TOPICS IN ALGEBRAIC TOPOLOGY, A VARIATION ON THE NOTION OF GROTHENDIECK TOPOLOGY

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Here is an attempt to continue the discussion from the last tutorial, yet not complete it. Recall that we had a topologically enriched category Man of n-manifolds (without boundary, for simplicity) and embeddings, and that our goal is to interpret the Taylor approximations $\mathcal{T}_k\colon \mathcal{P}_{\mathrm{Man}} \to \mathcal{P}_{\mathrm{Man}}$ of presheaves over Man in An as a purely ∞-categorical construction. Classically, instead, one has to work with presheaves and "homotopy sheaves" with values in Top (or in Kan).

1. Conventions

All manifolds are smooth and have some dimension $n \geq 1$ that we always omit. Top is the category of compactly generated weakly Hausdorff spaces. Kan is the (strict) category of Kan complexes, i.e. a full subcategory of simplicial sets. An is the ∞-category of Kan complexes (that are now called "animae"). So objects of Kan and of An are the same, but Kan is a strict category, An is an ∞-category. There is however a functor (of ∞-categories) $\mathrm{Kan} \to \mathrm{An}$, and we will use this to pass from Kan to An (and from presheaves with values in Kan to ∞-presheaves with values in An).

Similarly taking singular sets gives a (lax) monoidal functor $\mathrm{Top} \to \mathrm{Kan}$.

The category Man of n-manifolds and embeddings is first enriched in Top; taking singular sets we get a category enriched in Kan, and taking the coherent nerve yields an ∞-category.

We always denote by $\mathcal{C}$ an ∞-category. Our main example will be precisely Man, but we try to keep our discussion as general as possible.

2. Definition of sieve

First, we try to give a sensible definition of sieve and Grothendieck topology that generalises [HTT 6.2.2].

Definition 2.1. Let $x \in \mathcal{C}$ be an object. A *sieve* (S, F) on x is an ∞-category S with a functor of ∞-categories $F\colon S \to \mathcal{C}/x$ to the comma category over x. The functor F is required to be a cartesian fibration [HTT Definition 2.4.2.1]:

- F is an inner fibration;
- given $t \in S$ with $F(t) = (z \to x)$ (an object in $\mathcal{C}/x$ is an arrow in $\mathcal{C}$ with target x) and a morphism $(y \to x) \to (z \to x)$ in $\mathcal{C}/x$ (which is really a triangle $z \to y \to x$ in $\mathcal{C}$), among all lifts $s \to t$ of $(y \to x) \to (z \to x)$ there is one which is "cartesian" (i.e. it is a final object in an appropriate category of all possible lifts of $(y \to x) \to (z \to x)$ ending at t).

We will be sometimes sloppy and write that S, rather than the couple (S, F), is a sieve.

Example 2.2. The identity functor $\mathcal{C}/x = \mathcal{C}/x$ makes $\mathcal{C}/x$ (better, $(\mathcal{C}/x, Id_{\mathcal{C}/x})$) into a sieve over x. It is a standard fact that an identity functor is always a cartesian fibration: all lifting problems one can think of have a (set-theoretically) *unique* solution!

More generally, if $S \subseteq \mathcal{C}/x$ is a sieve in the sense of Lurie [HTT Definition 6.2.2.1], then S is also a sieve in the sense of Definition 2.1. Viceversa, if $S \subseteq \mathcal{C}/x$ is a sieve in the sense of Definition 2.1, then it is also a Lurie-sieve.

Lurie's conditions for a subcategory $S \subseteq \mathcal{C}/x$ to be a sieve are:

- S is a full subcategory of $\mathcal{C}/x$: this can be rephrased as saying that $S \to \mathcal{C}/x$ is an inner fibration;

- if $(z \to x)$ is an object in S and $(y \to x) \to (z \to x)$ is a morphism in $\mathcal{C}/x$, then $(y \to x)$ lies again in S: this can be rephrased as saying that the inclusion $S \to \mathcal{C}/x$ is a cartesian functor.

Essentially, Definition 2.1 is the same as Lurie's one, but without the restriction that S must be a full subcategory of $\mathcal{C}/x$.

3. Constructions with sieves

Sieves can be pulled back along morphisms in $\mathcal{C}$.

Definition 3.1. Let $F\colon S \to \mathcal{C}/x$ be a sieve over x and let $f\colon y \to x$ be a morphism in $\mathcal{C}$. There is a "postcomposition functor" $f \circ -\colon \mathcal{C}/y \to \mathcal{C}/x$. We define the sieve f^*F as the fibre product

$$f^*F := S \times_{\mathcal{C}/x} \mathcal{C}/y.$$

The functor $f^*F \to \mathcal{C}/y$ is projection on the second coordinate.

Exercise: prove or recall that a pullback of a cartesian fibration is again a cartesian fibration.

Sieves can be extended. Classically, if $S \subseteq \mathcal{C}/x$ is a sieve in the sense of Lurie, then an extension of S is a bigger full subcategory $S \subseteq T \subseteq \mathcal{C}/x$, such that also T is a sieve.

Definition 3.2. Given two sieves $F\colon S \to \mathcal{C}/x$ and $G\colon T \to \mathcal{C}/x$, we say that T extends S if there is a functor $H\colon S \to T$ making the obvious triangle over $\mathcal{C}/x$ commute. We will also say that S is a refinement of T.

Here the commutativity can be interpreted strictly, or rather as further piece of structure (a natural *equivalence* between F and TH). It starts being apparent that sieves over x form rather a new infinity category than merely a poset by the "being finer than" relation. Compare this with coverings of topological spaces, which are classically only "ordered" by refinement.

This observation must play a key role when trying to define Cech cohomology for an object $x \in \mathcal{C}$ with coefficients in a sheaf over $\mathcal{C}$, as a sort of "colimit" on all coverings by refinement. But one thing at a time, first let us define what a morphism of sieves is.

Definition 3.3. Let (S,F) and (T,G) be sieves on $x \in \mathcal{C}$. A morphism of sieves $H\colon (S,F) \to (T,G)$ over $\mathcal{C}/x$ is a triangle in Cat_∞ whose horn Λ^2_2 is obtained using $S, T, \mathcal{C}/x, F, G$ in the only sensible way. Since we are sloppy, we use the letter H also to denote the functor $S \to T$ which is really only one side of this triangle (the side missing in the horn).

We obtain a category $\mathfrak{Siev}/x$ of sieves over x.

The previous definition can be globalised to the entire category $\mathcal{C}$.

The last important construction we have is the following.

Definition 3.4. Let $(S,F) \in \mathfrak{Siev}/x$ and $(T,G) \in \mathfrak{Siev}/y$. A morphism of sieves $H\colon (S,F) \to (T,G)$ is given by a morphism $f\colon x \to y$ in $\mathcal{C}$ and by a functor $H\colon S \to T$ and by a filling of the following square in Cat_∞

$$\begin{array}{ccc} S & \xrightarrow{H} & T \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} \\ \mathcal{C}/x & \xrightarrow{f\circ -} & \mathcal{C}/y. \end{array}$$

We obtain a $\mathfrak{Siev}(\mathcal{C})$ of sieves over all objects of $\mathcal{C}$.

Note that the assignment $[F\colon S \to \mathcal{C}/x] \mapsto x$ gives a functor $\pi\colon \mathfrak{Siev}(\mathcal{C}) \to \mathcal{C}$: this functor assigns to every sieve the object that the sieve is supposed to be covering. The existence of pullbacks is saying that this canonical map π is again a cartesian fibration.

Definition 3.5. Let $F\colon S \to \mathcal{C}/x$ be a sieve over x, and denote by $\sigma\colon \mathcal{C}/x \to \mathcal{C}$ the "source functor" sending $(y \to x) \mapsto y$. Recall that there is a functor $\mathcal{C}/-\colon \mathcal{C} \to \mathrm{Cat}_\infty$ sending y to the comma category $\mathcal{C}/y$.

Then we have a composition of functors

$$S \xrightarrow{F} \mathcal{C}/x \xrightarrow{\sigma} \mathcal{C} \xrightarrow{\mathcal{C}/-} \mathrm{Cat}_\infty.$$

We call this composition $F^{\mathrm{Cat}_\infty}\colon S \to \mathrm{Cat}_\infty$. For any $\mathcal{D} \in \mathrm{Cat}_\infty$ denote by $\kappa_{\mathcal{D}}\colon S \to \mathrm{Cat}_\infty$ the constant functor with value $\mathcal{D}$. Then there is an obvious natural transformation of functors $\alpha\colon F^{\mathrm{Cat}_\infty} \Rightarrow \kappa_{\mathcal{C}/x}$: for all $s \in S$ we have to construct a functor from $F^{\mathrm{Cat}_\infty}(s) = \mathcal{C}/\sigma(F(s))$ to $\mathcal{C}/x$, and we take the "compose with $F(s)$" functor.

In the following we will consider general functors $\mathfrak{B}\colon S \to \mathrm{Cat}_\infty$, where (S, F) is a sieve on x, together with a natural transformation $\beta\colon \mathfrak{B} \Rightarrow F^{\mathrm{Cat}_\infty}$. Note then that, for any object $s \in S$, we obtain a map of ∞-categories $\beta_s\colon \mathfrak{B}(s) \to F^{\mathrm{Cat}_\infty}(s) = \mathcal{C}/\sigma(F(s))$, and it will make sense to ask whether $(\mathfrak{B}(s), \beta_s)$ is a sieve on $\sigma(F(s))$.

Definition 3.6. A couple $(\mathfrak{B}, \beta)$ as above is *good* if for all $s \in S$ the couple $(\mathfrak{B}(s), \beta_s)$ is a sieve on $\sigma(F(s))$ (in particular, β_s is a cartesian fibration).

Note that, given a good couple $(\mathfrak{B}, \beta)$, we can construct a functor, sloppily also called $\mathfrak{B}\colon S \to \mathfrak{Siev}(\mathcal{C})$, such that the following square commutes

$$\begin{array}{ccc} S & \xrightarrow{F} & \mathcal{C}/x \\ \big\downarrow{\scriptstyle \mathfrak{B}} & & \big\downarrow{\scriptstyle \sigma} \\ \mathfrak{Siev}(\mathcal{C}) & \xrightarrow{\pi} & \mathcal{C}. \end{array}$$

Check that this is indeed an equivalent description of what a good couple is.

4. Axioms for Grothendieck topologies

A Grothendieck topology τ on $\mathcal{C}$ will now be a "collection" of sieves over $\mathcal{C}$. Again we have to agree what it means to be a collection. After Definition 3.3, I guess that the most natural choice is to take, for all objects $x \in \mathcal{C}$, a full subcategory τ_x of $\mathfrak{Siev}/x$, such that these full subcategories satisfy some conditions. After Definition 3.4, however, it seems reasonable to define τ directly as a full subcategory of $\mathfrak{Siev}(\mathcal{C})$.

Definition 4.1. A collection τ of sieves over $\mathcal{C}$ is a full subcategory $\tau \subseteq \mathfrak{Siev}(\mathcal{C})$.

Not every collection will be a Grothendieck topology! There are some axioms to be satisfied, which should be the analogues of the classical axioms (adapted to the ∞-setting by Lurie). Here they are. So we declare a collection τ to be a topology if it satisfies the following axioms.

4.1. **Axiom 1. Each sieve $\mathcal{C}/x = \mathcal{C}/x$ on $x \in \mathcal{C}$ is in τ.** This clearly corresponds to the classical axiom that $\mathcal{C}/x \subseteq \mathcal{C}/x$, seen as full subcategory, is a sieve on $\mathcal{C}/x$. Now $\mathcal{C}/x \to \mathcal{C}/x$ is seen as the identity cartesian fibration.

4.2. **Axiom 2. The topology is closed under pullbacks.** Classically this axiom says that if $S \subseteq \mathcal{C}/x$ is a τ-sieve on x and $f\colon y \to x$ is any morphism in $\mathcal{C}$, then the pullback $f^*S \subseteq \mathcal{C}/y$ is a τ-sieve on y. The pullback is classically defined as the sieve spanned, as a full subcategory of $\mathcal{C}/y$, by all arrows $z \to y$ such that one composition (hence any composition) $z \to y \xrightarrow{f} x$ is an object in S. Rephrased, one can consider the "composition with f" functor $f \circ -\colon \mathcal{C}/y \to \mathcal{C}/x$ and define f^*S as the preimage of S along this functor. Now the preimage is a form of fibre product, so we have, classically,

$$f^*S = S \times_{\mathcal{C}/x} \mathcal{C}/y.$$

Our new version of the axiom requires, again, that if $S \in \tau$, then $f^*S \in \tau$.

4.3. **Axiom 3. Coverings of coverings are coverings.** This is the most difficult axiom to generalise. Classically, we start with two sieves (S, F) and (T, G) on x, and we know that the first is in τ. Classically, to conclude that the second is in τ, we require that for all $s \in S$ of the form $s\colon y \to x$ the sieve s^*T is in τ. The problem now is that in our setting $s \in S$ is not immediately a morphism with target x; rather $F(s) \in \mathcal{C}/x$ is such. A mild requirement could be that for all $s \in S$ the sieve $F(s)^*T$ is in τ, but this seems not to use the sieve S enough, rather only its image in $\mathcal{C}/x$.

What we do is to consider an S-parametrised family of sieves over $\mathcal{C}$. This is precisely what a good functor gives us (see Definition 3.6.

The axiom becomes the following. Let $(S, F) \in \tau$ be a τ-sieve on x and let (T, G) be any sieve on x. Let $(\mathfrak{B}, \beta)$ be a good functor; in particular $\mathfrak{B}\colon S \to \mathrm{Cat}_\infty$ is a functor. Suppose that for all $s \in S$ the functor $\beta_s\colon \mathfrak{B}(s) \to \mathcal{C}/\sigma(F(s))$ makes $\mathfrak{B}(s)$ not only into a sieve (which follows from goodness), but also a sieve in τ. Finally, suppose that there is a commutative square in $Fun(S, \mathrm{Cat}_\infty)$ (in particular, there is an upper horizontal arrow such that the square can be filled)

$$\begin{array}{ccc} \mathfrak{B} & \dashrightarrow & \kappa_T \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\kappa_G} \\ F^{\mathrm{Cat}_\infty} & \xrightarrow{\alpha} & \kappa_{\mathcal{C}/x}. \end{array}$$

Then $T \in \tau$.

ASIDE: Maybe there is a direct way to convert a good functor $(\mathfrak{B}, \beta)$ into a sieve over x by using straightening/unstraightening: this should give the "universal" T to which we want to apply the axiom, and then, possibly, we need a further containment axiom stating that if $H\colon (S, F) \to (T, G)$ is a morphism in $\mathfrak{Siev}/x$ and $(S, F) \in \tau$, then also $(T, G) \in \tau_x$. The stated axiom should generalise both of these two.

ASIDE: Axiom 3 can be rephrased as follows: if (S, F) is τ-sieve on x and (T, G) is any sieve on x, then we can consider two functors $S \to \mathfrak{Siev}(\mathcal{C})$. The first is a good functor $\mathfrak{B}$, i.e. making the following diagram commute

$$\begin{array}{ccc} S & \xrightarrow{F} & \mathcal{C}/x \\ \downarrow{\scriptstyle\mathfrak{B}} & & \downarrow{\scriptstyle\sigma} \\ \mathfrak{Siev}(\mathcal{C}) & \xrightarrow{\pi} & \mathcal{C}. \end{array}$$

The second functor is constant equal to (T, G), so it makes the following diagram commute.

$$\begin{array}{ccc} S & \xrightarrow{F} & \mathcal{C}/x \\ \downarrow{\scriptstyle\kappa_T} & & \downarrow{\scriptstyle\kappa_x} \\ \mathfrak{Siev}(\mathcal{C}) & \xrightarrow{\pi} & \mathcal{C}. \end{array}$$

Suppose that there is a natural transformation of functors $\mathfrak{B} \Rightarrow \kappa_T$ in $Fun(S, \mathfrak{Siev}(\mathcal{C}))$ which is compatible, along π and F, with the obvious natural transformation of functors $\sigma \Rightarrow \kappa_x$ in $Fun(\mathcal{C}/x, \mathcal{C})$. Then $T \in \tau$.

Is there a simpler way to state this axiom?

5. Some examples of applications of the axioms

We analyse two examples showing that the axioms are meaningful. In the following we denote, for a topology τ and for $x \in \mathcal{C}$, by τ_x the full subcategory of $\mathfrak{Siev}/x$ spanned by sieves which are in τ. Note that the inclusion $\mathfrak{Siev}/x \subset \mathfrak{Siev}(\mathcal{C})$ is not fully faithful in general, hence also the inclusion $\tau_x \subseteq \tau$ is not fully faithful in general. We have actually that $\mathfrak{Siev}/x$ is the fibre over x of the target map $\pi\colon \mathfrak{Siev}(\mathcal{C}) \to \mathcal{C}$.

Example 5.1. Let τ be a Grothendieck topology; if $H\colon (T',G') \to (T,G)$ is a morphism in $\mathfrak{Siev}/x$ and $(T',G') \in \tau_x$, then also $(T,G) \in \tau_x$. To see this, consider the functor $\mathfrak{B}\colon \mathcal{C}/x \to \mathfrak{Siev}(\mathcal{C})$ given on objects by the formula

$$\mathfrak{B}(f\colon y \to x) = f^*S = S \times_{\mathcal{C}/x} \mathcal{C}/y$$

There is a natural transformation from $\mathfrak{B}$ to $\kappa_S\colon \mathcal{C}/x \to \mathfrak{Siev}(\mathcal{C})$, given objectwise by projection on the first coordinate in the last formula. Further composition with κ_H gives a natural transformation from $\mathfrak{B} \Rightarrow \kappa_T$, and one can check that this does the job.

To create with one sentence the maximum possible confusion, we note that the previous example is saying the following: $\tau_x \subseteq \mathfrak{Siev}/x$ is a Lurie-co-sieve, that is, $(\tau_x)^{op} \subseteq (\mathfrak{Siev}/x)^{op}$ satisfies the property for being a sieve in the sense of Lurie. Recall that Lurie defines a sieve on a category $\mathcal{D}$ to be a full subcategory $\mathcal{D}^{(0)}$ such that the inclusion $\mathcal{D}^{(0)} \subset \mathcal{D}$ is a cartesian fibration.

It is not true (and shouldn't be true) in general that if $H\colon (T',G') \to (T,G)$ is any morphism in $\mathfrak{Siev}(\mathcal{C})$ and $(T',G') \in \tau$, then (T,G) is also in τ: the example restricts to the case in which (T',G') and (T,G) are sieves over the same object x and H is a morphism over the identity of x. Roughly speaking, we want that extending a covering of x gives a new covering of x, but we don't want that if $f\colon y \to x$ is a map and we τ-cover y with (T',G'), then every covering (T,G) on x receiving a map from (T',G') is automatically in τ! Think of y being the emptyset, then we could take T' to be the empty sieve as well and we don't want to conclude that every sieve on x is in τ!

Draw the relevant diagrams and convince yourself that the example above does not generalise to the following statement: if $H\colon (T',G') \to (T,G)$ is any morphism in $\mathfrak{Siev}(\mathcal{C})$ and $(T',G') \in \tau$, then (T,G) is also in τ.

Example 5.2. Let (S,F) and (T,G) be two sieves over x. If both S,T are in τ_x, then also the "intersection" (or product) sieve $S \cap T = S \times_{\mathcal{C}/x} T$ is in τ_x. To see this, define $\mathfrak{B}\colon S \to \mathfrak{Siev}(\mathcal{C})$ by $s \mapsto F(s)^*T$, which is a τ-sieve over $\sigma(F(s))$. The natural transformation $\mathfrak{B} \Rightarrow \kappa_{S\cap T}$ filling the square is given on objects of S as follows.

Let $s \in S$: to construct a functor $F(s)^*T \to S \times_{\mathcal{C}/x} T$ it suffices to construct functors $F(s)^*T \to S$ and $F(s)^*T \to T$ which are compatible over $\mathcal{C}/x$. Recall that $F(s)^*T = \mathcal{C}/\sigma(F(s)) \times_{\mathcal{C}/x} T$, where the functor $\mathcal{C}/\sigma(F(s)) \to \mathcal{C}/x$ is given by $F(s) \circ -$.

The functor $F(s)^*T \to T$ is easily constructed: we take the projection to T, using the last formula. The other functor $F(s)^*T \to S$ is more involved. First we use the projection to $\mathcal{C}/\sigma(F(s))$ to map $F(s)^*T$ to $\mathcal{C}/\sigma(F(s))$.

Now we use that $S \to \mathcal{C}/x$ is a cartesian fibration: if $(z \to \sigma(F(s)))$ is an object in $\mathcal{C}/\sigma(F(s))$, then composing with $F(s)$ gives an arrow $(z \to x) \to F(s)$ in $\mathcal{C}/x$, and we can pullback s, which evidently lies over $F(s)$, to another object $s' \in S$ lying over $(z \to x)$. We then set the functor $\mathcal{C}/\sigma(F(s)) \to S$ by sending $(z \to y) \mapsto s'$.

The two functors $F(s)^*T \to S$ and $F(s)^*T \to T$ constructed are compatible over $\mathcal{C}/x$, so we get a functor $F(s)^*T \to S \times_{\mathcal{C}/x} T$.

Check the details!

Note that the previous example can be generalised to the following statement: τ_x, as full subcategory of $\mathfrak{Siev}/x$, is closed under finite products. (the example discusses the case of products of two objects). Recall that $\mathfrak{Siev}/x$ is admits all small limits, in particular the (coherent) nerve of $\mathfrak{Siev}/x$ is weakly contractible. **Is also the coherent nerve of τ_x contractible? This would follow from τ_x being cofiltered, which would in turn follow from τ_x being closed under finite limits. Are equalizers in $\mathfrak{Siev}/x$ of objects in τ_x automatically in τ_x?**

In light of the previous examples, one can give the following definition.

Definition 5.3. Given a collection of sieves $\Xi \subset \mathfrak{Siev}(\mathcal{C})$, we define $\tau(\Xi)$ as the *smallest* topology on $\mathcal{C}$ containing all these sieves, i.e., the intersection of all topologies that contain these sieves. Note that a topology τ is a full subcategory of $\mathfrak{Siev}(\mathcal{C})$, such that τ is equal to the essential image of the inclusion $\tau \to \mathfrak{Siev}(\mathcal{C})$ (this needs a little check, but follows from the pullback axiom). Hence talking of "intersection of topologies" makes perfectly sense.

6. Definition of sheaf

Given a topology τ on $\mathcal{C}$, we consider presheaves $\mathfrak{F}$ on $\mathcal{C}$ with values in the ∞-category An: if you prefer, you can replace An by any ∞-category which is complete and cocomplete (just to be sure).

Given a presheaf $\mathfrak{F}$ on $\mathcal{C}$, an object $x \in \mathcal{C}$ and a sieve (S, F) on x, we have two obvious functors $S^{op} \to \mathrm{An}$: the first is the constant functor with value $\mathfrak{F}(x)$, the second is the functor $s \mapsto \mathfrak{F}(\sigma(F(s)))$. There is a natural transformation between the functors, given by applying $\mathfrak{F}$ to the natural transformation between the two functors $S \to \mathcal{C}$ given by $\sigma(F)$ and x respectively. The first, naive (and unfortunately not very useful) definition of sheaf is the following.

Definition 6.1. A naive sheaf on $(\mathcal{C}, \tau)$ is a presheaf $\mathfrak{F}$ such that the following holds: for each $x \in \mathcal{C}$ and each sieve $(S, F) \in \tau_x$, the following composition is an equivalence.

$$\mathfrak{F}(x) \to \lim_{s \in S^{op}} \mathfrak{F}(x) \to \lim_{s \in S^{op}} \mathfrak{F}(\sigma(F(s))).$$

Here the first map is given by the very definition of limit, and the second map is given by the natural transformation described above.

We introduce the notation $\mathfrak{F}(S) := \lim_{s \in S^{op}} \mathfrak{F}(\sigma(F(s)))$, and think of it as the "space of sections of $\mathfrak{F}$ over the sieve S".

Why is this definition too naive? Look at the following example (or first try to think yourself what can go wrong).

Example 6.2. Let $(S, F) \in \tau_x$; then we can construct a new sieve $(S \sqcup S, F)$ over x, which is essentially given by taking two disjoint copies of S and mapping both of them to $\mathcal{C}/x$ along F. There are two inclusions $S \hookrightarrow S \sqcup S$ and a fold map $S \sqcup S \to S$, all being morphisms in $\mathfrak{Siev}/x$.

If $\mathfrak{F}$ is a sheaf, then we must have $\mathfrak{F}(x) \simeq \mathfrak{F}(S)$, but also $\mathfrak{F}(x) \simeq \mathfrak{F}(S \sqcup S)$. The latter is easily identified with $\mathfrak{F}(S)^2$, and it is easy to check that not only the animae $\mathfrak{F}(S)$ and $\mathfrak{F}(S)^2$ must be abstractly equivalent, but really the diagonal map $\mathfrak{F}(S) \to \mathfrak{F}(S)^2$ and both projections $\mathfrak{F}(S)^2 \to \mathfrak{F}(S)$ must be equivalences. This happens only if $\mathfrak{F}(x) \simeq \mathfrak{F}(S) \simeq *$.

Sad conclusion: a naive sheaf is equivalent to the point presheaf $\mathfrak{F} \equiv *$.

The previous example shows that we should relax the condition that the canonical map $\mathfrak{F}(x) \to \mathfrak{F}(S)$ is an equivalence. Observe that in the case discussed in the example we have that $\mathfrak{F}(S)$ is a retract of $\mathfrak{F}(S \sqcup S)$; so if $\mathfrak{F}(x) \simeq \mathfrak{F}(S)$, then at least $\mathfrak{F}(x)$ is a retract of $\mathfrak{F}(S \sqcup S)$. These considerations lead us to the following two definitions.

Definition 6.3. A weak sheaf $\mathfrak{F}$ is a presheaf such that for all $(S, F) \in \tau_x$ the natural map $\mathfrak{F}(x) \to \mathfrak{F}(S)$ is the inclusion of a retract.

Definition 6.4. A strong sheaf $\mathfrak{F}$ is a presheaf such that for all $(S, F) \in \tau_x$ there exists a refinement $(T, G) \in \tau_x$ (i.e. there is a morphism $(T, G) \to (S, F)$ in $\tau_x \subseteq \mathfrak{Siev}/x$), such that the natural map $\mathfrak{F}(x) \to \mathfrak{F}(T)$ is an equivalence.

Check that a strong sheaf is also a weak sheaf.

The other possible definition of sheaf comes from what we expect the sheafification functor should look like. Given a presheaf $\mathfrak{F} \in \mathcal{P}_{\mathcal{C}}$, we would like to mimic the classical definition (for presheaves over topological spaces) and define its sheafification $Sh(\mathfrak{F})$ as the presheaf whose value at $x \in \mathcal{C}$ is

$$Sh(\mathfrak{F})(x) = \mathrm{colim}_{(S,F) \in (\tau_x)^{op}} \mathfrak{F}(S).$$

That is, first we take sections of $\mathfrak{F}$ over S, then we refine S and pass to the colimit.

Definition 6.5. A genuine sheaf $\mathfrak{F}$ is a presheaf such that any of the following equivalent conditions is satisfied:

- the canonical map $\mathfrak{F} \to Sh(\mathfrak{F})$ is an equivalence (i.e., objectwise equivalence): that is, for all $x \in \mathcal{C}$ the canonical map $\mathfrak{F}(x) \to \mathrm{colim}_{(S,F) \in (\tau_x)^{op}} \mathfrak{F}(S)$ is an equivalence;
- for all $x \in \mathcal{C}$ and all $(T, G) \in \tau_x$ the canonical map $\mathfrak{F}(x) \to \mathrm{colim}_{(S,F) \in (\tau_x/(T,G))^{op}} \mathfrak{F}(S)$ is an equivalence;

- for all $x \in \mathcal{C}$ there exists $(T, G) \in \tau_x$ such that the canonical map $\mathfrak{F}(x) \to \operatorname{colim}_{(S,F)\in(\tau_x/(T,G))^{op}} \mathfrak{F}(S)$ is an equivalence;

Check that the three conditions are indeed equivalent!
Is it true that a strong sheaf is also genuine, and that a genuine sheaf is also weak?
Is it true that sheafification is a left Bousfield localisation on the category $\mathcal{P}_{\mathcal{C}}$? Is it even true that the canonical map $Sh(\mathfrak{F}) \to Sh(Sh(\mathfrak{F}))$ is an equivalence?

7. Weiss topologies

At some point we want to deal with Weiss topologies on Man. Recall that $\mathrm{Disc}_k \subset \mathrm{Man}$ is the full subcategory spanned by objects of the form $\coprod_h \mathbb{R}^n$ with $0 \le h \le k$. Recall that for a presheaf $\mathfrak{F} \in \mathcal{P}_{\mathrm{Man}}$ we have defined $\mathcal{T}_k\mathfrak{F}$ as the composition of the restriction $U_k\colon \mathcal{P}_{\mathrm{Man}} \to \mathcal{P}_{\mathrm{Disc}_k}$ with the right Kan extension $Ran_{\iota_k}\colon \mathcal{P}_{\mathrm{Disc}_k} \to \mathcal{P}_{\mathrm{Man}}$. This definition makes perfect sense in the context of ∞-categories, and is even easier than the original one using model categories and derived mapping spaces and so on, at least for me.

Our problem was to find an ∞-categorical analogue of the Weiss topologies. This was hard because the topology τ_k on Man (as a topologically enriched category) has a very set-theoretic and homotopy-dependent definition.

Recall that a sieve S on $x \in \mathrm{Man}$, in the sense of set-theoretic subpresheaf of $\mathrm{Man}(-, x)$, is in τ_k if for all subset $J \subset x$ of cardinality $\le k$ there is $y \in \mathrm{Man}$ and $f\colon y \to x$ such that the image of the embedding f hits all points of J.

In this set-theoretic sense, there is a particular example of a sieve in τ_k over x: the collection of all embeddings $y \to x$, for varying y, that factor through the embedding of a manifold $z \in \mathrm{Disc}_k$ in x. First, this is a sieve on x: if $f\colon y \to x$ factors through some $z \to x$, then so does every $y' \to x$ obtained by first embedding $y' \to y$ and then $y \to x$ along f. Second, this sieve is in τ_k, since for every collection of $\le k$ points in x it is easy to find $\le k$ discs that embed into x and hit those points. Third, this sieve is usually non-trivial. For example every embedding $y \to x$ in the sieve tends to be (as a continuous map) null-homotopic.

Note that for $y \to x$ to lie in the given sieve, we ask the *existence* of $z \to x$ through which $y \to x$ factors, but not the *choice*. Indeed in the classical sense, the morphism $y \to x$ can only *either be in the sieve, or not*, but if it is, it doesn't make sense to ask *how $y \to x$ is in the sieve.* This additional information can be given with our brand new notion of sieve.

Definition 7.1. We regard Man as an ∞-category and define, for all objects $x \in \mathrm{Man}$, a the sieve $S_k^x \in \mathfrak{Siev}/x$. An *object* in S_k^x is a triangle $y \to z \to x$ in Man, with $y \in \mathrm{Man}$ and $z \in \mathrm{Disc}_k$. Such triangles span a full subcategory of $Fun(\Delta^2, \mathrm{Man})$, and we call this category S_k^x. The projection $F_k^x\colon S_k^x \to \mathrm{Man}/x$ is given by forgetting z.

Check that F_k^x is a cartesian fibration.

Now we would like to define τ_k as the topology generated by the sieves S_k^x for x varying in Man.

Example 7.2. If $f\colon x \to x'$ is an embedding, then $f^*S_k^{x'}$ can be described as the category of commutative squares in Man of the following form, where $y \in \mathrm{Man}$ and $z \in \mathrm{Disc}_k$:

$$\begin{array}{ccc} y & \longrightarrow & x \\ \big\downarrow & & \big\downarrow f \\ z & \longrightarrow & x'. \end{array}$$

Again, check that this description is correct and that this full subcategory of $Fun(\Delta^1 \times \Delta^1, \mathrm{Man})$ is a sieve on x', where the projection sends the above square to $y \to x'$.

There is a functor from S_k^x to $f^*S_k^{x'}$, which expands a triangle $y \to z \to x$ to a square as above by composing $z \to x$ with f to obtain a map $z \to x'$, and then forgets the "antidiagonal" map $z \to x$.

This shows that $f^*S_k^{x'}$ can be witnessed to be a sieve in τ_k both because it is a pullback of a generator, and because it is an extension of another generator.

We now note that if $\mathfrak{F}$ is obtained as right Kan extension of a presheaf defined over Disc_k, then for all $x \in \mathrm{Man}$ we have

$$\mathfrak{F}(x) = \lim_{(z \to x) \in (\mathrm{Disc}_k / x)^{op}} \mathfrak{F}(z) = \lim_{(y \to z \to x) \in (S_k^x)^{op}} \mathfrak{F}(z) = \mathfrak{F}(S_k^x).$$

The first equality is by definition of right Kan extension, the second is because the objects of the form $(z = z \to x)$ span a *final* subcategory of S_k^x which is isomorphic to Disc_k/x, hence the same objects span an *initial* subcategory of $(S_k^x)^{op}$, which is isomorphic to $(\mathrm{Disc}_k/x)^{op}$ and is equally good to compute the limit.

This is pointing in the right direction: if $\mathfrak{F}$ is obtained as right Kan extension of a presheaf defined over Disc_k, then it behaves like a sheaf at least for the sieves S_k^x of the topology τ_k.

Can we conclude that $\mathfrak{F}$ is a weak/genuine/strong sheaf for τ_k?

8. Exercise

Now one should check whether or not it is true that there is a commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}_{\mathrm{Man}} & \xrightarrow{Sh_{\tau_k}} & Sh(\mathrm{Man}, \tau_k) & \xrightarrow{\subseteq} & \mathcal{P}_{\mathrm{Man}} \\
\| & & \simeq \downarrow U_k & & \| \\
\mathcal{P}_{\mathrm{Man}} & \xrightarrow{U_k} & \mathcal{P}_{\mathrm{Disc}_k} & \xrightarrow{Ran_{\iota_k}} & \mathcal{P}_{\mathrm{Man}}.
\end{array}$$

Actually, we still have to check that $Sh_{\tau_k} \colon \mathcal{P}_{\mathrm{Man}} \to \mathcal{P}_{\mathrm{Man}}$ really lands in the subcategory $Sh(\mathrm{Man}, \tau_k)$ of sheaves...

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