signs, so we'd like $d$ to be even.

**Question 1.6.** *Check that the requirement imposes that $r_{i,j}$ sits in degree $-d(i+j-1)$, so all coefficients have even degree as well.*

The ring $L$ is endowed with a canonical grading with $c_{i,j}$ sitting in degree $2(i + j - 1)$, becoming a graded-commutative ring. So we choose $d = -2$; we could have chosen $d = 2, 4, 0, -6$ etc. as well; but the choice $d = -2$ is the only one making Quillen's isomorphism $\pi_*(MU) \cong L$ into a graded isomorphism, so it is the only choice having a geometric meaning.

We saw that $MU$ has a similar, universal property as $L$: every complex orientation $\theta \in E^2(\mathbb{C}P^\infty)$ on an $\mathbb{E}-\infty$ ring spectrum $E$ corresponds to exactly one map of ring spectra $MU \to E$, yielding a morphism $MU^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^\infty)$ along which the canonical complex orientation $\theta_{MU} \in MU^2(\mathbb{C}P^\infty)$ is sent precisely to $\theta$.

**Question 1.7.** *Let $E$ be a $\mathbb{E}_\infty$ ring spectrum, and denote by $R$ the graded-commutative ring $\pi_* E = E^*(*)$. Which of the following sets are in canonical bijection with each other?*

1. *Formal group laws $f(x,y)$ on R, with $x, y, f(x,y)$ homogeneous of degree $-2$.*
2. *Graded ring homomorphisms $L \to R$.*
3. *Complex orientations $\theta \in E^2(\mathbb{C}P^\infty$.*
4. *(Homotopy classes of) ring spectra maps $MU \to E$.*

## 2. Formal group laws, purely algebraically

Here are some examples of formal group laws:

- $x + y \in \mathbb{Z}[[x,y]]$; this is the fgl associated with the complex oriented cohomology theory $HZ$. More generally $x + y \in R[[x,y]]$ is associated with $HR$, where $R$ is a commutative ring (concentrated in degree 0);
- $x + y + \beta xy \in R[[x,y]]$, for $R = \mathbb{Z}[\beta^{\pm 1}]$ being a Laurent ring in one variable $\beta$ of degree 2; this comes from $KU$.
- $\sum_{i,j\geq 0} c_{i,j} x^i y^j \in L[[x,y]]$, the universal example, corresponding to $MU$.

In the non graded setting, for all commutative ring $R$ we have the formal group laws $x + y$, $x + y + \beta xy$ with $\beta \in R^*$, and the generic example has just some "infinite" form $\sum_{i,j\geq 0} r_{i,j} x^i y^j$. In the rest of the section, let us consider non-graded commutative rings and formal group laws on them.

**Question 2.1.** *Let $R$ be a commutative ring (without grading), and assume that $R$ has no nilpotent elements. What are the formal group laws on $R$ of "polynomial type", i.e. of the form $f(x,y) \in R[x,y] \subset R[[x,y]]$?*

Recall that two formal group laws $f(x,y), f'(x,y) \in R[[x,y]]$ are *isomorphic* if there is a power series $b(t) \in R[[t]]$ of the form $b(t) = b_0 t + b_1 t^2 + b_2 t^3 + \dots$ such that $b_0 \in R^*$ and such that

$f(b(x), b(y)) = b(f'(x, y))$. If $b(t)$ can be chosen with $b_0 = 1$, then $f$ and $f'$ are said to be strictly isomorphic.

**Question 2.2.** *Prove or disprove: for $\beta, \beta' \in R^*$ the fgls $x + y + \beta xy$ and $x + y + \beta' xy$ are isomorphic, by constructing an explicit $b(t)$ as above, or by finding an obstruction. In the answer was positive, are $x + y + \beta xy$ and $x + y + \beta' xy$ also strictly isomorphic?*

**Question 2.3.** *Prove that, for $R = \mathbb{Q}$, the fgls $x+y$ and $x+y+xy$ are strictly isomorphic, by constructing an explicit $b(t)$ as above.*

In fact, all formal group laws over $\mathbb{Q}$ are *uniquely, strictly isomorphic* to each other (see Lurie's notes on chromatic homotopy theory, this follows essentially from Lemma 10 in Lecture 2).

Instead, over $\mathbb{F}_p$ there are non-isomorphic fgls; for example $x+y$, which has height $\infty$, and $x+y+xy$, which has height 1.

**Question 2.4.** *Fix a prime $p$ and an integer $n \geq 0$. Is there a formal group law over $\mathbb{F}_p$ of height exactly $n$?*

For this last question, look here:
https://mathoverflow.net/questions/124048/what-do-formal-group-laws-of-height-geq-3-look-like

## 3. Germs of bundles of Lie groups

Let $G$ be a complex Lie group (e.g. $(\mathbb{C}, +)$, or $(\mathbb{C}^*, \cdot)$, or $GL(n)$ or $O(n)$), and denote by $e \in G$ the neutral element. Let $n$ denote the dimension (over $\mathbb{C}$) of $G$, and let $z_1, \dots, z_n$ be the coordinates of a local chart $\phi\colon U \to \mathbb{C}^n$ defined on a small neighbourhood of $e$, such that $\phi(e) = 0$. If you wish, just focus on the case $n = 1$.

There is a smaller neighbourhood $V \subset U$ of $e$ such that the multiplication $\mu\colon G \times G \to G$ restricts to a map $V \times V \to U$, and can thus be read in coordinates.

**Question 3.1.** *How does $\mu$ look like in coordinates? Focus first on the case $n = 1$, and recall that a 1-dimensional Lie group is always commutative.*

Now suppose that $\psi\colon U \to \mathbb{C}^n$ is another chart centred at $e$, with coordinates $w_1, \dots, w_n$. Then the coordinate change allows us to pass from the description of $\mu\colon V \times V$ using the chart $\phi$, to the description of the same map using the chart $\psi$.

**Question 3.2.** *How does the coordinate change look like?*

Suppose on the contrary that we have just one chart $\psi$, but we have a self-map $b\colon G \to G$ of Lie groups. We don't assume that $b$ is an automorphism of groups, just an endomorphism.[3].

**Question 3.3.** *How does b look like in coordinates?*

Note: in characteristic $p$, an endomorphism of a formal group law may be much more complicated! For example $b(t) = t^p$ is an endomorphism of the formal group law $x + y$.

Suppose now that we have a complex variety $X$ and a *family*[4] of groups $\pi\colon \mathcal{E} \to X$, i.e. $\mathcal{E}$ is also a complex variety, $\pi$ is holomorphic, each fibre of $\pi$ is a complex Lie group and all relevant maps are holomorphic, namely the multiplication map $\mu\colon \mathcal{E} \times_X \mathcal{E} \to \mathcal{E}$ and the neutral element section $e\colon X \to \mathcal{E}$.

**Question 3.4.** *Prove or disprove: the inverse map $(-)^{-1}\colon \mathcal{E} \to \mathcal{E}$ is then automatically holomorphic.*

We are actually going to study the behaviour of $\mathcal{E}$ near the $e$-section $e(X) \subset \mathcal{E}$; so it suffices that $\mathcal{E}$ looks like a bundle of groups near the $e$-section; think of any neighbourhood of $e(X) \subset \mathcal{E}$ as being the total space of a bundle of germs of Lie groups over $X$. The following should be the example to keep in mind.

[3]Actually it suffices, for our purposes, to assume that $b\colon V \to V$ is defined only on a neighbourhood $V$ of $e \in G$, and that it commutes with multiplication on a subneighbourhood, where every meaningful formula is defined

[4]If you want, think of a bundle; probably we want to consider bundles with exceptional fibres, or something of this kind

**Example 3.5.** Let $X = \mathbb{C}$ and let $\mathcal{E} = \mathbb{C} \times \mathbb{C} \setminus \{(\beta, w) | \beta w = -1\}$. The projection $\pi \colon \mathcal{E} \to X$ is given by $(\beta, w) \mapsto \beta$. The $e$-section is given by $e(\beta) = (\beta, 0)$, and the fibrewise multiplication is given by

$$\mu((\beta, x), (\beta, y)) = x + y + \beta xy.$$

**Question 3.6.** *Prove or disprove: the above data define a family of Lie groups over $X = \mathbb{C}$, all fibres being commutative complex Lie groups of dimension 1.*

Note that $\pi^{-1}(0)$ is isomorphic to $\mathbb{C}$, but $\pi^{-1}(\beta)$ is isomorphic to $\mathbb{C}^*$.

Let $A \subset X$ be an open set on which $\mathcal{E}$ is "trivial near the $e$-section"[5]. By this we mean that there is a Lie group $G$ with unit $e_G$ and a neighbourhood $U$ of $e_G$, such that a neighbourhood of $e(A)$ in $\mathcal{E}$ can be identified with $A \times U$, in such a way that $e \colon A \to \mathcal{E}$ corresponds to the obvious section $(-, e_G) \colon A \to A \times U$, and that the multiplication $\mu$ agrees fibrewise with the multiplication of $G$, at least when restricted, for a smaller neighbourhood $V \subset U \subset G$ of $e_G$, to the subspace $(A \times V) \times_A (A \times V)$ of $(A \times U) \times_A (A \times U)$.

**Question 3.7.** *How does the multiplication $\mu \colon (A \times V) \times_A (A \times V) \to A \times U$ read in local coordinates? Express it in terms of the ring $\mathcal{O}(A)$ of holomorphic functions defined on $A$.*

A *germ of gauge transformation* along $A$ is then a selfmap $b \colon A \times V$ which fixes the $e$-section $e(A)$ and is compatible with the multiplication $\mu$ near $e(V) \subset A \times V$

**Question 3.8.** *How does $b$ look like in coordinates? Express it in terms of the ring $\mathcal{O}(A)$.*

Again, in characteristic $p$ the situation is more complicated than just multiplying by an invertible element of $\mathcal{O}(A)$.

The last question is the following.

**Question 3.9.** *Is there a meaningful way to connect the following two facts about the formal group law $x + y + \beta xy$ over the ring $\mathbb{Z}[\beta]$?*

- *If we basechange from $\mathbb{Z}$ to $\mathbb{C}$ we get a family of groups as in the example above, and exactly one of them, namely for $\beta = 0$, is* globally additive *(isomorphic to $\mathbb{C}$), but generically (for $\beta \neq 0$) we obtain a group which is* globally multiplicative *(isomorphic to $\mathbb{C}^*$)*
- *If we basechange from $\mathbb{Z}$ to $\mathbb{F}_p$ we get a family of formal group laws as well; exactly one of them, namely for $\beta = 0$, has height $\infty$, but generically (for $\beta \neq 0$) we obtain a formal group law of height 1.*

*Email address*: `anbi@math.ku.dk`

[5]The letter $A$ is not chosen randomly: it is the first letter of "affine", so $A$ should be thought as an affine subscheme of $X$, if you are more familiar with schemes than with complex manifolds