Topics in Geometry 2021—Exercises

Niels Martin Møller and Andrea Bianchi

September 9, 2021

Week 1 — deadline on Tuesday, September 14

Exercise 1. Let c be the circle of radius r in $\mathbb{R}^2 \subset \mathbb{R}^3$ centered at the point $p = (0, R.0)$, with $R > r > 0$, and let $\mathcal{S}$ be the surface of revolution obtained by rotating c around the x axis.

(i) Find sufficient conditions on R and r so that the following holds: the mean curvature at each point $p \in \mathcal{S}$ is different from 0.[1]

(ii)* Is there a closed subsurface $\mathcal{S} \subset \mathbb{R}^3$ of genus ≥ 2, such that the mean curvature is non-zero at each point of $\mathcal{S}$?

Exercise 2. Let $\mathcal{S}$ be an abstract 2-surface. Two Riemannian metrics g_1 and g_2 on $\mathcal{S}$ are *conformally equivalent* if there is a positive function $\lambda\colon \mathcal{S} \to \mathbb{R}$ such that $\lambda g_1 = g_2$ (in typical notation $\lambda = \Omega^2$). Conformal equivalence is an equivalence relation on Riemannian metrics on $\mathcal{S}$, and a *conformal structure* on $\mathcal{S}$ is an equivalence class of such metrics.

(i) Give an example of a surface $\mathcal{S}$ and two non-conformally equivalent metrics g_1 and g_2 on $\mathcal{S}$.

An *almost complex structure* on $\mathcal{S}$ is a smooth choice of an endomorphism $J_p\colon T_p\mathcal{S} \to T_p\mathcal{S}$ for all $p \in \mathcal{S}$, such that $J_p^2 = -\mathrm{Id}_{T_p\mathcal{S}}$.

(ii) What does the word smooth mean in the previous definition? Of which bundle over $\mathcal{S}$ is J a section?

(iii) Show that an almost complex structure induces an orientation on $\mathcal{S}$.

An almost complex structure J is compatible with a Riemannian metric g on $\mathcal{S}$ if J_p is an isometry of $(T_p\mathcal{S}, g_p)$ for all $p \in \mathcal{S}$.

[1] The mean curvature is only defined up to a sign, because..., so being non-zero is well-defined independently of...

(iv) Show that two metrics g_1 and g_2 which are compatible with the same almost complex structure J on $\mathcal{S}$ are conformally equivalent.

(v) Conversely, suppose that $\mathcal{S}$ is oriented and let g be a Riemannian metric on $\mathcal{S}$. Show that there exists precisely one almost complex structure J on $\mathcal{S}$ which is compatible both with the orientation and with the metric g.

This shows that, for $\mathcal{S}$ oriented, an orientation-compatible almost complex structure and a conformal structure are equivalent structures on $\mathcal{S}$.

Let now $\underline{x} = \underline{x}(u, v) \colon U \subset \mathbb{R}^2 \to V \subset \mathcal{S}$ be a local parametrisation of $\mathcal{S}$, and assume that $\mathcal{S}$ is endowed with a Riemannian metric g, which in coordinates reads $g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, where E, F, G are real-valued functions of u, v. Recall that $\underline{x}$ is an *isothermal parametrisation* if $E \equiv G$ and $F \equiv 0$.

(vi) Assume that $\mathcal{S}$ is oriented and let $\underline{x}$ be a local, orientation-preserving and isothermal parametrisation. Let J be the almost complex structure on $\mathcal{S}$ associated with g. How does J read in local coordinates?

(vii) Conclude: if $\underline{x} \colon U \to V \subset \mathcal{S}$ and $\underline{x}' \colon U' \to V$ are local parametrisations of the same open subset of $\mathcal{S}$, then $\underline{x}^{-1} \circ \underline{x}'$ is a holomorphic map.

Thus the following two statements are one a reformulation of the other (and are both true!):

- Every oriented Riemannian surface $(\mathcal{S}, g)$ admits an atlas of isothermal charts;
- Every surface $\mathcal{S}$ endowed with an almost complex structure J admits an atlas promoting it to a Riemann surface.

Topics in Geometry 2021—Exercises

Niels Martin Møller and Andrea Bianchi

September 15, 2021

Week 2 — deadline on Tuesday, September 21

Exercise 2.1. Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular surface diffeomorphic to an (open) Möbius band.

(i) Show that there is a point $p \in \mathcal{S}$ with vanishing mean curvature (why is *vanishing mean curvature* a well-defined notion?).

(ii)* Find an example of a Möbius band $\tilde{\mathcal{S}} \subset \mathbb{R}^3$ such that for all $p \in \tilde{\mathcal{S}}$ the Gauss curvature does not vanish.

(iii)* Find an example of a Möbius band $\tilde{\mathcal{S}} \subset \mathbb{R}^3$ such that no point $p \in \tilde{\mathcal{S}}$ is umbilical.

Exercise 2.2. Let $\mathcal{S} \subset \mathbb{R}^3$ be a closed, embedded, orientable[1] surface of genus different from 1. Prove that $\mathcal{S}$ must have an umbilical point.

(i) Define a vector-up-to-sign field as a continuous assignment $p \mapsto \pm w_p$ of a couple of opposite vectors in $T_p\mathcal{S}$ for all $p \in \mathcal{S}$: formally, this is a section of the *fibre bundle* $T\mathcal{S}/\pm 1$, obtained by identifying fibrewise opposite vectors; the fibre of this fibre bundle is $T_p\mathcal{S}$, which is topologically again a 2-plane, but geometrically more a cone. Convince yourself that this is indeed a fibre bundle with local trivialisations.

(ii) For all $p \in \mathcal{S}$ we can decompose $T_p\mathcal{S}$ as an orthogonal direct sum of the two eigenlines of the shape operator, corresponding to the two distinct eigenvalues $k_1(p) < k_2(p)$ (the principal curvatures). Prove that there is a continuous vector-up-to-sign field $\pm w$ on $\mathcal{S}$ withouot zeroes, assigning to each $p \in \mathcal{S}$ a couple of opposite, non-zero eigenvectors $\pm w_p$ for the maximal eigenvalue $k_2(p)$ (why can the *maximal eigenvalue* be continuously defined on all $\mathcal{S}$?).

[1] Bonus exercise: a closed subsurface of $\mathbb{R}^3$ is automatically orientable!

(iii) Prove that $\chi(\mathcal{S}) = 0$: you can either define a suitable notion of index for vector-up-to-sign fields and prove a version of Poincare-Hopf, or you can consider the double cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$, containing all couples of the form $(p.v)$ where $v \in \{\pm w_p\}$ and estimate $\chi(\tilde{\mathcal{S}}) = 2\chi(\mathcal{S})$ using the classical Poincare-Hopf.

(iv) Find an example of a torus in $\mathbb{R}^3$ with no umbilical point.

Exercise 2.3. Let $M \subset \mathbb{R}^4$ be an embedded 3-manifold.

(i) Define the shape operator $W_p : T_p \to T_p$ with the help of a unit vector field N which has values in $\mathbb{R}^4$ and is normal to M (this is only locally defined, and there are locally 2 possible choices for N).

(ii) Define an umbilical point of M as a point $p \in M$ for which $W_p = \lambda \mathrm{Id}_{T_pM}$. Check that the theorem of Hopf still works: if M is connected and all points of M are umbilical, then M is contained in a hyperplane or a sphere.

Exercise 2.4. For each integer $n \in \mathbb{Z}$ find a vector field w on $\mathbb{R}^2$ having an isolated zero at $0 \in \mathbb{R}^2$ of index n: write an explicit formula for w, depending on n, and draw some pictures.[2]

[2] You don't need to attach the pictures, if this is problematic!

Topics in Geometry 2021—Exercises

Niels Martin Møller and Andrea Bianchi

September 24, 2021

Week 3 — deadline on Tuesday, September 28

Exercise 3.1.

Let $\mathcal{S}$ be an oriented surface [1], and let g_1 and g_2 be two conformal metrics on $\mathcal{S}$, i.e. $g_1 = \lambda g_2$ for some smooth function $\lambda\colon \mathcal{S} \to \mathbb{R}^+$. In this exercise we prove that the laplacian $\Delta\colon \Omega^0(\mathcal{S}) \to \Omega^0(\mathcal{S})$ is the same with respect to the metrics g_1 and g_2. Here $\Omega^0(\mathcal{S}) = C^\infty(\mathcal{S})$ denotes the vector space of smooth functions (0-forms) on $\mathcal{S}$. In order to be precise during the exercise, we denote the two a priori different Laplacians by Δ_{g_i}, for $i = 1, 2$.

(i) Look up the definition of the Hodge star operator $*_{g_i,p}\colon T_p^*(\mathcal{S}) \to T_p^*(\mathcal{S})$ on the cotangent space of each $p \in \mathcal{S}$, and compare $*_{g_i,p}$ with (the dual of) J_p, where J is the almost complex structure associated with both g_1 and g_2. Conclude that $*_{g_1}$ and $*_{g_2}$ are the same map $\Omega^1(\mathcal{S}) \to \Omega^1(\mathcal{S})$.

(ii) Look up also the definition of the star operator $*_{g_i,p}\colon \Lambda^2(T_p^*(\mathcal{S})) \to \mathbb{R}$. Is this operator the same for g_1 and g_2?

(iii) Recall that the Laplacian on 0-forms is defined[2] as $*d*d$, where $d\colon \Omega^0(\mathcal{S}) \to \Omega^1(\mathcal{S})$ is the usual differential (which doesn't even need a metric to be defined). Prove that $\Delta_{g_1} \equiv \lambda \Delta_{g_2}$.

(iv) Show that in two dimensions, being a harmonic function is conformally invariant.

(v)* Liouville's Theorem for harmonic functions on Euclidean space states: If $u : \mathbb{R}^n \to \mathbb{R}$ satisfies $\Delta_{\delta_{ij}} u = 0$ and $u \geq 0$ everywhere, then $u = 0$. Here δ_{ij} denotes the (coefficients of the) Euclidean metric on $\mathbb{R}^n$.

Show that Liouville's theorem fails on hyperbolic space $\mathbb{H}^n$, by giving examples of bounded harmonic functions there.

[Hint: First do the $n = 2$ case in the upper half-plane model $\mathbb{H}^2 = (\mathbb{R} \times \mathbb{R}_+, \frac{1}{y^2}\delta_{ij})$, by using Part (iv) and picking a suitable meromorphic function $f : \mathbb{C} \to \mathbb{C}$.]

[1] We assume orientability and fix once and for all an orientation on $\mathcal{S}$ for simplicity

[2] At least this is a possible definition

Exercise 3.2. Let $\mathcal{S}$ be an oriented, closed surface of genus g with a Riemannian metric g [3]. Let $\gamma\colon S^1 \to \mathcal{S}$ be a smooth immersion, possibly with self-intersections, and assume for simplicity that γ is self-transverse and has no triple point [4]. A *bigon* for γ is a choice of two disjoint arcs $[a,b],[c,d] \subset S^1$ such that $\gamma(a) = \gamma(c)$, $\gamma(b) = \gamma(d)$, $\gamma([a,b])$ and $\gamma([c,d])$ are embedded arcs in $\mathcal{S}$, disjoint away from their endpoints, and bounding together a topological disc in $\mathcal{S}$. A *monogon* for γ is a choice of one arc $[a,b] \subset S^1$ such that $\gamma(a) = \gamma(b)$, γ is otherwise injective on $[a,b]$ and the closed curve $\gamma[a,b] \subset \mathcal{S}$ bounds a disc in $\mathcal{S}$. Note that if γ admits a bigon or a monogon, then we can homotope γ to a new immersed, self-transverse curve with no triple point $\gamma'\colon S^1 \to \mathcal{S}$, so that γ' has fewer self-intersections than γ. You can assume without proof the following theorem[5]:

> **Theorem** If γ is immersed, self-transverse and not embedded, but it is isotopic to an embedded curve, then γ admits at least one bigon or one monogon.

(i) Let δ be an embedded, non-nullhomotopic curve in $\mathcal{S}$ and let δ' be a (geodesic) curve in the homotopy class of δ that minimizes the length (not only locally). Prove that δ' is embedded.

(ii) Find an example[6] of a surface $\mathcal{S}$ with a Riemannian metric g, of an embedded, non-nullhomotopic curve δ in $\mathcal{S}$, and of an immersed, but not embedded geodesic $\delta'\colon S^1 \to \mathcal{S}$, such that δ and δ' are homotopic to each other, δ' is strictly locally minimising the length in the space of curves homotopic to δ[7], but δ' is not a global minimum of the length in the space of curves homotopic to δ.

Exercise 3.3. Let $A = S^1 \times [0,1]$ be the standard annulus, with local coordinates (u,v), u only defined locally. Use the standard orientation on A.

(i) Convince yourselves that both ∂_u and ∂_v are well-defined vector fields on A, giving for each $p \in A$ a basis of the tangent plane T_pA.

(ii) Consider an immersion $\iota\colon A \to \mathbb{R}^3$. Use the vectors fields $\iota_*(\partial_u)$, $\iota_*(\partial_v)$ and the oriented normal N to ι to define a map $e_\iota\colon A \to GL_3(\mathbb{R})$.

(iii) Use that $\pi_1(GL_3(\mathbb{R})) \cong \mathbb{Z}_2$, generated for example by a 360 degree gradual rotation around the x axis, and find two immersions $\iota_1, \iota_2\colon A \to \mathbb{R}^3$ which are not isotopic.

[3] Sorry for using the letter g twice!

[4] This can be achieved by small perturbations of γ.

[5] Very brief sketch of proof: by small perturbation make the homotopy $H\colon S^1 \times I \to \mathcal{S}$ smooth and self-transverse. Analyse then what happens at the finitely many times $t \in I$ for which $\gamma_t = H(-,t)$ is not immersed or not self-transverse.

[6] Here you can also be qualitative, e.g. make a picture, you don't have to find explicit formulas.

[7] In other words, every small perturbation of δ' has length bigger than δ'.

Topics in Geometry 2021—Exercises

Niels Martin Møller and Andrea Bianchi

September 30, 2021

Week 4 — deadline on October 5th

Exercise 4.1. *"The Artist's Problem (Il Problema dell'Artista)"*
Jesper Grodal's artist friend from primary school has been poking membranes with a stick (see Figure) and asks questions about special surfaces he read about: "*Is it a minimal surface? It looks like a pseudosphere to me, so is it a constant negative Gauß curvature surface too?*".

Figure 1: How to artist: Poke a stick into a membrane.

(i) Help the artist by (more generally) classifying the complete surfaces $S \subseteq \mathbb{R}^3$ with both constant mean and Gauß curvatures. I.e. suppose that there exist constants $c_1, c_2 \in \mathbb{R}$ such that for all points $p \in S$ holds $H(p) = c_1$ and $K(p) = c_2$.

(ii)* What happens in Part (i) if we allow $\partial S \neq \emptyset$ or a possible singular point? (As needed for the artist's work.)

Exercise 4.2 (3.2 plus bonus). *In the entire exercise we only consider homotopy, and not regular homotopy/isotopy/homotopy through immersions as equivalence relation between curves. I have improved the definitions of monogon and bigon to two definitions that make the theorem true.* Let $\mathcal{S}$ be an oriented, closed surface of genus g with a Riemannian metric g [1]. Let $\gamma\colon S^1 \to \mathcal{S}$ be a smooth immersion, possibly with self-intersections, and assume for simplicity that γ is self-transverse [2]. A *bigon* for γ is a choice of two disjoint arcs $[a, b], [c, d] \subset S^1$ such that $\gamma(a) = \gamma(c)$, $\gamma(b) = \gamma(d)$, and such that the induced closed curve

$$\gamma\colon ([a, b] \cup [c, d])/\{a \equiv c, b \equiv d\} \to \mathcal{S}$$

obtained by glueing the restrictions of γ to $[a, b]$ and $[c, d]$ is null-homotopic. A *monogon* for γ is a choice of one arc $[a, b] \subset S^1$ such that $\gamma(a) = \gamma(b)$ and the induced curve

$$\gamma\colon [a, b]/\{a \equiv b\} \to \mathcal{S}$$

is null-homotopic[3]. Note that if γ admits a bigon or a monogon, then we can homotope γ to a new immersed, self-transverse curve $\gamma'\colon S^1 \to \mathcal{S}$, so that γ' has fewer self-intersections than γ. You can assume without proof the following theorem:

> **Theorem** If γ is immersed, self-transverse and not embedded, but it is homotopic to an embedded curve, then γ admits at least one bigon or one monogon.

(i) Let $\alpha\colon S^1 \to \mathcal{S}$ be a *multiple of a simple closed curve*, i.e. there is an integer $k \geq 2$ and another embedded curve $\alpha'\colon S^1 \to \mathcal{S}$ such that α is the composition

$$S^1 \xrightarrow{\cdot k} S^1 \xrightarrow{\alpha'} \mathcal{S}.$$

Suppose that α' is not null-homotopic. Prove that α is neither null-homotopic nor homotopic to an embedded curve, by exhibiting a small perturbation of α which is immersed, self-transverse and has no monogons and no bigons.

(ii) Prove that a closed, non-constant geodesic on $\mathcal{S}$ is always a multiple of a self-transverse (but possibly not embedded) geodesic.

(iii) Let δ be an embedded, non-nullhomotopic curve in $\mathcal{S}$ and let δ' be a geodesic curve in the homotopy class of δ that minimizes the length in the entire homotopy class of δ. Prove that δ' is embedded.

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[1] Sorry again for using the letter g twice!

[2] This can be achieved by small perturbations of γ.

[3] We don't require anymore that either induced curve is injective and bounds a disc in $\mathcal{S}$.

(iv) Find an example[4] of a surface $\mathcal{S}$ with a Riemannian metric g, of an embedded, non-nullhomotopic curve δ in $\mathcal{S}$, and of an immersed, but not embedded geodesic $\delta' : S^1 \to \mathcal{S}$, such that δ and δ' are homotopic to each other, δ' is strictly locally minimising the length in the space of curves homotopic to δ[5], but δ' is not a global minimum of the length in the space of curves homotopic to δ.

Exercise 4.3 In this exercise we only consider non-constant geodesics. We study the existence of *non-periodic geodesics* on closed (hence, in particular, complete) connected Riemannian manifolds M. A *periodic* geodesic $\gamma : \mathbb{R} \to M$ is a geodesic that factors through a quotient $\mathbb{R} \to \mathbb{R}/\mathbb{Z}\ell \cong S^1$, for some $\ell > 0$.

(i) Let M be the standard, round sphere S^n: prove that all geodesics are periodic.

(ii) Find an example of M not simply connected, such that all geodesics on M are periodic.

(iv) Is it true that if M has the property that all geodesics are periodic, then M must be a round sphere? Give a motivated answer! The footnote contains a hint[6].

(v) Consider the torus $\mathbb{R}^2/\mathbb{Z}^2$ with the Euclidean metric. State and prove a characterisation of closed geodesics on the torus containing the words "rational slope".

(vi)* Prove that any Riemannian surface $\mathcal{S}$ of genus 1 admits a non-periodic geodesic; this is for example in contrast with the genus 0, round case.

[4] Here you can also be qualitative, e.g. make a picture, you don't have to find explicit formulas.

[5] In other words, every small perturbation of δ' has length bigger than δ'.

[6] Hint: Fubini-Study.

Topics in Geometry 2021—Exercises

Niels Martin Møller and Andrea Bianchi

October 7, 2021

Week 5 — deadline on October 12th

Exercise 5.1 (4.3 expanded). The aim of this exercise is to show that every Riemannian metric g on the torus $T := \mathbb{R}^2/\mathbb{Z}^2$ admits a non-periodic geodesic. As we have seen, this is in contrast with what happens on other manifolds, that admit metrics all of whose geodesics are periodic. In the entire exercise *geodesic* means *non-constant geodesic*.

Identify $\pi_1(T)$ with $\mathbb{Z}^2$ in the canonical way. Since $\pi_1(T)$ is an abelian group, the set of homotopy classes of curves in T is in natural bijection with $\mathbb{Z}^2$; e.g. the homotopy class of null-homotopic curves corresponds to $(0,0)$. For $(c,d) \in \mathbb{Z}^2$ and a curve $\tilde{\gamma}$ in $\mathbb{R}^2$ we write $\tilde{\gamma} + (c,d)$ for the translate curve by (c,d).

Use the theorem from the lecture to find, for all (a,b) with $(a,b) \neq (0,0)$, a closed geodesic $\gamma_{(a,b)}$ on T in the homotopy class (a,b) of minimal length $\ell_{(a,b)} > 0$.

(i) Prove that for all integer $k \in \mathbb{Z} \setminus \{0\}$, the curve $\gamma_{(a,b)}$ run k times[1] is a length minimiser in the homotopy class (ka,kb), by showing that anyway any curve in the homotopy class (ka,kb) has length at least $|k|\ell_{(a,b)}$.

Lift $\gamma_{(a,b)}$ to a map $\tilde{\gamma}_{(a,b)} \colon \mathbb{R} \to \mathbb{R}^2$, such that $\tilde{\gamma}_{(a,b)}(0) \in [0,1]^2$ (why is it possible?), and reparametrise $\tilde{\gamma}_{(a,b)}$ by arc length (so in the following $\tilde{\gamma}_{(a,b)}$ is assumed to be parametrised by arc length).

(ii) Prove that $\tilde{\gamma}_{(a,b)}$ minimises the length between any two of its points, i.e., for all $s \leq t \in \mathbb{R}$, the Riemannian distance between $\tilde{\gamma}_{(a,b)}(s)$ and $\tilde{\gamma}_{(a,b)}(t)$ is precisely the length of the geodesic segment $\tilde{\gamma}_{(a,b)}|_{[s,t]}$. Hint: prove first that this is true for t of the form $s + k\ell_{(a,b)}$.

In particular $\tilde{\gamma}_{(a,b)}$ is proper and injective. In the following we also consider $\tilde{\gamma}_{(a,b)}$ as a closed subset of $\mathbb{R}^2$.[2]

[1] For k negative, this means that we run $-k$ times in the opposite direction.

[2] I'm also using that $\mathbb{R}^2$ is a complete Riemannian manifold, by Hopf-Rinow, since geodesics can be extended for arbitrary times (this is true on the torus, hence on the plane by lifting).

(iii) Prove that for (a, b) and (c, d) such that $ad - bc \neq 0$, the two geodesics $\tilde{\gamma}_{(a,b)}$ and $\tilde{\gamma}_{(c,d)}$ intersect transversely precisely in one point of $\mathbb{R}^2$: prove that at least one intersection is needed by studying the behaviour of the two geodesics for $t \to \pm\infty$ (Hint: the geoddesics are each contained in a (Euclidean) strip of different slopes), and use (i) to prove that there is at most one intersection. Don't forget to exclude tangentiality by a suitable argument!

(iv) We say that $\tilde{\gamma}_{(a,b)}$ intersects $\tilde{\gamma}_{(c,d)}$ *from right* if, supposing $\tilde{\gamma}_{(a,b)}(s) = \tilde{\gamma}_{(c,d)}(t)$, we have that $\tilde{\gamma}'_{(a,b)}(s), \tilde{\gamma}'_{(c,d)}(t)$ is an oriented basis of $\mathbb{R}^2$. Prove that $\tilde{\gamma}_{(a,b)}$ intersects $\tilde{\gamma}_{(c,d)}$ *from right* if and only if $ad - bc > 0$, i.e. if (a, b) and (c, d) form an oriented basis of $\mathbb{R}^2$.[3]

(v) With the conventions of the previous point, prove that in fact every couple of translates of $\tilde{\gamma}_{(a,b)}$ and $\tilde{\gamma}_{(c,d)}$ by elements of $\mathbb{Z}^2$ intersect transversely in precisely one point, and a similar characterisation holds about which one comes *from right*

Take now a sequence (a_n, b_n) with $a_n, b_n > 0$ and such that a_n/b_n converges to an irrational number, say π.

(vi) Prove that, up to passing to a subsequence, we can assume that $\tilde{\gamma}_{(a_n,b_n)}(0)$ converges to a point $p \in [0, 1]^2$ and that $\tilde{\gamma}'_{(a_n,b_n)}(0)$ converges to a unit vector $v \in T_p\mathbb{R}^2$.[4]

Consider the geodesic $\tilde{\gamma}_\infty : \mathbb{R} \to \mathbb{R}^2$ starting at p with velocity v, and let $\gamma_\infty : \mathbb{R} \to T$ be the induced geodesic on the torus. Our aim is to prove that γ_∞ is not periodic. By uniform continuity of solutions of geodesic equations for arbitrary finite times, for all small $\varepsilon > 0$ and all big $t > 0$ there is $n > 0$ such that $\tilde{\gamma}_\infty$ and $\tilde{\gamma}_{(a_n,b_n)}$ are at distance at most ε for all times in $[-t, t]$.

(vii) Prove that $\tilde{\gamma}_\infty$ is length minimising for all $s, t \in \mathbb{R}$. In particular, it is not periodic. This exludes the case that γ_∞ is periodic, descending to a closed geodesic in the homotopy class $(0, 0)$.

(viii) Suppose that γ_∞ descends to a closed geodesic on T in the homotopy class (a_∞, b_∞). Choose n as above large enough, so that for a small $\varepsilon > 0$ (how small should it be?) and for some $t > \ell_{(a_\infty,b_\infty)}$, the geodesics $\tilde{\gamma}_\infty$ and $\tilde{\gamma}_{(a_n,b_n)}$ are at distance at most ε for all times in $[-2t, 2t]$. Suppose also that n is big enough so that $ab_n - ba_n \neq 0$ (why?). Choose (a', b') such that $ab' - ba'$ and $a'b_n - b'a_n$ are both non-zero and have the same sign (why does such (a', b') exist?). Find a translate $\tilde{\gamma}_{(a',b')} + (c, d)$ of the geodesic $\tilde{\gamma}_{(a',b')}$ that intersects $\tilde{\gamma}_{(a_\infty,b_\infty)}$ on a point of $\tilde{\gamma}_{(a_\infty,b_\infty)}|_{[0,t]}$. Conclude that $\tilde{\gamma}_{(a',b')} + (c, d)$ intersects also $\tilde{\gamma}_{(a_n,b_n)}$ on a point of $\tilde{\gamma}_{(a_n,b_n)}|_{[0,2t]}$. Find a contradiction using (v).

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[3] Somehow, (a, b) is the average speed of $\tilde{\gamma}_{(a,b)}$ and (c, d) is the average speed of $\tilde{\gamma}_{(c,d)}$; but this is of course no argument!

[4] Here we mean unit vector with respect to the metric g.

(ix)* Find a proof of the same theorem that works also for the 3-dimensional torus!

Exercise 5.2 Let M be a connected smooth manifold, $p \in M$ and U a neighbourhood of p.

(i) Prove that there exists a smooth map $\varphi\colon M \times M \to M$ and a compact neighbourhood $K \subset U$ of p with the following properties:

- for all $x \in M$, $\varphi(x, -)\colon M \to M$ is a diffeomorphism fixing $M \setminus U$ pointwise;
- $\varphi(x, -)$ is the identity of M for $x \in M \setminus U$;
- $\varphi(x, p) = x$ for $x \in K$.

Let ΛM be one of the two following spaces (you are free to choose the one you like most!):

- $\Lambda(M) = C^0(S^1, M)$ is the space of all continuous maps from S^1 to M, with compact-open topology;
- $\Lambda(M) = W^{1,2}(S^1, M)$ is the Hilbert manifold of all continuous maps from S^1 to M admitting a weak derivative of finite L_2-norm[5]

There is a map $e\colon \Lambda M \to M$ sending a function $\gamma\colon S^1 \to M$ to $\gamma(1)$, where $1 \in S^1$ is the basepoint. Viceversa, there is a map $c\colon M \to \Lambda M$ sending $p \in M$ to the constant function $\gamma_p\colon S^1 \to p \in M$.

(ii) Prove that e is a locally trivial fibre bundle map, and c is a section of this bundle. Prove that the fibre of e over p is homeomorphic to the loop space ΩM.[6]

(iii)* Suppose that there is a deformation retraction of ΛM onto its subspace $c(M)$. Prove that ΩM is weakly contractible (all of its homotopy groups vanish). Prove that then M must be also weakly contractible; how can we conclude that M is in fact contractible?

(iv)* Prove that a *closed* connected manifold M of dimension $n \geq 1$ is not contractible (Hint: prove that there is a non-nullhomotopic map $M \to S^n$; don't forget the non-orientable case!).

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[5] The norm is only defined by patching local norms on charts, so it is not quite canonical; but its being finite or infinite is a well-definied property of a continuous function $S^1 \to M$.

[6] Define this space, either using continuous functions or using H^1 functions

Topics in Geometry 2021—Exercises

Niels Martin Møller and Andrea Bianchi

October 14, 2021

Week 6 — deadline on October 26th

Exercise 6.1

(i) Look up the definition of *smooth* Hilbert manifold M: charts take values in open subspaces of some Hilbert spaces, and transition functions of charts must be smooth and invertible, with smooth inverse. What does it mean for a map $U \to V$ of open sets of Hilbert spaces $U \subset H_1$, $V \subset H_2$ to be smooth? What is a Taylor approximation in the context of Hilbert spaces?

(ii) Define, for a point p in a Hilbert manifold M, the tangent space T_pM: it is a topological vector space, but it doesn't have a canonical scalar product upgrading it to a Hilbert space, unless M is endowed with an atlas whose transition maps are *isometries* of open subspaces of Hilbert spaces. Show that, given a chart $\psi\colon U \subset M \to H$, for all $p \in U$ we can identify T_pM with H as a topological vector space.

A Riemannian metric on a Hilbert manifold M is a choice of positive definite scalar product g_p on each tangent space T_pM, upgrading T_pM to a Hilbert space, and such that two additional requirements holds. For a chart $\psi\colon U \to H$ and for $p \in U$, we can compare the two scalar products on T_pM, one being g_p, the other coming from the chart-induced identification $T_p \cong H$. The two additional requirements are the following, explain what they mean:

(iii) *The assignment $p \mapsto g_p$ is smooth.*

(iv) *For each chart, the two mentioned scalar products on T_pM are equivalent/commensurable.*

From now on, let M be a Hilbert manifold with a fixed Riemannian metric, and let $f\colon M \to \mathbb{R}$ be a smooth function.

(v) For $p \in M$, define the continuous linear functional[1] $D_pf\colon T_pM \to \mathbb{R}$. Use the Riemannian metric to define $\mathrm{grad} f \in T_pM$.

[1] This could be defined also before fixing a Riemannian metric on the Hilbert manifold!

Now all terms in the Palais-Smale condition for f should have a precise meaning! In the following we fix for simplicity a Hilbert manifold with a single chart $0 \in U \subset H$, for some Hilbert space H. We also put a Riemannian metric on U, i.e. a smooth family of scalar products g_p on $H \cong T_pU$ for $p \in U$. Note again that g_p is not assumed to coincide with the scalar product of H: otherwise the following part of the exercise would be just the local study of a *flat* Riemannian Hilbert manifold, and not a generic one! Assume that $f\colon U \to \mathbb{R}$ has an isolated minimum at 0, and for simplicity assume $f(0) = 0$ and $f(p) > 0$ for $p \neq 0 \in U$. Assume that f has the Palais-Smale condition. Fix $\delta > 0$ such that $\bar{B}_H(0, 2\delta) \subset U$, where we consider here the closed ball of radius 2δ with respect to the *chart metric* (the metric of H as a Hilbert space).

(vi) Let $p_1, p_2, \dots$ be a sequence of points in $\bar{B}_H(0, 2\delta)$. Use commensurability of g to prove that $|D_{p_i}f|_H \to 0$ iff $|D_{p_i}|_g \to 0$.

(vii) Use smoothness of f (in fact, it suffices that f is of class C^2) to prove the following: there exists a $0 < \varepsilon < \delta$ such that for all $p, v \in H$ with $|p|_H = \varepsilon$ and $|v|_H \leq \frac{1}{2}\varepsilon$ the following inequality holds:

$$|f(p+v) - f(p) - D_pf(v)| \leq |v|_H^{3/2}$$

(viii) Let $S = \partial B_H(0, \varepsilon)$, with ε as above.We want now to prove that $\inf\{f(p)|p \in S\} > 0$, where it is clear that this inf is ≥ 0. Suppose by absurd that there are points $p_1, p_2, \dots$ in S with $f(p_i) \to 0$. Prove using (vii) that $|D_{p_i}f|_H \to 0$. Use (vi) and the Palais-Smale condition on f to find a subsequence p_{i_j} with a limit (which clearly must lie in S). Find a contradiction.

Conclusion: the Palais-Smale condition allows us, as done in the lecture, to find a small sphere S around an isolated minimum of f, such that $\inf f(S)$ is strictly bigger than the value attained at the isolated minimum. This is much easier if we are working with a finite dimensional Hilbert manifold (i.e. a plain manifold)!

Exercise 6.2

(i) Find a closed, orientable 3-manifold M with a Riemannian metric g, such that there is an embedded, closed geodesic $\gamma\colon S^1 \to M$ with the following property: γ minimises the length among curves in its homotopy class, but the closed curve γ' obtained by running twice along γ is not a minimiser of length in its homotopy class (although it is also a geodesic). Hint: what property of $\pi_1(M)$ makes the exercise easy?

(ii) Prove that there is a Riemannian metric g on the 3-dimensional torus $T = S^1 \times S^1 \times S^1$ such that there is an embedded curve γ minimising the length in the homotopy class of $(1, 0, 0) \in \pi_1 T \cong \mathbb{Z}^3$ (use that π_1 is abelian and represent *free* homotopy classes by elements of π_1), but γ' obtained by running twice along γ is not a length minimiser in the homotopy class $(2,0,0)$, not even locally. Hint: embed a solid cylinder $D^2 \times S^1$ in the torus, use (i) to put a suitable metric on this solid cylinder, extend the metric on the rest of the torus, and argue that the core γ of the solid cylinder has the desired properties.

Exercise 6.3

(i) Let M be a Riemannian manifold, let $p, q \in M$, and let $\alpha\colon (-\varepsilon,\varepsilon) \times [0,1]$ be a (smooth) variation of geodesics from p to q: each curve $\alpha(s,-)$ is a geodesic with $\alpha(s,0) = p$ and $\alpha(s,1) = q$. Prove that all geodesics $\alpha(s,-)$ have the same length, for $s \in (-\varepsilon,\varepsilon)$.

Consider now the sphere S^n, for $n \geq 2$, with the standard round metric g, let p, q be two antipodal points, and let $\gamma\colon [0,1] \to S^n$ be an embedded geodesic arc connecting p and q (half maximal circle).

(ii) Look up the definition of Jacobi field, in particular the differential equation that a Jacobi field J along γ should satisfy, involving the Riemann tensor $R = R_g$. Show that there exists a Jacobi field J along γ with $J \not\equiv 0$, but $J(0) = 0 \in T_pS^n$ and $J(1) = 0 \in T_qS^n$.

(iii) Let $\eta\colon S^n \to \mathbb{R}$ be a smooth function with the following properties: $\eta \equiv 0$ on γ, $\eta > 0$ away from γ, and all partial derivatives of all orders of η vanish when evaluated at points of γ.[2] Consider the Riemannian metric $g' = (1+\eta)g$ on S^n. Prove that γ is the unique length-minimising geodesic between p and q for g'.

(iv) Prove that J is a Jacobi field also for g' (Hint: the Riemann tensor $R_{g'}$ can be computed in terms of g' and its derivatives, and it suffices to prove that $R_{g'} \equiv R_g$ along γ).

Conclusion: in the metric g', there is no non-trivial variation of γ through geodesics connecting p and q, althouogh γ admits non-trivial Jacobi fields vanishing at the endpoints.

Exercise 6.4

Regard S^2 as the quotient of $[0,1]^2$ where we identify $(0,t) \sim (1,t)$ for all $t \in [0,1)$, and we collapse $0 \times [0,1]$ to a single point p, and we collapse $0 \times [0,1]$ to a single point q. You can think of p and q as being the north and south pole.

Let M be a closed Riemannian manifold, and let $f\colon S^2 \to M$ be a non-nullhomotopic continuous map; denote $P = f(p)$ and $Q = f(q)$. For each

[2] Use the old trick with e^{-1/x^2} to produce such η.

continuous map $g\colon S^2 \to M$ with g homotopic to f consider the number $\tilde{E}(g) = \sup_{0\leq t\leq 1} E(g(-,t))$, where E is the energy functional on closed curves (attaining value ∞ on curves that are not of class H^1), and $g(-,t)$ is the restriction of g to the *parallel* of S^2 at latitude t. Prove that $\tilde{E}(g) \geq \varepsilon$ for some $\varepsilon > 0$ depending only on the metric of M. The argument should be: if for all t the curve $g(-,t)$ has energy less than ε, then it is contained in a small, convex ball of M, then $g(-,t)$ nullhomotopes onto $g(0,t) \in M$ by convex interpolation, and these null-homotopies are continuous in t, so that at the end g itself is null-homotopic. Formalise this argument.

Topics in Geometry 2021—Exercises

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Week 7 — deadline on November 4th

Exercise 7.1 (Fourier Series and Poincare inequality).

(i) Use Fourier series on the unit circle to suggest a new definition (no need to link it to other definitions) of the Sobolev space

$$H^1(\mathbb{S}^1, \mathbb{R}) := \{f \in L^2 : f' \in L^2\}$$

via conditions on the Fourier coefficients of functions.

(ii) Use this Fourier series picture to show that H^1-functions are continuous[1].

(iii) Use Fourier series to show the Poincare inequality for $f \in H^1(\mathbb{S}^1, \mathbb{R})$:

$$\|f - \bar{f}\|_{L^2(\mathbb{S}^1,\mathbb{R})} \leq C\|f'\|_{L^2(\mathbb{S}^1,\mathbb{R})},$$

where $\bar{f} := \frac{1}{2\pi}\int_{\mathbb{S}^1} f(s)ds$ denotes the average.

(iv) Give the sharp constant C in the Poincare inequality, and find all functions which satisfy equality for the sharp constant.

Exercise 7.2 (Compact Rectangular Boxes in Hilbert Space).

(i) Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal system[2] in an infinite-dimensional Hilbert space H. Suppose that $\{d_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers satisfying

$$\sum_{k=1}^{\infty} |d_k|^2 < \infty.$$

[1]In fact, of course, much more is true: they are $C^{0,1/2}$, i.e. 1/2-Hölder.
[2]Without loss of generality, you can assume it is an orthonormal basis: why?

Prove that the corresponding "infinite-dimensional rectangular box"

$$B := \left\{ \sum_{k=1}^{\infty} a_k e_k : a_k \in \mathbb{R}, |a_k| \leq d_k \right\} \subseteq H$$

is a compact subset.

(ii) Use (i) and Fourier series on the circle to show that the inclusion

$$H^1(\mathbb{S}^1, \mathbb{R}) \hookrightarrow L^2(\mathbb{S}^1, \mathbb{R})$$

is a compact linear operator (the image of the closed unit ball is compact).

Exercise 7.3 (A counterexample by Weierstrass).

In 1856 Lejeune Dirichlet held a series of lectures about the Dirichlet principle; as his time it was common to assume that a non-negative functional defined on a space in a natural way (usually in the context of a problem coming from physics) admits an absolute minimum.

But in the work "*Über das sogenannte Diritchlet'sche Princip*", presented in 1870, Karl Weierstrass suggested the following counterexample to the validity of such arguments in general[3]:

$$E(u) := \int_{-1}^{1} \left| x \frac{du}{dx} \right|^2 dx,$$

considered as a functional defined on

$$\mathscr{C} := \{u \in C^1([-1,1], \mathbb{R}) : u(-1) = -1,\ u(1) = 1\}.$$

We define $\kappa := \inf_{u \in \mathscr{C}} E(u)$.

(i) Show that $\kappa = 0$. *Hint: Dilations/rescalings of the function* $\arctan(\cdot)$ *are your best friends.*

(ii) Show that there is no function in $\mathscr{C}$ which attains the infimum κ.

(iii) Exhibit a function $u_0 \in L^\infty([-1,1])$ which is continuous on some neighborhoods of the endpoints, with $u_0(\pm 1) = \pm 1$, and such that the statement $E(u_0) = \kappa = 0$ is well-defined (and true) using weak derivatives. Discuss the relation to your solution to (i).

[3] See `https://michaelcweiss.files.wordpress.com/2020/04/weierstrass-example.pdf`, though it is written in German.