

# Categories and Topology

University of Copenhagen Lecture Notes



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Lecture notes by Jesper Grodal, with input and parts by Simon Gritschacher, Rune Haugseng, Piotr Pstrągowski, Lukas Woike, Andrea Bianchi  
[....?]

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN

*Email address:* [jg@math.ku.dk](mailto:jg@math.ku.dk)

*URL:* <http://www.math.ku.dk/~jg>



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## Introduction

This course will be an introduction to the interplay between category theory and topology, which permeates much of modern mathematics. The prerequisite is a basic understanding of classical homotopy theory, as presented for instance in Chapters 1-5 of Hatcher’s book [Hat02].

We will discuss categorical and combinatorial models for topological spaces, and conversely show how we can enhance our view of our favorite categories (be it groups, rings, modules, or something more exotic) by considering them as “topological” objects that, in a sense, can be glued together as if they were actual spaces.

We first want a better understanding of the *category* of spaces, from a homotopy point of view. Recall that we started our quest in algebraic topology with the purpose of understanding all topological spaces, and all continuous maps between them, up to a suitable notion of homotopy. This led us to consider, for any two spaces  $X$  and  $Y$ , the *space*

$$\text{map}(X, Y)$$

of continuous maps between them, equipped with the compact open topology.

If we want to model topological spaces in some combinatorial way, we should require the model to be strong enough to answer the following two questions:

- |    |  |
|----|--|
| Q1 | (1) Given two objects (“spaces”) $X$ and $Y$ , are they weakly <sup>1</sup> homotopy equivalent?                           |
| Q2 | (2) For two objects (“spaces”) $X$ and $Y$ , what is the weak homotopy type of the topological space of maps between them? |

We would furthermore like to be able to do this via some canonical comparison, and in particular to be able to say when two models for homotopy theory are “the same”. We have already noticed that both the category of topological spaces and the category of chain complexes are endowed with a certain notion of “homotopy”, but how are they both an instance of a general notion of “homotopy theory”? What does “being a homotopy theory” mean? Could there be a homotopy theory of homotopy theories?<sup>2</sup>

We have already strived in AlgTopII to answer these questions: we showed that every topological space is weakly homotopy equivalent to a CW complex, and that two CW complexes are homotopy equivalent if and only if they are weakly homotopy equivalent. With respect to maps, we showed at least that any map between CW complexes can be approximated by a cellular map, giving us some combinatorial control of the maps between CW complexes. And in HomAlg, we studied how we

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<sup>1</sup>The use of the word “weak” will become clear during the course; you are now invited to read “weakly homotopy equivalent” as “equivalent”, and later “weak homotopy type” as “equivalence class”.

<sup>2</sup>The answer is yes.

can calculate derived functors of Hom via projective or injective resolutions, and showed in particular that the specific choice of these resolutions do not matter. How are these things related?

Our first step in this story, carried out in Chapter 2, is to show that we can model topological spaces, considered up to weak equivalence, in a complete combinatorial way, using something called *simplicial sets* and denoted  $\mathbf{sSet}$ .<sup>3</sup> The rationale for the use of simplicial sets is an expansion of the fact, known from earlier courses, that homotopy theory is controlled by CW complexes. Simplicial sets are similar to simplicial complexes or CW complexes, but they only remember the combinatorics of the cell attachments, and not the point-set topology.

There are a few important and technical steps in making this work out—e.g., we’ll see that just like there are preferred spaces called CW-complexes there will also be preferred simplicial sets called Kan complexes (named after Dan Kan); this parallel is mild, and in fact Kan complexes resemble, in the context of chain complexes over a ring, more the injective resolutions than the projective ones (whereas CW complexes resemble more projective resolutions). We give the rudiments of this theory in Chapter 2 (which underpins everything that follows later in the course), and point the reader to the many existing references of the full story. One important player that we introduce here is also the *nerve construction*, which associates to any category a simplicial set (and hence a topological space via the above mentioned correspondence). The strength of this construction is two-fold: on the one hand we associate a geometric object with any category, on the other hand we can try to model simplicial sets (and hence topological spaces) by categories, and writing down categories, functors, and natural transformations, is often much simpler and less ambiguous than writing down spaces, maps, and homotopies.

The next thing that will concern us, is to give a first taste of what is a general “homotopy theory”, via the theory of model categories. Remember that by a “homotopy theory” we very roughly mean somewhere where we can give answers to questions (1) and (2). We will not spend a long time on the general theory of model categories, and we will just give enough definitions to indicate how topological spaces, simplicial sets and chain complexes, as well as certain other categories constructed from them<sup>4</sup>, can be brought into this framework. The theory of model categories was first introduced by Daniel Quillen in the late 1960’s, and often provides a good concrete way of working with any particular, classical context in which a notion of “homotopy” is defined, for instance spaces or chain complexes. A more modern definition of homotopy theory is via the theory of infinity-categories, which is the most setting where (1) and (2) make sense: roughly speaking, an infinity-category is the datum of certain objects, called “homotopy types”, together with, for any two homotopy types, a simplicial set of “maps” between them, satisfying certain composition rules. The fact that the notion of an  $\infty$ -category is so general will enable us to do homotopy theory in new settings (e.g., new categories constructed from our old categories) without

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<sup>3</sup>For the moment, think of  $\mathbf{sSet}$  as just a category whose objects and morphisms have a combinatorial description

<sup>4</sup>For instance, we will consider categories of certain diagrams of spaces

the need to make arbitrary choices,<sup>5</sup> and without the need to prove many technical lemmas that, using more classical methods, would be needed (but sometimes we are just not able to prove them, or to give a clear proof!). We will get to an example in a moment. We will not get to  $\infty$ -categories on a technical level in this course, but we will keep this viewpoint in mind from the very beginning.<sup>6</sup>

Developing simplicial sets, model categories, and their basic properties will occupy the first half of this course. In the second half we will be concerned with putting these tools in use: we will study “derived/homotopy” limits and colimits, and localization techniques.

More precisely, a question that will concern us is how to describe the objects in the categories of topological spaces, chain complexes, or the like, from simpler “bits”, so that if we know the simpler bits we can glue the simpler bits together and get information about what we started with.

This will involve studying functors

$$F : \mathcal{I} \rightarrow \text{Top}$$

i.e., functors which to each object  $i \in \mathcal{I}$  assigns a topological space, and to each morphism in  $\mathcal{I}$  assigns a continuous map of topological spaces.<sup>7</sup> We want to understand how we can glue the pieces (the topological spaces obtained by evaluating the functor on objects of  $\mathcal{I}$ ) together, as dictated by the functor, in such a way that the resulting output is suitably homotopy invariant. What this means exactly will need to be made precise! This will be the homotopy colimit construction

$$\text{hocolim}_{\mathcal{I}} F$$

which will generalize the homotopy pushout construction that we saw in AlgTopII.<sup>8</sup>

Note that ideally we would want to be able to make a much more general statement. We would want to say that the entire *category* Top of topological spaces is glued together from certain pieces, smaller categories. For this we may want to consider a functor

$$F : \mathcal{I} \rightarrow \text{Cat}$$

where for each  $\beta \in \mathcal{I}$  we assign a category, e.g., a subcategory of the category of topological spaces, and for each morphism in  $\mathcal{I}$  we assign a functor.<sup>9</sup>

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<sup>5</sup>think of the choice of projective/injective resolutions when defining derived functors in homological algebra, with the following need to prove carefully the independence of the result, up to natural isomorphism, from the choice

<sup>6</sup>Just as a teaser: on a technical level, an  $\infty$ -category will be a simplicial set satisfying an extra condition, which sounds rather innocuous; but the notion is general enough so that we can view a single space as an  $\infty$ -category, the category of all spaces as an  $\infty$ -category, and category of all  $\infty$ -categories as an  $\infty$ -category as well!

<sup>7</sup>E.g.  $\mathcal{I}$  could be a pushout diagram, a group viewed as category with one object, or the natural numbers viewed as an ordered set.

<sup>8</sup>There is also a dual “co-gluing” construction  $\text{holim}$ , generalizing homotopy pull-backs.

<sup>9</sup>We ignore for the point of this discussion any set-theoretic issues, arising from the fact that the objects of a category may not be a set, as with Top—these issues can usually be fixed by adding a few right words...

We are however left with a big problem: How can we “do homotopy theory” inside  $\text{Cat}$ ? Given two categories  $X$  and  $Y$ , can we speak of the space  $\text{map}(X, Y)$ ? The point is that we have viewed  $\text{Top}$ , not just as a category, but as a place where we can do homotopy theory, with notions of weak equivalences, and a notion of mapping space between any two objects, as in (1) and (2), i.e., an  $\infty$ -category; and now, voila, see that we should instead have considered  $F$  from before as functor to a suitable  $\infty$ -category of  $\infty$ -categories.

$$F : \mathcal{I} \rightarrow \text{Cat}_\infty$$

Such a thing  $\text{Cat}_\infty$  exists! But we are digressing, and getting *way* ahead of ourselves....

Our concrete topics in the second half will be to learn about two things. In Chapter 5 we will develop rudiments of homotopy colimits and homotopy limits of topological spaces or simplicial sets. This will include a useful spectral sequence for calculating the homology of a homotopy colimit from its pieces, vastly generalizing the Mayer-Vietoris exact sequence in homology you learned about back in AlgTop1. There is a dual sequence for homotopy limits (such as, for example, homotopy fixed-points), which is also super useful. We will also state and prove, using the previous machinery, the so-called Quillen’s Theorems A and B, which will provide a very concrete criterion for when a functor between categories induces a homotopy equivalence between nerves, and more generally (Theorem B) identify the homotopy fiber. This is perhaps the most useful single tool in showing that maps induce homotopy equivalences. We will see some examples.

In Chapter 6 we will then move on localization and completion. These are, in essence, versions of the known constructions from algebra. The key idea for localization is that we want to invert something, classically just elements in a ring, and in general a collection of morphisms in a category, and study the resulting object. The most basic (but already super useful!) case is just inverting a prime  $p$ , and we will see how the different primes and the rational information can be glued together to recover the whole space in the so-called *Sullivan arithmetic square*. More generally we can try to invert any suitable collection of maps, that we want to treat as equivalences. We are then naturally led to the question: When does the localized category again form a homotopy theory? Here the methods from the first part come in useful.

Completion, à la Bousfield–Kan, is in good cases a special kind of localization. And it has the additional advantage that it naturally comes with a tower of fibrations whose limit is the completed space. This will again allow us to describe maps into the completed space via simpler pieces, which e.g., sets up an “Adams style” spectral sequence calculating homotopy classes of maps from homological information.

If time allows, we will round off the course with a special chapter where we try to use some of the techniques mentioned so far to sketch a proof of the so-called Sullivan conjecture, which says that for a finite  $p$ -group  $P$  and for a finite  $P$ -CW complex  $X$ , fixed points and homotopy fixed-points agree at a prime  $p$ , in the precise sense that we have a homotopy equivalence

$$(X^P)_p^\wedge \rightarrow (X_p^\wedge)^{hP}$$



where  $(-)\hat{p}$  denotes Bousfield–Kan  $p$ -completion, and  $(-)^{hP}$  denotes homotopy fixed-points. This is surprising, since homotopy fixed-points a priori depends on much weaker properties of the space than actual fixed-points, and this theorem, proved by Miller and Lannes in the 1980s, has led to many subsequent results.

### About the origin of these notes

These notes are used for the course Categories and Topology, taught for many years each fall at University of Copenhagen. The first installment ran in 2009 taught by Alexander Berglund, Antonio Diaz, and Richard Hepworth (all postdocs at the time) and myself (JG). The basic outline of the course, and the first hand-written notes, trace back to this early iteration. E.g., traces of Berglund’s lectures on simplicial sets (I think again inspired by a course by Torsten Ekedahl) can still be found in Chapter 2.

I started typing these notes around 2015, in the beginning only for the second half of the course (the first being taught by Ib Madsen using a different set of notes). Rune Haugseng taught the second half of the course in 2017, where he greatly expanded the written notes on Chapters 4, 5, and 6, and those chapters are hence joint work. [ ...]

Piotr Pstrągowski gave some guest lectures in 2019, made a handwritten outline of the part on classification of fibrations, which was then expanded and included in the notes by Lukas Woike. [ ...]

Simon Gritschacher co-taught the courses in 2020 and much expanded the material in Chapter 1. [ ...]

Lukas Woike co-taught the course in 2021 and made many additions and improvements. [ ...]

Andrea Bianchi co-taught the course in 2022 and ... [ ...]

TA’s who contributed with exercises: [ ...].

Also thanks for sending corrections and suggestions: Max Fischer, Peter Patzt, [ ...]

Please report errors and suggestions to [jg@math.ku.dk](mailto:jg@math.ku.dk) Thanks! :) Cookies may be on the line.

### Other suggested literature

These notes have been inspired by many sources, which you are also encouraged to consult. Here is a sample:

#### BASIC CATEGORY THEORY

- Section 2 in Dwyer–Spalinski
- MacLane: Categories for the working mathematician
- Emily Riehl’s books [Rie17] and (the more advanced) [Rie14].

#### SIMPLICIAL HOMOTOPY THEORY:

- Ib Madsen and Irakli Patchkoria: Notes from last year(s).
- Goerss–Jardine: Simplicial homotopy theory [GJ99] (1999)
- Gabriel and Zisman: Categories of fractions in homotopy theory. (1967)

- May: Simplicial objects in algebraic topology (1967?) [May67]
- Book in progress by Joyal–Tierney: Simplicial homotopy theory
- Introduction to infinity this-and-that: Jacob Lurie: Keredon (notes downloadable from his homepage.)

MODEL CATEGORIES:

- Dwyer–Spalinski: Homotopy theories and Model categories.
- Hovey: Model categories.
- Quillen: the original thing.

HOMOTOPY (CO)LIMITS AND LOCALIZATIONS AND COMPLETIONS OF SPACES:

In the second part of the course we will cover two important topics in homotopy theory, namely homotopy (co)limits and localizations and completions of spaces.

The main source for both of these topics still remains the “yellow monster” [BK72] by Bousfield and Kan. Other useful references for homotopy colimits include:

HOMOTOPY (CO)LIMITS:

- The paper [DS95] by Dwyer–Spalinsky is an excellent reference for explaining the basic philosophy, and it does model categories as well.
- The notes by Dan Dugger “A primer on homotopy colimits”, which can be found on his homepage <http://pages.uoregon.edu/ddugger/>, are also highly recommended.
- The book by Goerss–Jardine [GJ99] is a more modern introduction to parts of the yellow monster.
- There is also the more recently published (but quite classical in its outlook) *More concise algebraic topology* [MP12] by May and Ponto (which at 514 pages should perhaps be called “Less concise algebraic topology”).
- Riehl’s book [Rie14] gives a nice abstract account of homotopy limits and colimits — our discussion of derived functors is essentially taken from here (though much of it is originally due to [Dwy+04]).

LOCALIZATIONS AND COMPLETIONS OF SPACES:

Some references for localization and completion

- Sullivan’s MIT notes [Sul05] is a good informal introduction to localization and completion.
- Another recommended introduction is Neisendorfer’s notes [Nei09].
- There’s also a lot of classical material summarized in May–Ponto [MP12].
- The simplicial viewpoint on localizations is explained in Bousfield–Kan [BK72].
- General existence of localizations with respect to a homology theory was proved by Bousfield in [Bou75].
- The book of Dror–Farjoun [DF92] explains in detail how to localize with respect to a map.
- A general version of the arithmetic square is given in [DDK77] (also correcting a point in Bousfield–Kan).
- The paper [Dwy04] by Dwyer contains a more advanced survey of localizations with many examples.

- The notes on *Rational and  $p$ -adic homotopy theory* by Thomas Nikolaus contain a good discussion of  $p$ -completion that we also benefited greatly from.



## CHAPTER 1

# Categories

The goal of this chapter is to introduce a bit of basic category theory with a view towards the later chapters. There are many sources for this material; a readable source for some of it is [DS95, Sec. 2].

### 1.1. Basic categorical definitions

def:category

DEFINITION 1.1.1. By a category  $\mathcal{C}$  we mean a class of *objects*  $\text{Ob}(\mathcal{C})$  and a class of *morphisms*  $\text{Mor}(\mathcal{C})$  (aka *maps* or *arrows*) together with assignments

source

target

(1)  $d_1 : \text{Mor}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$ , the *source* map.

(2)  $d_0 : \text{Mor}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$ , the *target* map.

(3)  $s_0 : \text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$ , the *identity* map.

(4)  $\circ : \text{Mor}(\mathcal{C}) \times_{\text{Ob}(\mathcal{C})} \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$  called *composition* and written  $(f, g) \mapsto g \circ f$ , where we denote by  $\text{Mor}(\mathcal{C}) \times_{\text{Ob}(\mathcal{C})} \text{Mor}(\mathcal{C})$  the class of pairs of morphisms  $\{(f, g) \in \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{C}) \mid d_0 f = d_1 g\}$ .

composition

These are required to satisfy the well-known identities:

- If  $d_0(f) = d_1(g)$  and  $d_0(g) = d_1(h)$  then  $(h \circ g) \circ f = h \circ (g \circ f)$ . (Associativity.)
- $f \circ s_0(d_1(f)) = f = s_0(d_0(f)) \circ f$ . (Identity)

We also write  $f : x \rightarrow y$  for a morphism  $f \in \text{Mor}(\mathcal{C})$  with  $d_1(f) = x$  and  $d_0(f) = y$ ,  $1_x$  for  $s_0(x)$ , and  $\text{Hom}_{\mathcal{C}}(x, y)$  for the collection of morphisms  $f$  with  $d_1 f = x$  and  $d_0 f = y$ . Some authors prefer the notation  $\mathcal{C}(x, y)$  for  $\text{Hom}_{\mathcal{C}}(x, y)$ .

We will assume that all categories  $\mathcal{C}$  satisfy the condition that  $\text{Hom}_{\mathcal{C}}(x, y)$  is a set. (This condition is referred to in the literature by saying that  $\mathcal{C}$  is *locally small*.)

REMARK 1.1.2. As notational remark, note that in the definition of source and target in (1) and (2), it is  $d_1$  rather than  $d_0$  that denotes the source, as we view it as “forgetting” the target.

Also note that in the definition of composition in (4), the order gets reversed. This is due to the well-established, but in hindsight probably regrettable, convention of composing functions (of sets) from right to left.

DEFINITION 1.1.3. A category  $\mathcal{C}$  is called *small* if it has a set of objects and a set of morphisms. We will often use the letter  $\mathcal{I}$  for indexing categories (i.e. categories used to describe the shape of a diagram), and indexing categories are assumed to be small.

allcategories

EXAMPLE 1.1.4. Some (very) small categories  $\mathcal{I}$  we shall consider are:

- Any set, considered as giving the objects of a category with no non-identity morphisms.

- The pushout category  $(\bullet \leftarrow \bullet \rightarrow \bullet)$  and the pullback category  $(\bullet \rightarrow \bullet \leftarrow \bullet)$ , each having three objects and two non-identity morphisms.
- The equalizer category  $\bullet \rightrightarrows \bullet$ .
- The category  $[n]$ , consisting of the  $n + 1$  numbers from 0 to  $n$ , considered as a totally ordered set, i.e. “ $[n] = (0 < 1 < \dots < n) = (\bullet \rightarrow \bullet \dots \rightarrow \bullet)$ ”
- The simplex category  $\Delta$ , with objects  $[0], [1], [2], \dots$ . A morphism  $f : [n] \rightarrow [m]$  is an order-preserving function of sets  $(\{0, \dots, n\}, \leq) \rightarrow (\{0, \dots, m\}, \leq)$ .
- The category  $\mathcal{B}G$ , with one object and a group  $G$  as morphisms. (We might even slip into the notation of denoting this category just as  $G$ .)
- The category  $\mathcal{E}G$  with  $G$  as objects, and a unique morphism between any two objects.
- The natural numbers  $\mathbb{N}$ , considered as a totally ordered set  $(\bullet \rightarrow \bullet \rightarrow \dots)$ .
- *Poset categories*, i.e., categories with at most one morphism between any two objects. One of our favorite examples of poset categories will be the poset of a certain class of subgroups (finite,  $p$ -groups, etc.) of a fixed group  $G$ .
- The *orbit category*  $\mathcal{O}_{\mathcal{F}}(G)$  for  $G$  a group, and  $\mathcal{F}$  a set of subgroups closed under conjugation. This has objects transitive  $G$ -sets with isotropy subgroup in  $\mathcal{F}$ , and morphisms  $G$ -equivariant maps. ( $\mathcal{F}$  is called a *collection* of subgroups, and if it is furthermore closed under passage to subgroups a *family* of subgroups.)
- So-called EI-categories, i.e., categories where every Endomorphism is an Isomorphism. Groups, poset categories, and orbit categories, are examples of EI-categories. EI-categories have an associated poset with the same underlying class of objects, by declaring  $[x] \leq [y]$  if there exists a non-trivial map  $x \rightarrow y$  — this provides a filtration of an EI-category by subcategories, and often allows proofs by induction.

EXAMPLE 1.1.5. Some larger categories we shall consider are:

- The category of all groups and group homomorphisms.
- The category of all  $R$ -modules and  $R$ -module maps.
- The category of topological spaces and all continuous maps.
- The category of chain complexes of  $R$ -modules and chain maps of those.

One central issue is when two categories should be considered “the same”. To attack this question we first need to view our categories (including our very large ones) as objects of some larger category. The first step in this process is to define morphisms between categories. These are called functors.

**def:functor**

DEFINITION 1.1.6. Given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , a *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  assigns to each object  $x$  of  $\mathcal{C}$  an object  $Fx$  of  $\mathcal{D}$  and to each morphism  $f : x \rightarrow y$  of  $\mathcal{C}$  a morphism  $Ff : Fx \rightarrow Fy$  of  $\mathcal{D}$ , such that for each object  $x$  of  $\mathcal{C}$

$$F(1_x) = 1_{Fx}$$

and for each pair of morphisms  $(f, g)$  whose composite  $g \circ f$  is defined in  $\mathcal{C}$

$$F(g \circ f) = F(g) \circ F(f).$$

EXAMPLE 1.1.7. A functor  $F: [n] \rightarrow [m]$  is precisely an order preserving function  $(\{0, \dots, n\}, \leq) \rightarrow (\{0, \dots, m\}, \leq)$ . In other words, a morphism  $f: [n] \rightarrow [m]$  in  $\mathbf{\Delta}$  is exactly a functor (where  $[n]$  and  $[m]$  are seen as categories).

REMARK 1.1.8. We can alternatively write the data of a small category  $\mathcal{C}$  in the following way. Setting  $X_0 = \text{Ob}(\mathcal{C})$  and  $X_1 = \text{Mor}(\mathcal{C})$ , the maps of Definition 1.1.1 are encoded in the following diagram

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_1} \end{array} X_1$$

with  $d_0 s_0 = 1$ ,  $d_1 s_0 = 1$ , together with a composition operation  $\circ: X_1 \times_{X_0} X_1 \rightarrow X_1$ , where  $X_1 \times_{X_0} X_1 = \{(f, g) \in X_1 \times X_1 \mid d_0 f = d_1 g\}$ , subject to the associativity and unitality conditions as in Definition 1.1.1.

Let  $\mathcal{D}$  be another small category and write  $Y_0 := \text{Ob}(\mathcal{D})$  and  $Y_1 := \text{Mor}(\mathcal{D})$ . Then a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  consists of maps  $X_0 \rightarrow Y_0$  and  $X_1 \rightarrow Y_1$  making the following diagrams commute

$$\begin{array}{ccc} X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_1} \end{array} X_1 & & X_1 \times_{X_0} X_1 \xrightarrow{\circ} X_1 \\ \downarrow & & \downarrow \\ Y_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_1} \end{array} Y_1 & & Y_1 \times_{Y_0} Y_1 \xrightarrow{\circ} Y_1. \end{array}$$

DEFINITION 1.1.9. Given two functors  $F, G: \mathcal{C} \rightarrow \mathcal{D}$ , a *natural transformation*  $\tau: F \Rightarrow G$  is an assignment  $\text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$  which assigns to each object  $x$  of  $\mathcal{C}$  a morphism  $\tau_x: Fx \rightarrow Gx$  of  $\mathcal{D}$  such that for every morphism  $f: x \rightarrow y$  of  $\mathcal{C}$  the following diagram in  $\mathcal{D}$  commutes

$$\begin{array}{ccc} Fx & \xrightarrow{\tau_x} & Gx \\ \downarrow Ff & & \downarrow Gf \\ Fy & \xrightarrow{\tau_y} & Gy \end{array}$$

in the sense that  $Gf \circ \tau_x = \tau_y \circ Ff$  as morphisms in  $\mathcal{D}$ .

DEFINITION 1.1.10 (Functor categories). Let  $\mathcal{I}$  be a small category and  $\mathcal{C}$  any (locally small) category; we write  $\mathcal{C}^{\mathcal{I}}$  (or sometimes  $\text{Fun}(\mathcal{I}, \mathcal{C})$ ) for the category of functors from  $\mathcal{I}$  to  $\mathcal{C}$ . It has as *class* of objects the functors  $F: \mathcal{I} \rightarrow \mathcal{C}$ , and as morphisms from  $F$  to  $F'$  the *set* of natural transformations.  $(\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, F'))$  is still a set since  $\mathcal{I}$  is assumed small and  $\mathcal{C}$  locally small.)

DEFINITION 1.1.11 (Subcategories). A *subcategory*  $\mathcal{D}$  of a category  $\mathcal{C}$  is a choice of subclasses of the classes of objects and morphisms of  $\mathcal{C}$  such that for any morphism  $f: x \rightarrow y$  in  $\mathcal{D}$ , the objects  $x$  and  $y$  also lie in  $\mathcal{D}$  and such that the morphisms in  $\mathcal{D}$  are closed under composition and contain all identities of objects in  $\mathcal{D}$ . This turns a subcategory  $\mathcal{D}$  of  $\mathcal{C}$  into a category in its own right and the inclusion of  $\mathcal{D}$  into  $\mathcal{C}$  into a functor. We call a subcategory *full* if for all objects  $x, y$  in  $\mathcal{D}$  we have an equality of sets  $\text{Hom}_{\mathcal{D}}(x, y) = \text{Hom}_{\mathcal{C}}(x, y)$ , i.e. the inclusion functor induces a bijection on morphisms sets.

REMARK 1.1.12. Given a category  $\mathcal{C}$ , we obtain a full subcategory of  $\mathcal{C}$  by specifying any subclass of the objects of  $\mathcal{C}$  (this determines the morphism of this subcategory already because we assume it is full). We refer to this as the full subcategory spanned by these objects. For example, we will use sometimes the full subcategory  $\Delta_{\leq n}$  of  $\Delta$  spanned by the objects  $[\ell]$  with  $\ell \leq n$ .

## 1.2. The Yoneda lemma

sec:yoneda

There are a couple of easy-to-prove, but frightfully useful things to say concerning the special functors  $\mathcal{C} \rightarrow \text{Set}$  of the form  $\text{Hom}_{\mathcal{C}}(x, -)$  and  $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$  of the form  $\text{Hom}_{\mathcal{C}}(-, x)$ , called (co)representable functors (and (co)represented by  $x$ ).

The most famous is the ‘‘Yoneda lemma’’ which has as a corollary that the ‘‘Yoneda embedding  $\mathfrak{y} : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$ , sending  $x$  to  $\text{Hom}_{\mathcal{C}}(-, x)$  is fully faithful. This allows us to study  $\mathcal{C}$  via  $\text{Set}^{\mathcal{C}^{\text{op}}}$ , which usually has much better formal properties.

yoneda

LEMMA 1.2.1 (Yoneda). *Let  $F : \mathcal{C} \rightarrow \text{Set}$  be a functor. Then we have isomorphism of sets, with source the set of natural transformations*

$$\text{Nat}(\text{Hom}_{\mathcal{C}}(x, -), F(-)) \xrightarrow{\cong} F(x),$$

given by  $\Phi \mapsto \Phi_x(\text{id}_x)$ .

The inverse is given by associating with any  $u \in F(x)$  the natural transformation  $\text{Hom}_{\mathcal{C}}(x, -) \rightarrow F(-)$  given by sending  $f \in \text{Hom}_{\mathcal{C}}(x, y)$  to  $F(f)(u) \in F(y)$ .

Similarly for contravariant functors  $F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ , we have

$$\text{Nat}(\text{Hom}_{\mathcal{C}}(-, y), F(-)) \cong F(y)$$

PROOF. Let  $\Phi \in \text{Nat}(\text{Hom}_{\mathcal{C}}(x, -), F(-))$  and consider the commutative diagram of sets

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(x, x) & \xrightarrow{\text{Hom}_{\mathcal{C}}(x, f)} & \text{Hom}_{\mathcal{C}}(x, y) \\ \downarrow \Phi_x & & \downarrow \Phi_y \\ F(x) & \xrightarrow{F(f)} & F(y). \end{array}$$

It shows that  $\Phi_y(f) = F(f)(\Phi_x(\text{id}_x))$ , i.e., that the natural transformation  $\Phi$  is determined by  $\Phi_x(\text{id}_x) \in F(x)$ . Furthermore any element  $u \in F(x)$  determines a natural transformation  $\text{Hom}_{\mathcal{C}}(x, -) \rightarrow F(-)$  by sending  $f \in \text{Hom}_{\mathcal{C}}(x, y)$  to  $F(f)(u) \in F(y)$ , proving the covariant version of the Yoneda lemma. The proof of the contravariant version follows by symmetry.  $\square$

The above immediately implies:

COROLLARY 1.2.2 (Yoneda embedding).

$$\text{Nat}(\text{Hom}_{\mathcal{C}}(-, x), \text{Hom}_{\mathcal{C}}(-, y)) \cong \text{Hom}_{\mathcal{C}}(x, y)$$

In particular the functor  $\mathfrak{y} : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$  is a fully faithful embedding of  $\mathcal{C}$  (called the Yoneda embedding)  $\square$



REMARK 1.2.3. Note our show-off use of the Japanese character  $\mathfrak{Y}$ , the first character of Yoneda; you can of course just write  $y$ , if you wish.

REMARK 1.2.4. We shall later prove the (equally easy) density theorem, which says that any object in  $\text{Set}^{\mathcal{C}^{\text{op}}}$  is a colimit of objects from  $\mathcal{C}$  via  $\mathfrak{Y}$ , which conversely can reduce the study of  $\text{Set}^{\mathcal{C}^{\text{op}}}$  to the study of  $\mathcal{C}$ .

REMARK 1.2.5. Note that we have already encountered (co)representable functors in  $\text{HomAlg}$ , in formulas such as  $(-)^G = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$  or  ${}_p(-) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, -)$  and in  $\text{AlgTopII}$ , in formulas such as  $\pi_n(-) = [S^n, -]_{pt}$  or  $H^n(-; M) = [-, K(M, n)]$ .

REMARK 1.2.6. The functor category  $\mathcal{P}(\mathcal{C}) = \text{Set}^{\mathcal{C}^{\text{op}}}$  is often called presheaves on  $\mathcal{C}$  (or Set-valued presheaves, if one wants to be precise...), and the Yoneda embedding is hence written  $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ .

### 1.3. Adjoint functors

An extremely useful concept in category theory is that of adjoint functors. There are different ways of formulating the definition of an adjoint functor pair, at least one of which you may be familiar with. But let us begin by building a baby-adjunction which will illustrate the general concept.

ex: baby

EXAMPLE 1.3.1. Suppose that  $A$  is a set and  $X$  is a topological space. There are two obvious ways to endow  $A$  with a topology: We can give  $S$  the discrete topology (every subset is open) and denote the resulting space by  $A^\delta$ . We can also equip  $A$  with the indiscrete topology (the only open subsets are  $\emptyset$  and  $A$ ) and denote the resulting space by  $A^0$ . It is clear, then, that *every* map of sets  $A^\delta \rightarrow X$  is continuous, and that *every* map of sets  $X \rightarrow A^0$  is continuous too. Using the language of categories, we can phrase this more formally as follows: Define functors

$$(-)^\delta: \text{Set} \rightarrow \text{Top}$$

$$(-)^0: \text{Set} \rightarrow \text{Top}$$

and a functor

$$U: \text{Top} \rightarrow \text{Set}$$

which sends a topological space  $X$  to its underlying set  $UX$ , i.e., it “forgets the topology of  $X$ ” (therefore also called a *forgetful* functor). We observed above that for any set  $A$  and any space  $X$  there are bijections

$$\text{Hom}_{\text{Top}}(A^\delta, X) \cong \text{Hom}_{\text{Set}}(A, UX)$$

and

$$\text{Hom}_{\text{Top}}(X, A^0) \cong \text{Hom}_{\text{Set}}(UX, A),$$

and in fact these bijections are *natural* in  $A$  and  $X$ .

The observation of the preceding example is formalized by the notion of an adjoint functor pair, where it will say that  $(-)^\delta$  and  $(-)^0$  are respectively left and right adjoint of the forgetful functor  $U: \text{Top} \rightarrow \text{Set}$ .

DEFINITION 1.3.2. Given categories  $\mathcal{C}$  and  $\mathcal{D}$  an *adjunction* from  $\mathcal{C}$  to  $\mathcal{D}$  consists of a pair of functors

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

and for each pair of objects  $x \in \mathcal{C}$  and  $y \in \mathcal{D}$  a bijection of sets

$$\varphi_{x,y}: \text{Hom}_{\mathcal{D}}(Fx, y) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(x, Gy)$$

that is natural in both  $x$  and  $y$ . We call  $F$  a *left-adjoint* for  $G$  and  $G$  a *right-adjoint* for  $F$  and often write  $F \dashv G$ .

REMARK 1.3.3. Naturality of the bijections  $\{\varphi_{x,y}\}$  can be expressed by saying that

$$\varphi: \text{Hom}_{\mathcal{D}}(F(-), -) \Rightarrow \text{Hom}_{\mathcal{C}}(-, G(-))$$

is a natural isomorphism of functors  $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{Set}$ .

As a notational remark, we will usually omit the subscript on  $\varphi_{x,y}$  and simply write  $\varphi$  for any component of the natural transformation  $\varphi$ .

EXAMPLE 1.3.4. In Example 1.3.1 we showed that  $(-)^{\delta} \dashv U$  and  $U \dashv (-)^0$ . In particular,  $U$  is both a right-adjoint (with left-adjoint  $(-)^{\delta}$ ) and a left-adjoint (with right-adjoint  $(-)^0$ ).

There are other, equivalent ways of talking about adjunctions. For this observe that if  $F$  is a left adjoint for  $G$ , then the adjunction provides a natural transformation  $\eta: 1 \Rightarrow GF$  with

$$\eta_x := \varphi(1_{Fx}) \in \text{Hom}_{\mathcal{C}}(x, GFx).$$

Naturality follows from the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(Fx, Fx) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(x, GFx) \\ \downarrow (Ff)_* & & \downarrow (GFf)_* \\ \text{Hom}_{\mathcal{D}}(Fx, Fy) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(x, GFy) \\ \uparrow (Ff)^* & & \uparrow f^* \\ \text{Hom}_{\mathcal{D}}(Fy, Fy) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(y, GFy) \end{array}$$

which shows (by chasing  $1_{Fx}$  and  $1_{Fy}$  through the diagram) that  $GF(f) \circ \eta_x = \eta_y \circ f$  for every morphism  $f: x \rightarrow y$ . Similarly, we get a natural transformation  $\varepsilon: FG \Rightarrow 1$  by setting

$$\varepsilon_y := \varphi^{-1}(1_{Gy}).$$

These natural transformations are called the *unit* and *co-unit* of the adjunction. They encode the natural bijection  $\varphi$  in the following fashion:

$$(1.3.1) \quad \varphi(f) = Gf \circ \eta_x, \quad \varphi^{-1}(g) = \varepsilon_y \circ Fg.$$

The first identity follows by chasing  $1_{Fx}$  through the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Fx, Fx) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathcal{C}}(x, GFx) \\ \downarrow f_* & & \downarrow (Gf)_* \\ \mathrm{Hom}_{\mathcal{D}}(Fx, y) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathcal{C}}(x, Gy) \end{array}$$

and the second identity follows similarly.

Unit and co-unit can be used to give the following equivalent characterization of an adjoint functor pair.

PROPOSITION 1.3.5. *An adjunction*

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

with  $F$  left-adjoint to  $G$  determines and is determined by natural transformations  $\eta: 1 \Rightarrow GF$  and  $\varepsilon: FG \Rightarrow 1$  satisfying the triangle identities

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow = & \downarrow \varepsilon F \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow = & \downarrow G\varepsilon \\ & & G. \end{array}$$

REMARK 1.3.6. Here, by definition,  $F\eta$  is the natural transformation whose components are  $(F\eta)_x := F(\eta_x)$  and  $\varepsilon F$  is the natural transformation whose components are  $(\varepsilon F)_x := \varepsilon_{Fx}$ .

PROOF OF PROPOSITION 1.3.5. We have already proved almost half of the proposition. Let us check that  $\eta$  and  $\varepsilon$ , when they come from an adjunction  $F \dashv G$ , indeed satisfy the triangle identities. This follows simply from (1.3.1) as

$$1_{Fx} = \varphi^{-1}(\eta_x) = \varepsilon_{Fx} \circ F(\eta_x),$$

and

$$1_{Gy} = \varphi(\varepsilon_y) = G\varepsilon_y \circ \eta_{Gy}.$$

Conversely, suppose we are given the natural transformations  $\eta$  and  $\varepsilon$  and they satisfy the triangle identities. Then define

$$\varphi: \mathrm{Hom}_{\mathcal{D}}(Fx, y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(x, Gy) : \psi$$

by  $\varphi(f) := Gf \circ \eta_x$  and  $\psi(g) := \varepsilon_y \circ Fg$ . These maps are natural, because  $\eta$  and  $\varepsilon$  are. Now for every  $g: x \rightarrow Gy$  we have

$$\varphi(\psi(g)) = G(\varepsilon_y \circ Fg) \circ \eta_x = G\varepsilon_y \circ GFg \circ \eta_x = G\varepsilon_y \circ \eta_{Gy} \circ g = g.$$

In the third equality we used naturality of  $\eta$ , and in the last equality we used the triangle identity. Hence,  $\varphi \circ \psi = \mathrm{id}$ . Similarly, one shows that  $\psi \circ \varphi = \mathrm{id}$ .  $\square$

Often the formal triangle identities of units and counits are easier to manipulate than the explicit bijections of Hom-sets. This also gives a concrete way of checking that two functors are adjoint, by specifying units and counits.

Adjoint functors have proved an amazingly useful concept. A reason is, as we shall return to in Section 1.7, that left adjoints preserve colimits and right adjoints preserve limits (and in fact, modulo set-theoretic issues, this characterizes left and right adjoints). Seeing that a functor is an adjoint is usually easier than verifying that it preserves (co)limits.

Another important point to mention is that adjoints are unique up to natural isomorphism. Sometimes one can prove that two functors are naturally isomorphic by showing that they are a left (or right) adjoint for the same functor.

**PROPOSITION 1.3.7.** *Any two left-adjoints  $F$  and  $F'$  of a functor  $G: \mathcal{D} \rightarrow \mathcal{C}$  are naturally isomorphic. The analogous statement for right-adjoints holds as well.*

**PROOF.** One can prove this using the Yoneda lemma 1.2.1. By definition of adjunctions we have for every morphism  $f: x \rightarrow y$  of  $\mathcal{C}$  a commutative diagram of natural transformations between functors  $\mathcal{D} \rightarrow \text{Set}$

$$\begin{array}{ccccc} \text{Hom}_{\mathcal{D}}(Fy, -) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(y, G(-)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{D}}(F'y, -) \\ \downarrow (Ff)^* & & \downarrow f^* & & \downarrow (F'f)^* \\ \text{Hom}_{\mathcal{D}}(Fx, -) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(x, G(-)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{D}}(F'x, -). \end{array}$$

By Lemma 1.2.1 the composite natural transformation in the top row is induced by a unique isomorphism  $\tau_y: F'y \rightarrow Fy$  and, likewise, the composition of the natural transformations in the bottom row is induced by a unique isomorphism  $\tau_x: F'x \rightarrow Fx$ . Moreover, there is a unique arrow  $F'x \rightarrow Fy$  which induces the natural transformation obtained by either going through the top right or the bottom left corner of the diagram. Both  $\tau_y \circ F'f$  as well as  $Ff \circ \tau_x$  induce this natural transformation, so they must be equal:  $\tau_y \circ F'f = Ff \circ \tau_x$ . This shows that  $\tau$  is a natural isomorphism  $F' \cong F$ .  $\square$

**1.3.1. Examples of adjoint functors.** We will now give some examples of types of adjoint functors that occur frequently. We urge the reader to think of which of their other favorite functors are part of adjoint pairs, and can also consult the classic [Mac71] or its modern counterpart [Rie17] for inspiration.

**EXAMPLE 1.3.8** (Free-forgetful adjunction). The functor  $R[-] : \text{Sets} \rightarrow R\text{-modules}$  given by  $X \mapsto RX$ , the free  $R$ -module on  $X$  has a right adjoint given by forgetful functor from  $R$ -modules to sets. I.e., the left adjoint is given by freely generating the extra structure. The unit is given  $\eta_X : X \rightarrow RX$  in  $\text{Set}$  given by inclusion and counit  $\epsilon_M : RM \rightarrow M$  given by evaluating a formal sum using the  $R$ -module structure on  $M$ .

Note that the same construction gives a functor  $R[-]$  from monoids into  $R$ -algebras, which again is left adjoint to the forgetful functor from  $R$ -algebras to monoids, forgetting the  $R$ -module structure and only remembering the multiplication.

**EXAMPLE 1.3.9** (Reflective subcategories). These are full subcategories where the inclusion  $i : \mathcal{C} \rightarrow \mathcal{D}$  has a left adjoint  $L$  (“the reflector”).

ex:reflective

An example to keep in mind is the inclusion of  $S$ -modules into  $R$ -modules via a map  $R \rightarrow S$ , where the left adjoint is given by “base-change”  $S \otimes_R (-)$ . A related example is the inclusion of groupoids into categories, which has a left adjoint where we formally adjoin inverses to all morphisms. Another example is the inclusion of abelian groups into all groups, where the left adjoint is abelianization, or torsion-free abelian groups into all abelian groups, where the left-adjoint quotients out by torsion. In all of these examples we are potentially adding some generators, suitably freely, and then modding out by some relations.

We may think of the unit of the adjunction  $\mathbb{1} \Rightarrow i \circ L : \mathcal{D} \rightarrow \mathcal{D}$  as a “localization”. Beware that as  $i \circ L$  is the composite of a left and a right adjoint, it may not itself have adjoints.

ex:coreflective

EXAMPLE 1.3.10 (Coreflective subcategories). These are full subcategories where the inclusion  $i : \mathcal{C} \rightarrow \mathcal{D}$  has a right adjoint  $R$  (“the coreflector”). The inclusion of groupoids into categories is one example, where the right adjoint is the groupoid core functor  $(-)^{\simeq}$  that assigns the subcategory of isomorphisms (in particular  $(-)^{\simeq}$  sends a monoid to its largest subgroup). Another example is torsion-abelian groups into abelian groups. A third example is the inclusions of topological spaces having the homotopy type of a CW complex into the homotopy category of all topological spaces, where the right adjoint is CW approximation. A related example is the subcategory  $k\text{Top}$  of  $\text{Top}$  given by spaces having the compactly generated topology, also called “Kelly spaces”, where a set is open iff its restriction every compact set is open. Here the right adjoint  $k$  is the functor which re-topologizes a space to make it compactly generated. We will work this out in Proposition 1.3.13 below.

We may think of the counit of the adjunction  $i \circ R \Rightarrow \mathbb{1} : \mathcal{D} \rightarrow \mathcal{D}$  as a “colocalization”, and repeat the warning that this may not have adjoints from earlier.

Note e.g., that the functor  $R[-] : \text{Groups} \rightarrow R\text{-algebras}$ , assigning to  $G$  its groupring  $RG$  has a right adjoint given by the group of units  $(-)^{\times}$ , which can again be viewed as the composite of two right adjoints: The forgetful functor from  $R$ -algebras to monoids, followed by the groupoid core functor  $(-)^{\simeq}$  to groups.

ex:homtensor

EXAMPLE 1.3.11 (Hom- $\otimes$ -adjunctions). In this example we give some examples of adjoint functors arising as “Hom- $\otimes$ -adjunctions”, a discussion we continue in Section 1.4.

- (1) The original Hom- $\otimes$ -adjunction: Let  $R$  be a commutative ring, and let  $A, B, C$  be  $R$ -modules. Then there are natural bijections

$$\text{Hom}_R(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_R(B, C)),$$

so  $- \otimes_R B \dashv \text{Hom}_R(B, -)$ . Note that in order to make sense of the right hand side we must view  $\text{Hom}_R(B, C)$  as an  $R$ -module and not just as a set. As an exercise you can convince yourself that the co-unit of the adjunction is given by evaluation

$$\begin{aligned} \text{Hom}_R(B, C) \otimes B &\rightarrow C \\ (f, b) &\mapsto f(b) \end{aligned}$$

while the unit takes the form

$$\begin{aligned} A &\rightarrow \text{Hom}_R(B, A \otimes B) \\ a &\mapsto (b \mapsto a \otimes b). \end{aligned}$$

**item:setht**

(2) Let  $X, Y, Z \in \text{Set}$ . Then there are natural bijections

$$\text{Hom}_{\text{Set}}(X \times Y, Z) \cong \text{Hom}_{\text{Set}}(X, \text{Hom}_{\text{Set}}(Y, Z)),$$

and co-unit and unit take much the same form as in the  $\text{Hom}$ - $\otimes$ -adjunction.

**ex:CGHaus**

(3) It is desirable to have the previous adjunction available also in the category  $\text{Top}$ . Let  $X, Y, Z$  be topological spaces. Define the *mapping space*

$$\text{map}(Y, Z)$$

to be the set  $\text{Hom}_{\text{Top}}(Y, Z)$  equipped with the *compact-open topology*: That is, a subbasis for the topology is the collection of sets

$$\{f \in \text{Hom}_{\text{Top}}(Y, Z) \mid f(K) \subseteq U\}_{K,U},$$

where  $K$  runs through compact subsets of  $Y$  and  $U$  runs through open subsets of  $Z$ . One can then show that there are natural bijections

$$\text{Hom}_{\text{Top}}(X \times Y, Z) \cong \text{Hom}_{\text{Top}}(X, \text{map}(Y, Z))$$

provided that  $X, Y$  are Hausdorff and  $Y$  is locally compact.

These point-set topological assumptions are of course unfortunate, as we do not get an adjunction on all of  $\text{Top}$ —note that even when  $Y$  and  $Z$  are locally compact,  $\text{map}(Y, Z)$  need not be. A way out is to restrict the category  $\text{Top}$  appropriately to what some refer to as a “convenient category of topological spaces”. One such category is the category  $\text{CGHaus}$  of all compactly generated Hausdorff spaces. As mentioned above it turns out that the full embedding  $\text{CGHaus} \hookrightarrow \text{Haus}$  into all Hausdorff spaces is coreflective, with right-adjoint  $k: \text{Haus} \rightarrow \text{CGHaus}$ . The category  $\text{CGHaus}$  is easily seen to have products given as  $k(X \times Y)$  (see also Proposition 1.8.5 below). With these modifications, one then obtains for all  $X, Y, Z \in \text{CGHaus}$  a bijection

$$\text{Hom}_{\text{CGHaus}}(X \times_k Y, Z) \cong \text{Hom}_{\text{CGHaus}}(X, k \text{map}(Y, Z)),$$

where  $(- \times_k -) = k(- \times -)$  is the categorical product and  $k(\text{map}(-, -))$  an “internal hom object” in  $\text{CGHaus}$ . We will elaborate on this in Proposition 1.3.13 and Section 1.4 below.

(4) There is also a basepointed version of the previous example: Let  $\text{CGHaus}_*$  be the category of compactly generated Hausdorff spaces equipped with a basepoint, and basepoint preserving maps between them. Let  $X, Y, Z \in \text{CGHaus}_*$ .<sup>1</sup> Then there is a natural bijection

$$\text{Hom}_{\text{CGHaus}_*}(X \wedge Y, Z) \cong \text{Hom}_{\text{CGHaus}_*}(X, k \text{map}_*(Y, Z)),$$

where  $\text{map}_*(Y, Z) \subseteq k \text{map}(Y, Z)$  is the subspace of basepoint preserving maps and

$$X \wedge Y := X \times_k Y / X \vee Y$$

<sup>1</sup>It is standard to write “ $x \in \mathcal{C}$ ” meaning that  $x$  is an object of the category  $\mathcal{C}$ .

is the *smash-product* of  $X$  and  $Y$  (see AlgTop II).

In particular, if  $Y = S^1$ , then  $X \wedge S^1 = \Sigma X$  is the reduced suspension of  $X$ , and  $\text{map}_*(S^1, Z) = \Omega Z$  is the loop space of  $Z$ . Consequently, we obtain an adjunction  $\Sigma \dashv \Omega$  on the category CGHaus $_*$ .

Let us end the section by, for completeness verifying the properties of the subspace  $k\text{Top}$  of  $\text{Top}$  claimed above, as it plays a role later, when discussing the so-called nerve functor—the impatient reader may charge ahead to the next section and only return to this later. Let us repeat the definition.

**DEFINITION 1.3.12.** Define  $k\text{Top}$ , the category of compactly generated topological spaces, to be the full subcategory of  $\text{Top}$  consisting of spaces  $X$  with the property that a subset  $A \subseteq X$  is open if and only if  $A \cap K$  is open in  $K$  for all compact subsets  $K \subseteq X$ .

prop:ktop

**PROPOSITION 1.3.13.** *Let  $k : \text{Top} \rightarrow k\text{Top}$  be the functor which re-topologizes a topological space  $X$ , by the topology declaring a subset  $U$  to be open in  $k(X)$  if  $U \cap K$  is open in  $K$  for all compact subsets  $K \subseteq X$ .*

*The functor  $k$  is right adjoint to the inclusion functor  $k\text{Top} \rightarrow \text{Top}$ , i.e., for  $X \in k\text{Top}$  and  $Y \in \text{Top}$  we have*

$$\text{Hom}_{\text{Top}}(X, Y) \cong \text{Hom}_{k\text{Top}}(X, k(Y)) = \text{Hom}_{\text{Top}}(X, k(Y)).$$

*Any CW complex is in  $k\text{Top}$ , and  $\text{Top}$  and  $k\text{Top}$  have the same compact subsets.*

**PROOF.** One verifies directly that this indeed defines a topology and that the functor  $k$  is idempotent (i.e.  $kkY \cong kY$  naturally in  $Y$ ). One also checks that if  $X$  is compactly generated and  $Y$  is any space then  $\text{Hom}_{\text{Top}}(X, Y) \cong \text{Hom}_{k\text{Top}}(X, kY)$ .

Namely  $\text{Hom}_{k\text{Top}}(X, kY) \subseteq \text{Hom}_{\text{Top}}(X, Y)$  since  $kY$  has more open sets than  $Y$ . Assume that  $f : X \rightarrow Y$  is continuous, and let  $U \subseteq Y$  be open in  $kY$ , i.e.,  $U \cap L$  open in  $L$  for all compact subsets  $L \subseteq Y$ . To see that  $f^{-1}(U)$  is open in  $X$  it is enough to see that  $f^{-1}(U) \cap K$  is open in  $K$  for all compact  $K$ , by the assumption  $X \in k\text{Top}$ . But  $f^{-1}(U) \cap K = (f|_K)^{-1}(U) = (f|_K)^{-1}(U \cap f(K))$ . And as  $f : X \rightarrow Y$  is continuous,  $f(K)$  is compact, and therefore  $f(K) \cap U$  is open in  $f(K)$  by assumption on  $U$ . But then  $f^{-1}(U) \cap K$  is open in  $K$  as desired, as  $f|_K$  is continuous.

That CW complexes are in  $k\text{Top}$  follows by the definition of topology on CW complexes (as finite subcomplexes are compact, and any compact subset lies in a finite subcomplex, see Hatcher [Hat02, Prop. A.1]). It follows by the definition that they have the same compact subsets.  $\square$

**PROPOSITION 1.3.14.** *The counit map  $kY \rightarrow Y$  in  $\text{Top}$  is a weak equivalence for any topological space  $Y$ .*

**PROOF.** It follows from the adjunction that  $\text{Hom}_{\text{Top}}(X, Y) = \text{Hom}_{\text{Top}}(X, kY)$  for any finite CW-complex  $X$  and arbitrary  $Y$  as any finite CW complex lies in  $k\text{Top}$  by Proposition 1.3.13.  $\square$

## 1.4. Monoidal and closed categories

ec:monoidalcat

Let us make a few additional remarks about the adjunctions of Example 1.3.11. Here the left-adjoint was some kind of product and the right-adjoint always some

kind of mapping object. Note however that the product was not always a categorical product, for example, the tensor product of modules is not their product, nor is the smash product of based spaces. The general framework into which these examples fit is that of *closed categories*. To explain what a closed category is one must first explain what a *monoidal category* is. Very briefly, a monoidal category consists of

- a category  $\mathcal{C}$
- a functor  $\square: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  (product), see also Proposition 1.6.1
- an object  $\mathbf{1} \in \mathcal{C}$  (unit)

together with specified natural isomorphisms

- $x\square(y\square z) \cong (x\square y)\square z$  (associativity)
- $x\square\mathbf{1} \cong x \cong \mathbf{1}\square x$  (unitality)

for all  $x, y, z \in \mathcal{C}$ . These isomorphisms are required to make certain “coherence diagrams” commute, which we will not spell out. For the purposes here, it suffices to know that a monoidal category is a category with some kind of product on it that behaves more or less how we expect it. Let us write  $(\mathcal{C}, \square, \mathbf{1})$  for a monoidal category, omitting the associativity and identity isomorphisms from the notation.

A *braiding* on a monoidal category  $(\mathcal{C}, \square, \mathbf{1})$  consists of natural isomorphisms  $x\square y \cong y\square x$  for all  $x, y \in \mathcal{C}$  which are again required to make certain diagrams commute. A *symmetric* monoidal category is a braided monoidal category subject to the condition that the double braiding  $x\square y \cong y\square x \cong x\square y$  is the identity for all  $x, y \in \mathcal{C}$ . Examples of symmetric monoidal categories are all of the above mentioned examples

$$(R\text{-Mod}, \otimes, R), \quad (\text{Set}, \times, \{*\}), \quad (\text{CGHaus}, \times, \{*\}), \quad (\text{CGHaus}_*, \wedge, S^0).$$

(Recall that indeed  $X \wedge S^0 \cong X \cong S^0 \wedge X$ ).

Now one calls a symmetric monoidal category  $(\mathcal{C}, \square, \mathbf{1})$  *closed* if for each  $x \in \mathcal{C}$  the functor

$$-\square x: \mathcal{C} \rightarrow \mathcal{C}$$

has a specified right-adjoint, which is usually denoted  $(-)^x$  (also called an “exponential”). And one says that  $\mathcal{C}$  is *cartesian closed* if  $\square = \times$  happens to be the categorical product.

Phrased in this language we can summarize Example 1.3.11 by saying that  $R\text{-Mod}$  and  $\text{CGHaus}_*$  are closed symmetric monoidal categories under tensor product and smash product, respectively, and  $\text{Set}$  and  $\text{CGHaus}$  are cartesian closed. Their “exponentials” are mapping objects like  $\text{map}(Y, Z)$  or  $\text{Hom}_R(B, C)$ . Note that these are just the Hom-sets of the respective categories equipped with some extra structure making them objects of this very same category! For this reason they are also referred to as “internal Hom-objects”.

### 1.5. Equivalence of categories

We have an obvious notion of isomorphism of categories: a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is an isomorphism, if there is a functor  $G: \mathcal{D} \rightarrow \mathcal{C}$  such that  $FG = 1$  and  $GF = 1$ . But in practice, categories are rarely isomorphic. *Equivalence* of categories is a much more flexible and useful notion than isomorphism of categories.



DEFINITION 1.5.1. A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is an *equivalence* if there exists a functor  $G: \mathcal{D} \rightarrow \mathcal{C}$  and natural isomorphisms  $GF \cong 1$  and  $FG \cong 1$ .

Any equivalence  $F: \mathcal{C} \rightarrow \mathcal{D}$  gives rise to an adjunction (which we may then call an *adjoint equivalence*), but not every adjunction is an equivalence. The following proposition makes the relationship precise, and at the same time gives us a criterion to check that a given functor is an equivalence.

equivalence

PROPOSITION 1.5.2. For a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  the following are equivalent:

- (1)  $F$  is an equivalence
- (2)  $F$  has a right-adjoint  $G$  such that unit  $\eta: 1 \Rightarrow GF$  and co-unit  $\varepsilon: FG \Rightarrow 1$  are natural isomorphisms
- (3)  $F$  has a left-adjoint  $G'$  such that unit  $\eta': 1 \Rightarrow FG'$  and co-unit  $\varepsilon': G'F \Rightarrow 1$  are natural isomorphisms
- (4)  $F$  is fully faithful and essentially surjective.

In the above, we say that  $F$  is *full* if for any two objects  $x, y$  of  $\mathcal{C}$  the map of sets induced by  $F$

$$(1.5.1) \quad \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

is surjective. We call  $F$  *faithful* if for any two objects  $x, y$  of  $\mathcal{C}$  the map (1.5.1) is injective. We say that  $F$  is *essentially surjective* if for every object  $y$  of  $\mathcal{D}$  there is an object  $x$  of  $\mathcal{C}$  and an isomorphism  $Fx \cong y$ .

PROOF OF PROPOSITION 1.5.2. We first show that (2)  $\Rightarrow$  (3). Suppose that  $F \dashv G$  is an adjunction and that unit  $\eta: 1 \Rightarrow GF$  and co-unit  $\varepsilon: FG \Rightarrow 1$  are natural isomorphisms. We claim that  $G$  is then also a left-adjoint for  $F$  with unit  $\eta' := \varepsilon^{-1}$  and co-unit  $\varepsilon' := \eta^{-1}$ . By Proposition 1.3.5 it is enough to check the triangle identities, that is, we must check that  $F\eta^{-1} \circ \varepsilon^{-1}F = 1$  and  $\eta^{-1}G \circ G\varepsilon^{-1} = 1$ . But clearly, this is equivalent to  $\varepsilon F \circ F\eta = 1$  and  $G\varepsilon \circ \eta G = 1$ , which are the triangle identities for the adjunction  $F \dashv G$ . This proves that  $G$  is a left-adjoint for  $F$ . In the same fashion we show that (3)  $\Rightarrow$  (2).

It is clear that (2)  $\Rightarrow$  (1). Next we show that (1)  $\Rightarrow$  (4). Let  $G: \mathcal{D} \rightarrow \mathcal{C}$  denote the inverse equivalence. The natural isomorphism  $FG \cong 1$  shows that for each object  $y$  of  $\mathcal{D}$  there is an isomorphism  $FGy \cong y$ . This proves that  $F$  is essentially surjective. To show that  $F$  is faithful and full let  $\theta: GF \cong 1$  be a natural isomorphism. For every morphism  $f: x \rightarrow y$  of  $\mathcal{C}$  there is a commutative diagram

$$\begin{array}{ccc} GFx & \xrightarrow[\cong]{\theta_x} & x \\ GFf \downarrow & & \downarrow f \\ GFy & \xrightarrow[\cong]{\theta_y} & y \end{array}$$

which shows that the map

$$\text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{F} \text{Hom}_{\mathcal{D}}(Fx, Fy) \xrightarrow{G} \text{Hom}_{\mathcal{C}}(GFx, GFy)$$

is given by  $f \mapsto GFf = \theta_y^{-1} \circ f \circ \theta_x$ . Since this map is a bijection,  $F$  is faithful. By symmetry,  $G$  is faithful, too. To show that  $F$  is full, let  $h \in \text{Hom}_{\mathcal{D}}(Fx, Fy)$ . Define

$f := \theta_y \circ Gh \circ \theta_x^{-1}$ . Then  $GFf = Gh$ , but since  $G$  is faithful, this implies that  $Ff = h$ . Hence,  $F$  is full.

To prove that (4)  $\Rightarrow$  (2) we *choose*<sup>2</sup> for every object  $y$  of  $\mathcal{D}$  an object  $Gy$  of  $\mathcal{C}$  and an isomorphism  $\varepsilon_y: FGy \cong y$ . If  $f: y \rightarrow y'$  is a morphism, then, since  $F$  is full and faithful, there is a unique morphism  $Gf: G(y) \rightarrow G(y')$  with  $FGf = \varepsilon_{y'}^{-1} \circ f \circ \varepsilon_y$ . By construction, the diagram

$$\begin{array}{ccc} FG(y) & \xrightarrow{\varepsilon_y} & y \\ \downarrow FG(f) & & \downarrow f \\ FG(y') & \xrightarrow{\varepsilon_{y'}} & y' \end{array}$$

commutes. By uniqueness,  $G(f \circ f') = G(f) \circ G(f')$ , hence  $G$  is a functor and  $\varepsilon: FG \Rightarrow 1$  is a natural isomorphism. In fact,  $F$  and  $\varepsilon$  induce bijections

$$\mathrm{Hom}_{\mathcal{C}}(x, Gy) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(Fx, FGy) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(Fx, y),$$

so  $G$  is a right-adjoint for  $F$  and  $\varepsilon$  is the counit of the adjunction. It remains to see that the unit of the adjunction  $\eta: 1 \Rightarrow GF$  is also a natural isomorphism, but this follows from the triangle identities.  $\square$

EXAMPLE 1.5.3. Part (4) of Proposition 1.5.2 provides a way to replace a category  $\mathcal{C}$  by an equivalent (possibly smaller) category  $\mathrm{sk}(\mathcal{C})$  in the following way: For every isomorphism class of objects of  $\mathcal{C}$  choose precisely one representative. Then take  $\mathrm{sk}(\mathcal{C})$  to be the full subcategory of  $\mathcal{C}$  on those representatives. By construction, the inclusion  $\mathrm{sk}(\mathcal{C}) \subset \mathcal{C}$  satisfies (4), so it is an equivalence. We call  $\mathrm{sk}(\mathcal{C})$  a *skeleton* of  $\mathcal{C}$ .

As a concrete example, take  $\mathcal{C}$  to be the category  $\mathrm{Vect}_k^{fd}$  of finite dimensional  $k$ -vectorspaces and  $k$ -linear maps. From Linear Algebra we know that a finite dimensional vector space is isomorphic to  $k^n$  for some  $n \in \mathbb{N}$ . The linear maps  $k^n \rightarrow k^m$  are precisely the  $m \times n$  matrices over  $k$ . So the category whose objects are  $0, 1, 2, \dots$  and whose Hom-sets are  $\mathrm{Hom}(n, m) = \mathrm{Mat}_{m \times n}(k)$  is equivalent to  $\mathrm{Vect}_k^{fd}$ . They are certainly not isomorphic, because the objects of  $\mathrm{Vect}_k^{fd}$  form a proper class.

EXAMPLE 1.5.4. [ Morita equivalence? ]

## 1.6. The category Cat of small categories

An important category for us will be the (big!!)<sup>3</sup> category  $\mathrm{Cat}$  of all small categories. It has objects small categories and morphisms functors. (It even has a bit of higher structure in that we can define 2-morphisms as natural transformations, but that shall not concern us here.)

Let's start with the following easy observation:

<sup>2</sup>In this step, we need a strong enough version of the axiom of choice: note indeed that there is a choice to be made for each object of  $\mathcal{D}$ , and these form a class that, possibly, is not a set!

<sup>3</sup>As the word "big" suggests,  $\mathrm{Cat}$  is not itself a small category, but it is still locally small, in the sense of Definition 1.1.1.

productcategory

PROPOSITION 1.6.1. *For two categories  $\mathcal{C}$  and  $\mathcal{D}$  in  $\text{Cat}$  the categorical product exists. The product category  $\mathcal{C} \times \mathcal{D}$  is the category whose objects are pairs  $(c, d)$ , where  $c$  is an object in  $\mathcal{C}$  and  $d$  is an object in  $\mathcal{D}$ ; a morphism  $(c, d) \rightarrow (c', d')$  in  $\mathcal{C} \times \mathcal{D}$  is a pair  $(f, g)$ , where  $f : c \rightarrow c'$  is a morphism in  $\mathcal{C}$  and  $g : d \rightarrow d'$  a morphism in  $\mathcal{D}$ .  $\square$*

EXERCISE 1.6.2. Explicitly describe the objects and morphisms in  $[n] \times [m]$ .

cat-map

PROPOSITION 1.6.3. *For small categories  $\mathcal{C}$  and  $\mathcal{D}$ , let  $\text{Fun}(\mathcal{C}, \mathcal{D})$  be the category with objects functors  $\mathcal{C} \rightarrow \mathcal{D}$  and morphisms natural transformations between them. Then for small categories  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  have a natural bijection*

$$\text{Hom}_{\text{Cat}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \xrightarrow{\cong} \text{Hom}_{\text{Cat}}(\mathcal{C}, \text{Fun}(\mathcal{D}, \mathcal{E}))$$

*inducing an isomorphism of functors  $\text{Cat}^{\text{op}} \times \text{Cat}^{\text{op}} \times \text{Cat} \rightarrow \text{Set}$ . In other words,  $\text{Fun}(\mathcal{D}, -)$  is right adjoint to  $- \times \mathcal{D}$  as endofunctors of  $\text{Cat}$ , i.e.,  $\text{Fun}(\mathcal{D}, \mathcal{E})$  is an internal hom functor in  $\text{Cat}$ , endowed with the cartesian monoidal structure  $\square = - \times - : \text{Cat} \times \text{Cat} \rightarrow \text{Cat}$ , in the language of Section 1.4.*

PROOF. This is an exercise in unraveling the definitions.  $\square$

Let us look at what homotopies look like, from a categorical point of view:

cat-homotopy

PROPOSITION 1.6.4. *For small categories  $\mathcal{C}, \mathcal{D}$ , the functor category  $\text{Fun}(\mathcal{C} \times [1], \mathcal{D})$  has objects that can be identified with triples  $(F, G, \kappa)$  where  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  are functors and  $\kappa : F \Rightarrow G$  a natural transformation.*

PROOF. The objects  $\mathcal{C} \times [1]$  are of the form  $(c, 0)$  and  $(c, 1)$ , and the morphisms generated by  $(c, 0) \xrightarrow{(f, id)} (d, 0)$ ,  $(c, 1) \xrightarrow{(f, id)} (d, 1)$ , and  $(c, 0) \rightarrow (c, 1)$ . Hence specifying a functor  $H : \mathcal{C} \times [1] \rightarrow \mathcal{D}$ , amounts to specifying functors  $H(-, 0) : \mathcal{C} \times 0 \rightarrow \mathcal{D}$ ,  $H(-, 1) : \mathcal{C} \times 1 \rightarrow \mathcal{D}$ , and  $H(c, 0 < 1) : H(c, 0) \rightarrow H(c, 1)$  such that

$$\begin{array}{ccc} H(c, 0) & \xrightarrow{H(c, 0 < 1)} & H(c, 1) \\ \downarrow H(f, 0) & & \downarrow H(f, 1) \\ H(d, 0) & \xrightarrow{H(d, 0 < 1)} & H(d, 1) \end{array}$$

commutes for all  $f : c \rightarrow d$ . Hence identifying  $F(-) = H(-, 0)$ ,  $G = H(-, 1)$  and  $\kappa_c = H(c, 0 < 1)$ , this is exactly the data  $(F, G, \kappa)$  as claimed.  $\square$

REMARK 1.6.5. From this point of view, natural transformations look like “categorical homotopies”, at least if we regard the category  $[1]$  as the counterpart of an interval. We shall return to this viewpoint. Note however now that we see an asymmetry in the above description, as natural transformations cannot in general be reversed! We need much work from higher algebra to properly deal with this issue.

## 1.7. Limits and colimits

sec:limcolim

DEFINITION 1.7.1 (Limits and colimits). Suppose  $\mathcal{I}$  is a small category. We write  $\delta$  for the constant-diagram (aka diagonal) functor  $\delta : \mathcal{C} \rightarrow \text{Fun}(\mathcal{I}, \mathcal{C})$ , induced by the constant functor  $\mathcal{I} \rightarrow *$ .

- (1) The *limit* of a functor  $F: \mathcal{I} \rightarrow \mathcal{C}$  is an object  $\lim_{\mathcal{I}} F$  in  $\mathcal{C}$  together with a morphism  $\delta(\lim F) \rightarrow F$  in  $\mathcal{C}^{\mathcal{I}}$  such that the induced map

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} F) \rightarrow \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta X, \delta \lim_{\mathcal{I}} F) \rightarrow \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta X, F)$$

is an isomorphism for every  $X \in \mathcal{C}$ . We can illustrate this as follows:

$$\begin{array}{ccc} X & & \\ & \searrow & \\ & \lim F & \longrightarrow Y_j \\ & \downarrow & \downarrow \\ & Y_i & \longrightarrow Y_k \end{array}$$

(Note: A dotted arrow from  $X$  to  $\lim F$  is labeled with  $\exists!$ )

- (2) The *colimit* of an object  $F \in \mathcal{C}^{\mathcal{I}}$  is an object  $\mathrm{colim}_{\mathcal{I}} F \in \mathcal{C}$  together with a morphism  $F \rightarrow \delta \mathrm{colim}_{\mathcal{I}} F$  such that the induced map

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{I}} F, X) \rightarrow \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta \mathrm{colim}_{\mathcal{I}} F, \delta X) \rightarrow \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \delta X)$$

is an isomorphism for every  $X \in \mathcal{C}$ .

For an arbitrary category  $\mathcal{C}$  the limit of over  $\mathcal{I}$  may or may not exist. However, if it does, it can be described in standard categorical terms:

**PROPOSITION 1.7.2.** *For a fixed  $F: \mathcal{I} \rightarrow \mathcal{C}$  the limit of  $F$  exists if and only if the functor  $\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta(-), F): \mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Set}$  is representable. If it exists,  $\lim_{\mathcal{I}} F$  is the object satisfying  $\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta(-), F) \cong \mathrm{Hom}_{\mathcal{C}}(-, \lim_{\mathcal{I}} F)$ , unique up to isomorphism.*

*Similarly  $\mathrm{colim}_{\mathcal{I}} F$ , if it exists, is the, up to isomorphism, unique object satisfying  $\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \delta(-)) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{I}} F, -)$*

**PROOF.** The fact that  $\lim_{\mathcal{I}} F$  and  $\mathrm{colim}_{\mathcal{I}} F$  have these properties is a consequence of the definition. Uniqueness follows from the Yoneda Lemma 1.2.1.  $\square$

The following also follows from the definitions.

**PROPOSITION 1.7.3.** *If every functor  $F: \mathcal{I} \rightarrow \mathcal{C}$  has a limit, then  $\lim_{\mathcal{I}}$  defines a functor  $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}$  which is right adjoint to the diagonal functor  $\delta$ .*

*Similarly if every functor  $F: \mathcal{I} \rightarrow \mathcal{C}$  has a colimit, then  $\mathrm{colim}_{\mathcal{I}}$  defines a functor  $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}$  which is left adjoint to the diagonal functor  $\delta$ .*  $\square$

**EXERCISE 1.7.4.** If  $F: \mathcal{I} \rightarrow \mathcal{C}$  is a functor, then we also have a functor  $F^{\mathrm{op}}: \mathcal{I}^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$ . Show that if an object  $x \in \mathcal{C}$ , together with a morphism  $\phi: \delta x \rightarrow F$  in  $\mathcal{C}^{\mathcal{I}}$ , is a limit of  $F$ , then the “same” object  $x$  in  $\mathcal{C}^{\mathrm{op}}$ , together with the morphism  $\phi^{\mathrm{op}}: F^{\mathrm{op}} \rightarrow (\delta x)^{\mathrm{op}}$  in  $(\mathcal{C}^{\mathrm{op}})^{\mathcal{I}^{\mathrm{op}}}$ , is a colimit of  $F^{\mathrm{op}}$ . In few words: passing to opposite categories transforms limits into colimits. And vice-versa.

**EXAMPLE 1.7.5.** Here are some special cases:

- (1) The colimit over the empty set is called an *initial object*. The limit over the empty set is called a *terminal object*.

- (2) A limit over a discrete category (a category with only identity morphisms) is called a (categorical) *product*. A colimit over a discrete category is called a (categorical) *coproduct*.
- (3) A colimit over  $\mathbb{N}$  is a standard *direct limit*. A limit over  $\mathbb{N}^{\text{op}}$  is an *inverse limit*.
- (4) Limits over  $(\bullet \rightarrow \bullet \leftarrow \bullet)$  are called *pull-backs*. Colimits over  $(\bullet \leftarrow \bullet \rightarrow \bullet) = (\bullet \rightarrow \bullet \leftarrow \bullet)^{\text{op}}$  are called *push-outs*.
- (5) Limits and colimits over  $(\bullet \rightrightarrows \bullet)$  are called *equalizers* and *coequalizers* respectively (note that  $(\bullet \rightrightarrows \bullet)$  is isomorphic to its own opposite).
- (6) Limits over  $\mathcal{BG}$  are called *fixed-points* (or invariants, if the target category is abelian). Colimits over  $\mathcal{BG}$  are called *orbits* (or coinvariants if the target category is abelian).
- (7) The limit over  $[n]$  is just evaluation at 0; the colimit is evaluation at  $n$  (as these are initial and terminal objects).

Note that the limit of course crucially depends on in which category the category is taken. The definition however immediately gives the following:

PROPOSITION 1.7.6. *Suppose  $\iota : \mathcal{C}_0 \rightarrow \mathcal{C}$  is an inclusion of a full subcategory. If a diagram  $\kappa : \mathcal{I} \rightarrow \mathcal{C}_0$  has a limit or colimit in  $\mathcal{C}$  which lies in  $\mathcal{C}_0$ , then it is also a limit or colimit in  $\mathcal{C}_0$ .*  $\square$

Simple examples, such as the inclusion of the abelian group  $\mathbb{Z}$  as a full subcategory of the category of all abelian groups, show that the assumption that the limit lies in  $\mathcal{C}_0$  is not automatic—

More generally, other functors that  $\iota$  that satisfy the conclusion of the above proposition are said to “reflect limits or colimits”.

We say a category  $\mathcal{C}$  is (co)complete if all (co)limits indexed by any small category  $\mathcal{I}$  exist. Many suitably big categories that you know such as  $\text{Set}$ ,  $\text{Top}$ ,  $\text{Ab}$ , ... are both complete and cocomplete. Other categories, such as many diagram categories, or categories arising as homotopy categories are not. A key to

lim-in-functor

PROPOSITION 1.7.7. *Given a functor  $F : \mathcal{J} \rightarrow \mathcal{C}^{\mathcal{I}}$  such that the composite  $F_i : \mathcal{J} \rightarrow \mathcal{C}^{\mathcal{I}} \xrightarrow{ev_i} \mathcal{C}$  has a limit for each  $i \in \mathcal{I}$ . Then  $F$  has a limit as well given by the “pointwise” functor  $i \mapsto \lim_{\mathcal{J}} F(i)$ .*

*In particular if  $\mathcal{C}$  is (co)complete then so is  $\mathcal{C}^{\mathcal{I}}$  and (co)limits are calculated “objectwise”.*

PROOF. As

$$\text{Fun}(\mathcal{J}, \mathcal{C}^{\mathcal{I}}) \cong \text{Fun}(\mathcal{J} \times \mathcal{I}, \mathcal{C}) \cong \text{Fun}(\mathcal{I}, \mathcal{C}^{\mathcal{J}})$$

by Proposition 1.6.3, the claim is immediate from the universal property.  $\square$

copn:colimcoeq

PROPOSITION 1.7.8 (A model for colimit). *Let  $\mathcal{C}$  be a category with all coproducts and coequalizers. Then  $\mathcal{C}$  has all colimits, and for  $F \in \mathcal{C}^{\mathcal{I}}$ ,*

$$\text{colim } F \cong \text{coeq} \left( \coprod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} F(i) \rightrightarrows \coprod_{k \in \text{Ob}(\mathcal{I})} F(k) \right).$$

where the two morphisms in the coequalizer are given on the factor indexed by  $f: i \rightarrow j$  by  $F(i) \xrightarrow{\text{id}} F(i) \rightarrow \coprod_k F(k)$  and  $F(i) \xrightarrow{F(f)} F(j) \rightarrow \coprod_k F(k)$ , respectively.

Dually for limits.

REMARK 1.7.9. If  $\mathcal{C} = \text{Set}$  we can describe this coequalizer as the quotient  $\coprod_{i \in \mathcal{I}} F(i) / \sim$  where  $\sim$  is the equivalence relation generated by identifying  $x \in F(i)$  with  $f(x) \in F(j)$  for all morphisms  $f: i \rightarrow j$  in  $\mathcal{I}$  and all  $x \in F(i)$ .

PROOF OF PROPOSITION 1.7.8. By the Yoneda Lemma 1.2.1 it is enough to show that the representing functors agree. Hence it is enough to check the isomorphism in  $\text{Set}$ , where it is seen directly, as remarked above.  $\square$

EXERCISE 1.7.10. Prove by hand that  $\text{Cat}$  is complete and cocomplete, extending Proposition 1.6.1. (We will later get a slick proof of this using how  $\text{Cat}$  sits inside the functor category of simplicial sets, which is again complete and cocomplete by Proposition 1.7.7.)

EXAMPLE 1.7.11 (Some examples in abelian groups).

- $\lim \left( 0 \rightarrow B \xleftarrow{f} A \right) = \ker(f)$  and  $\text{colim} \left( 0 \rightarrow B \xleftarrow{f} A \right) = B$ .
- For  $\mathcal{I} = \mathcal{B}G$ , a group  $G$  considered as a category with one object, and  $\mathcal{C} = \text{Ab}$ ,  $\lim_{\mathcal{B}G} M \cong M^G$ , the invariants, and  $\text{colim}_{\mathcal{B}G} M \cong M_G$ , the coinvariants.
- $\lim(\cdots \rightarrow \mathbb{Z}/p^3 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p) \cong \mathbb{Z}_p$ , the  $p$ -adic integers. The colimit is  $\mathbb{Z}/p$ .
- $\text{colim}(\mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p^3 \rightarrow \cdots) = \mathbb{Z}/p^\infty$ , the injective envelope of  $\mathbb{Z}/p$ . The limit is  $\mathbb{Z}/p$ .

EXERCISE 1.7.12. Try to give five examples of where a limit does NOT exist. In particular, can you come up with examples of categories, which do have products, but not more complicated limits? [HINT: Try to work out what a limit would look like in certain derived categories or homotopy categories; if you get stuck, try to look at Exercise 1.10.16]

REMARK 1.7.13 (Problems with limits and colimits). The problem with colimits in the context of homotopy theory can already be seen from the definition, where we are making “hard” identifications. The role of homotopy colimits will be replacing these hard identifications with soft identifications.

More precisely, in topological spaces (or simplicial sets), taking limits or colimits is in general *not homotopy invariant*. E.g.  $\text{colim}(* \leftarrow S^{n-1} \rightarrow D^n) \cong S^n$  but  $\text{colim}(* \leftarrow S^n \rightarrow *) \cong *$ . This diagram does in fact have a colimit in the homotopy category of topological spaces, namely the homotopy type of a point (verify this!). However, most of the time limits and colimits will not exist in the homotopy category of spaces. Exercise 1.10.16 provides an example of this.

A related problem occurs already in algebra: In abelian groups, limits and colimits do exist, but are in general *not exact functors*. E.g. the sequence  $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$  is exact, but taking fixed-points under the  $-1$  action, gives only a left-exact sequence  $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ . This is why we need homological algebra.

### 1.8. Adjoint functors and limits

As mentioned, one of the most useful properties of adjoint functors is that left adjoints commute with colimits, and right adjoints commute with limits. Slightly more formally we define:

DEFINITION 1.8.1. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor, and  $\kappa : \mathcal{I} \rightarrow \mathcal{C}$  a diagram such that the colimit  $\operatorname{colim}_{i \in \mathcal{I}} \kappa(i)$  exists in  $\mathcal{C}$ .

We say  $F$  commutes with (or preserves) the colimit of  $\kappa$  if  $F(\operatorname{colim}_{i \in \mathcal{I}} \kappa(i))$ , together with the canonical structure maps  $F(\kappa(j)) \rightarrow \operatorname{colim}_{i \in \mathcal{I}} F(\kappa(i))$  in  $\mathcal{D}$ , is a colimit of  $F \circ \kappa$ , i.e.,

$$\operatorname{colim}_{i \in \mathcal{I}} F(\kappa(i)) \xrightarrow{\cong} F(\operatorname{colim}_{i \in \mathcal{I}} \kappa(i))$$

We use the similar terminology when  $F$  commutes colimits of all  $\kappa$  indexed over a certain class of indexing categories  $\mathcal{I}$ , e.g., “ $F$  commutes with pushouts” or “ $F$  commutes with all small colimits”. Dually for limits.

Perhaps the first thing to observe is that we have already seen a functor that commutes with limits, namely the universal functor  $\operatorname{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \rightarrow \operatorname{Set}$ :

hom-limits

PROPOSITION 1.8.2.  $\operatorname{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \rightarrow \operatorname{Set}$  commutes with limits and the contravariant functor  $\operatorname{Hom}_{\mathcal{C}}(-, y) : \mathcal{C} \rightarrow \operatorname{Set}$  sends colimits to limits (i.e.  $\operatorname{Hom}_{\mathcal{C}}(-, y) : \mathcal{C}^{\operatorname{op}} \rightarrow \operatorname{Set}$  commutes with limits).

PROOF. This follows directly from the definition, as it spells out exactly what it means to be a limit or colimit in  $\mathcal{C}$ . We encourage the reader to go over this very carefully.  $\square$

The observation of Proposition 1.8.2 immediately implies that right adjoints preserve limits, and dually for left adjoints, which is the key to scary amount of results:

limits-and-limits

PROPOSITION 1.8.3. Suppose that

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}.$$

are adjoint functors with  $F$  left adjoint to  $G$ . Then  $F$  commutes with colimits and  $G$  commutes with limits.

PROOF. Let  $\kappa : \mathcal{I} \rightarrow \mathcal{C}$  be a diagram admitting a colimit  $\operatorname{colim}_{i \in \mathcal{I}} \kappa(i)$ . Then for all  $y \in \mathcal{D}$  we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}} \left( F(\operatorname{colim}_{i \in \mathcal{I}} \kappa(i)), y \right) &\cong \operatorname{Hom}_{\mathcal{C}} \left( \operatorname{colim}_{i \in \mathcal{I}} (\kappa(i)), Gy \right) \cong \lim_{i \in \mathcal{I}} \operatorname{Hom}_{\mathcal{C}} (\kappa(i), Gy) \\ &\cong \lim_{i \in \mathcal{I}} \operatorname{Hom}_{\mathcal{D}} (F(\kappa(i)), y), \end{aligned}$$

using Proposition 1.8.2. This shows that  $F(\operatorname{colim}_{i \in \mathcal{I}} \kappa(i))$  has the universal property of  $\operatorname{colim}_{i \in \mathcal{I}} F(\kappa(i))$  (which hence in particular exists).

The lim statement is similar.  $\square$

REMARK 1.8.4. Freyd's adjoint functor Theorem says that this is pretty close to being an if and only if, i.e., modulo a set theoretic condition, functors which commute with all colimits have a right adjoint (and the dual statement holds for functors commuting with limits). In fact, there are several different sufficient conditions which ensure that the converse holds, for instance the so-called solution set condition.

Recall that we mentioned reflective and coreflective subcategories in Examples 1.3.9 and 1.3.10. One of the nice features of them is that, unlike arbitrary full subcategories, they are both automatically complete and cocomplete if the ambient category and limits and colimits are related in a nice way to those of the ambient category.

reflective-limits

PROPOSITION 1.8.5. *Suppose  $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$  is the inclusion of a reflective subcategory, with  $L$  the left adjoint to  $i$ , and consider a diagram  $\kappa : \mathcal{I} \rightarrow \mathcal{C}_0$ .*

*If  $i\kappa$  has a limit  $\lim i\kappa$  in  $\mathcal{C}$ , then  $\lim \kappa$  exists in  $\mathcal{C}_0$  and  $i(\lim \kappa) \cong \lim i\kappa$ , i.e., "the limit is already in  $\mathcal{C}_0$ ".*

*If  $i\kappa$  has a colimit in  $\mathcal{C}$ , then  $\kappa$  also has a colimit in  $\mathcal{C}_0$ , given by  $L(\text{colim } i\kappa)$ .*

*Dually if  $\mathcal{C}_0$  is coreflective then colimits can be taken in  $\mathcal{C}$ , and limits are obtained by applying the right adjoint  $R$  to the limit in  $\mathcal{C}$ .*

PROOF. This follows from the existence of a unit  $1 \Rightarrow iL$  which is an isomorphism for each object of  $\mathcal{C}_0$ :

Applying the unit to  $\lim(i\kappa)$  provides a map  $\lim(i\kappa) \rightarrow iL(\lim(i\kappa))$  which gives a factorization

$$\delta(\lim(i\kappa)) \rightarrow \delta(iL(\lim(i\kappa))) \rightarrow i\kappa$$

since  $iLi \cong i$ . But this says that  $iL(\lim(i\kappa))$  also has the universal property of the limit in  $\mathcal{C}$ , so  $\lim(i\kappa) \xrightarrow{\cong} iL(\lim(i\kappa))$ , by uniqueness of limits in  $\mathcal{C}$ , as wanted.

Similarly for colimits we get a map  $\text{colim}(i\kappa) \rightarrow iL(\text{colim}(i\kappa))$  which now provides a map

$$\delta(\lim(i\kappa)) \rightarrow \delta(iL(\lim(i\kappa))) \rightarrow i\kappa$$

as  $iLi \cong i$ .

For colimits we likewise get maps  $\text{colim}(i\kappa) \rightarrow iL(\text{colim}(i\kappa))$  which for any  $Y_0 \in \mathcal{C}$  provides universal maps

$$\begin{array}{ccc} i\kappa & \longrightarrow & \delta(\text{colim}(i\kappa)) \longrightarrow \delta i(Y) \\ & & \downarrow \qquad \qquad \downarrow \cong \\ & & \delta(iL(\text{colim}(i\kappa))) \longrightarrow \delta iLi(Y) \end{array}$$

This shows that  $L(\text{colim}(i\kappa))$  satisfies the universal property of a colimit in  $\mathcal{C}_0$  as wanted.  $\square$

Here are some examples to keep in mind: The inclusion of abelian groups in all groups, where  $L$  is given by abelianization. Products agree, but the coproduct in abelian groups (=the cartesian product). is the abelianization of the coproduct in groups (=the free product). Another example is the inclusion of compact Hausdorff



spaces in all Hausdorff spaces, here  $L$  is the Stone-Cech compactification, and coproducts in compact Hausdorff spaces are obtained by disjoint union, and then applying Stone-Cech compactification.

In the dual case of coreflective subcategories a good example to keep in mind is  $k\text{Top} \hookrightarrow \text{Top}$  where products in  $k\text{Top}$  are given by first taking product in  $\text{Top}$  and then retopologizing.

### 1.9. Overcategories, Kan extensions, and the density theorem

**1.9.1. Over and under categories.** Given a functor between two categories there are a couple of natural related categories that one can examine, reminiscent of fibers and cofibers in homotopy theory.

subsec:overundercat

**DEFINITION 1.9.1 (Over and under categories).** Given a functor  $\phi : \mathcal{I} \rightarrow \mathcal{J}$  we can form, for each object  $j \in \mathcal{J}$ , the *overcategory*  $\phi \downarrow j$ , as the category with objects pairs  $(i, \phi(i) \rightarrow j) \in \text{Ob}(\mathcal{I}) \times \text{Mor}(\mathcal{J})$  and morphisms  $(i, \phi(i) \rightarrow j) \rightarrow (i', \phi(i') \rightarrow j)$  given by morphisms  $g : i \rightarrow i'$  such that the following triangle commutes in  $\mathcal{J}$

$$\begin{array}{ccc} \phi(i) & \xrightarrow{\phi(g)} & \phi(i') \\ & \searrow & \swarrow \\ & j & \end{array}$$

We also refer to this as the *slice* category, and also denoted  $\mathcal{I}_{/j}$ , suppressing  $\phi$  from the notation.

Similarly, we define the *undercategory* (or *coslice* category)  $j \downarrow \phi$  (or  $\mathcal{I}_{j/}$ ) as the category with objects pairs  $(i, j \rightarrow \phi(i)) \in \text{Ob}(\mathcal{I}) \times \text{Mor}(\mathcal{J})$  and morphisms given by morphisms  $g : i \rightarrow i'$  such that the dual diagram commutes.<sup>4</sup>

A special case is when  $\phi$  is the identity functor, where we get over and under categories in a more classical sense, which we then write  $\mathcal{I} \downarrow i$  and  $i \downarrow \mathcal{I}$ , for  $i \in \mathcal{I}$ .

**EXERCISE 1.9.2.** Prove that the undercategory under  $\phi$  can be described as a pull-back in  $\text{Cat}$  of an undercategory in  $\mathcal{J}$ :

$$j \downarrow \phi = \mathcal{I}_{j/} \cong \mathcal{I} \times_{\mathcal{J}} \mathcal{I}_{j/}$$

And similarly for overcategories.

subsec:kanext

**1.9.2. Left and right Kan extensions.** In the words of Mac Lane, “all concepts are Kan extensions” [Mac71, Ch. X]. We introduce them here and recover limits and colimits by taking  $\mathcal{J} = *$  in the definition below.

<sup>4</sup>Someone should come up with a good notation for slice and coslice categories:  $\phi \downarrow j$  is potentially confusing as in practice  $j$  may itself be an object in a functor category, and may e.g. already be denoted  $F$ : then one would write  $\phi \downarrow F$ , losing the apparent asymmetry;  $\mathcal{I}_{/j}$  can also create confusion, as it drops  $\phi$  from the notation, and  $\phi_{/j}$  is also confusing as it is not apparent that the notation refers to a category rather than say to a functor (the same issue occurs for  $\phi \downarrow j$ );  $\phi_{/j}$  is anyway not widely used. Maybe  $\mathcal{I}_{\phi/j}$  for the overcategory and  $\mathcal{I}_{\phi \setminus j}$  or  $\mathcal{I}_{j/\phi}$  for the undercategory?

DEFINITION 1.9.3 (Kan extensions). Consider the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{C} \\ \downarrow \phi & & \\ \mathcal{J} & & \end{array}$$

The *right Kan extension* of  $F$  along  $\phi$  is a functor  $\text{Ran}_\phi F : \mathcal{J} \rightarrow \mathcal{C}$  together with a natural transformation  $\text{Ran}_\phi F \circ \phi \Rightarrow F$ , that is terminal amongst such pairs of a functor and a natural transformation, i.e., for any  $G : \mathcal{J} \rightarrow \mathcal{C}$ , with a natural transformation  $G \circ \phi \Rightarrow F$  there exists a unique natural transformation  $G \Rightarrow \text{Ran}_\phi F$  making the diagram of natural transformations to  $F$  commute

$$\begin{array}{ccc} G \circ \phi & \xrightarrow{\quad\quad\quad} & \text{Ran}_\phi F \circ \phi \\ & \searrow & \swarrow \\ & F & \end{array}$$

Dually the *left Kan extension* of  $F$  along  $\phi$  is a functor  $\text{Lan}_\phi F : \mathcal{J} \rightarrow \mathcal{C}$  together with a natural transformation  $F \Rightarrow \text{Lan}_\phi F \circ \phi$ , that is initial amongst such functors, i.e., for any  $G : \mathcal{J} \rightarrow \mathcal{C}$ , with a natural transformation  $F \Rightarrow G \circ \phi$  there exists a unique natural transformation  $\text{Lan}_\phi F \Rightarrow G$  making the diagram of natural transformations to  $F$  commute.

We will also sometimes use the standard notation  $\phi_! F := \text{Lan}_\phi F$ .

Just as for limits, left and right Kan extensions may or may not exist! The following proposition explains how right and left Kan extension should be thought of as right and left induction, and generalizes the fact that limits and colimits are right and left adjoints of the diagonal functor.

PROPOSITION 1.9.4. *Let  $\mathcal{I}$  and  $\mathcal{J}$  be small categories. Assume that, for a fixed  $\phi$ , the right Kan extension exists for all  $F$  in  $\mathcal{C}^{\mathcal{I}}$ . Then the functor  $\text{Ran}_\phi : \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}^{\mathcal{J}}$  is right adjoint to the “restriction along  $\phi$ ” functor  $\text{Res}_\phi : \mathcal{C}^{\mathcal{J}} \rightarrow \mathcal{C}^{\mathcal{I}}$  given by precomposing with  $\phi$ .*

The proof amounts to writing down the definitions, and is left as an exercise.

In the important special case where  $\mathcal{C}$  is furthermore complete and cocomplete, the Kan extensions exist as well, and their calculation reduces to that of limits and colimits:

PROPOSITION 1.9.5. *Let  $\phi : \mathcal{I} \rightarrow \mathcal{J}$  be a functor between small categories, and assume that  $\mathcal{C}$  is complete. Then  $\text{Ran}_\phi : \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}^{\mathcal{J}}$  exists, and is calculated “pointwise” via the formula:*

$$\text{Ran}_\phi F(j) = \lim_{(j \rightarrow \phi(i)) \in \mathcal{I}_{j/}} F(i)$$

Similarly, if  $\mathcal{C}$  is cocomplete then  $\text{Lan}_\phi : \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}^{\mathcal{J}}$  exists and is given by

$$\text{Lan}_\phi F(j) = \text{colim}_{(\phi(i) \rightarrow j) \in \mathcal{I}_{/j}} F(i).$$

PROOF. Exercise. □

While the above formula is useful in many contexts, and generalize all the usual induction constructions from algebra (see the exercises), there are also many cases where we want to study Kan extensions but the assumptions do *not* apply, e.g., where  $\mathcal{I}$  and  $\mathcal{J}$  are large, and  $\mathcal{C}$  is a homotopy category, hence generally without limits and colimits. Indeed this is the setup we have when studying derived functors, where  $\mathcal{J}$  will be the homotopy category of  $\mathcal{I}$ , and  $\mathcal{C}$  some other homotopy category, and we want to study the “best approximation” of  $F$  by a homotopy invariant functor—we return to this formally in Chapter 4.

subsec:density

**1.9.3. The density theorem.** In the latter parts of the course, many of our categories of interest will occur as functor categories. With the notion of limits and colimits under our belt, we can state and prove the density theorem, which says that any functor in a functor category can be written as a colimit of representable functors. This reduces checking statements for all functors to checking it on representables, and seeing that class of functors for which the statement is true is closed under colimits.

fn:cat-of-elts

**DEFINITION 1.9.6** (Category of elements). For a given functor  $F \in \text{Set}^{\mathcal{I}}$  consider the category *category of elements* in  $F$ , denoted  $\text{el}(F)$  or  $\int^{\mathcal{I}} F$ , with objects the pairs  $(i, u_i \in F(i))$ , and morphisms from  $(i, u_i \in F(i))$  to  $(i', u_{i'} \in F(i'))$  given by those  $(i \rightarrow i') \in \mathcal{I}$  such that  $F(\tau)(u_i) = u_{i'}$ .

We will encounter a generalization of this category later in the so-called Grothendieck construction, where  $\text{Set}$  is replaced by  $\text{Cat}$ .

We can description of  $\text{el}(F)$  as  $(F_{\setminus *})^{\text{op}}$ , the opposite undercategory of  $* \in \text{Set}$  with respect to the functor  $F : \mathcal{I} \rightarrow \text{Set}$ . Via the Yoneda Lemma 1.2.1 it can also be described as follows: The objects of  $\text{el}(F)$  identify with a representable functor  $\text{Hom}(i, -)$  together with a natural transformation to  $F$ . More precisely, let  $\kappa : \mathcal{I}^{\text{op}} \rightarrow \text{Set}^{\mathcal{I}}$  be given by  $i \mapsto \text{Hom}_{\mathcal{I}}(i, -)$  then the category  $\text{el}(F)$  identifies with the opposite overcategory  $((\mathcal{I}^{\text{op}})_{/\kappa})^{\text{op}}$ . The objects are natural transformations  $\phi : \text{Hom}(i, -) \rightarrow F(-)$  and the morphisms are morphisms  $(\tau : i \rightarrow i') \in \mathcal{I}$  such that the following triangle commutes in  $\text{Set}^{\mathcal{I}}$

$$\begin{array}{ccc}
 \text{Hom}(i, -) & & \\
 \uparrow \tau^* & \searrow \phi & \\
 & & F \\
 & \nearrow \phi' & \\
 \text{Hom}(i', -) & & 
 \end{array}$$

By definition  $\text{el}(F)$  parametrizes approximations of  $F$  from the left by representable functors, and the next result, the density theorem, says that the colimit over these recovers the functor  $F$ , much like any module can be written as a cokernel of free modules. It will allow us to reduce certain verifications for simplicial sets (like proving that geometric realization preserves products) to a small family of cases, e.g. checking only for an  $n$ -simplex for all  $n$ .

colim-of-rep

PROPOSITION 1.9.7 (Density theorem). *Let  $\mathcal{I}$  be a small category and  $F : \mathcal{I} \rightarrow \text{Set}$  a functor. Then there is a natural isomorphism in  $\text{Set}^{\mathcal{I}}$*

$$\text{colim}_{(i,u_i) \in \text{el}(F)} \text{Hom}_{\mathcal{I}}(i, -) \xrightarrow{\cong} F$$

where  $\text{el}(F)$  is the category of elements, built from pairs  $(i, u \in F(i))$ . The isomorphism is induced by the natural transformations  $\text{Hom}_{\mathcal{I}}(i, -) \rightarrow F$ , for each object  $(i, u \in F(i))$  in  $\text{el}(F)$ , sending the element  $(\phi : i \rightarrow j) \in \text{Hom}_{\mathcal{I}}(i, j)$  to  $F(\phi)u \in F(j)$ .

PROOF. By the Yoneda embedding it is enough to see that we have a natural isomorphism after applying  $\text{Hom}_{\text{Set}^{\mathcal{I}}}(-, G)$ , for all  $G \in \text{Set}^{\mathcal{I}}$ . Indeed we have

$$\begin{aligned} \text{Hom}_{\text{Set}^{\mathcal{I}}} \left( \text{colim}_{(i,u_i) \in \text{el}(F)} \text{Hom}_{\mathcal{I}}(i, -), G(-) \right) &\cong \lim_{(i,u_i) \in \text{el}(F)} \text{Hom}_{\text{Set}^{\mathcal{I}}}(\text{Hom}_{\mathcal{I}}(i, -), G(-)) \\ &\cong \lim_{(i,u_i) \in \text{el}(F)} G(i) \cong \text{Hom}_{\text{Set}^{\mathcal{I}}}(F, G) \end{aligned}$$

where the last isomorphism amounts to unravelling the definitions: Giving a natural transformation between  $F$  and  $G$  amounts to, for each  $i$  and each element  $u_i \in F(i)$ , specifying an element  $\phi_i(u_i) \in G(i)$ , such that  $\phi_j(F(f)(u_i)) = G(f)(\phi_i(u_i))$  for all  $f : i \rightarrow j$ . But this is the same as specifying an element in  $\lim_{(i,u_i) \in \text{el}(F)} G(i) \subseteq \prod_i \text{Hom}(F(i), G(i))$ .  $\square$

REMARK 1.9.8. The proposition is often applied to an index category “ $\mathcal{I}$ ” of the form  $\mathcal{I}^{\text{op}}$ , i.e. in the context of presheaves. We leave it to the reader to explicitly state this dual version.

### 1.10. More exercises

#### Homological algebra background exercises:

e:projectives

EXERCISE 1.10.1. Consider a functor  $F : \mathcal{I} \rightarrow \text{Vect}_k$ , for some field  $k$ , where  $\mathcal{I} = (\bullet \rightarrow \bullet \leftarrow \bullet)$  is the pull-back category. What are the projective objects in this category? Calculate the (right) derived functors of the (left exact) functor  $\lim : \text{Vect}_k^{\mathcal{I}} \rightarrow \text{Vect}$ .

EXERCISE 1.10.2. Do Exercise 1.10.1 with  $\mathcal{I} = \mathcal{B}G$  instead.

EXERCISE 1.10.3. Let  $\mathcal{I} = \mathcal{B}C_n$ , the cyclic group of order  $n$ , and let  $M$  be a  $\mathbb{Z}C_n$ -module with trivial action. Calculate  $\text{colim}_{\mathcal{I}}^* M$  and  $\text{lim}_{\mathcal{I}}^* M$  as abelian groups. Now let  $M = \mathbb{F}_p$  and do the same calculation as rings.

[Hint: This being doable probably requires knowledge of group cohomology. Look up in a book on group cohomology if you get stuck. At the very least, you should know the answer.]

EXERCISE 1.10.4 (Group cohomology). Let  $\mathcal{I} = \mathcal{B}(Z/p)^r$ , calculate  $\text{lim}_{\mathcal{I}}^* \mathbb{F}_p$  as a ring.

[Hint: Did you ever calculate the  $\mathbb{F}_2$ -cohomology of  $\mathbf{RP}^{\infty}$ ? If not, now is a good time. Look up in a book on group cohomology if you get stuck. Again, at the very least, you should know the answer.]

EXERCISE 1.10.5 (Group cohomology). Let  $\mathcal{I} = \mathcal{B}D_8 = \mathcal{B}(\mathbb{Z}/4 \rtimes \mathbb{Z}/2)$ , calculate  $\lim_{\mathcal{I}}^* \mathbb{F}_p$  as a ring.

[Hint: You'll probably need to know the Lyndon-Hochschild-Serre spectral sequence. Try applying it to the central extension  $0 \rightarrow \mathbb{Z}/2 \rightarrow D_8 \rightarrow (\mathbb{Z}/2)^2 \rightarrow 0$ , and look at the extensions arising by pulling back to the 3 subgroups of order 2 in  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . Look up in a book on group cohomology if you get stuck.]

**Category exercises:**

Recall the examples in Example 1.1.4.

EXERCISE 1.10.6 (Structure of morphisms in  $\Delta$ ). Consider the following distinguished morphisms in the simplicial category  $\Delta$ :

- (1) For all  $n \geq 0$  and  $0 \leq i \leq n+1$ , the  $i^{\text{th}}$  face map (at  $n$ ), denoted  $d^i : [n] \rightarrow [n+1]$ , is the unique order preserving injection whose image does not contain  $i$ .
- (2) For all  $n \geq 1$  and  $0 \leq i \leq n-1$ , the  $i^{\text{th}}$  degeneracy map (at  $n$ ), denoted  $s^i : [n] \rightarrow [n-1]$ , is the unique order preserving surjection that sends  $i$  and  $i+1$  to  $i$ .

(Here we are slightly abusing the notation by ignoring  $n$  in the notations  $d^i$  and  $s^i$ . This is more convenient in computations.)

Show that any morphism  $\varphi : [n] \rightarrow [m] \in \text{Mor}(\Delta)$  factors uniquely as follows:

$$\varphi = d^{i_1} d^{i_2} \dots d^{i_t} s^{j_1} s^{j_2} \dots s^{j_t},$$

where  $0 \leq i_t < \dots < i_1 \leq m$  and  $0 \leq j_1 < \dots < j_t < n$ .

EXERCISE 1.10.7 (Structure of the orbit category  $\mathcal{O}_{\mathcal{F}}(G)$ ).

- (1) Find the automorphisms of an object  $G/H$  in  $\mathcal{O}_{\mathcal{F}}(G)$
- (2) Make a diagram of the orbit category  $\mathcal{O}_{\mathcal{F}}(\mathfrak{S}_4)$ , where  $\mathcal{F}$  is the collection of 2-subgroups the symmetric group on 4 letters.

EXERCISE 1.10.8 (Coverings and  $\mathcal{O}(G)$ ). Strengthen the classification of covering spaces as given in say Hatcher, to prove an equivalence of categories between the category of connected coverings of  $X$  (with the usual hypotheses) and the orbit category of the fundamental group.

EXERCISE 1.10.9. Let  $G = \mathbb{Z}/2$  and  $k$  a field of characteristic 2. What are the simple and projective objects in  $\text{Vect}^{\mathcal{O}(G)^{\text{op}}}$ ? Do the same exercise with  $G = \mathfrak{S}_3$ , and the category  $\mathcal{O}_{\mathcal{F}}(G)$ , where  $\mathcal{F}$  is the collection of 2-subgroups of  $G$ .

**Adjoints exercises:**

exercise:pi0

EXERCISE 1.10.10. Show that the inclusion functor  $\text{Set} \rightarrow \text{Cat}$  has a left adjoint. Describe it explicitly. (We call this functor  $\pi_0$ , the set of components of the category; if  $\pi_0(\mathcal{C})$  is a point, we call  $\mathcal{C}$  *connected*.)

EXERCISE 1.10.11. Show that the inclusion functor  $\text{Grpd} \rightarrow \text{Cat}$ , from groupoids to categories has a left adjoint, and describe it explicitly. (We call this functor  $\pi$ , the fundamental groupoid of  $\mathcal{C}$ .)

**Limits exercises:**

EXERCISE 1.10.12. Make the formula in Proposition 1.7.8 explicit in the categories  $\text{Set}$ ,  $\text{Set}_*$ ,  $\text{Ab}$ ,  $\text{Grp}$ , and  $\text{Ring}$ .

EXERCISE 1.10.13. Does the category of compact Hausdorff spaces have arbitrary colimits? What are they? Is this a special case of a more general result? [To formulate this you may need a definition: A full subcategory is called *reflective* if the inclusion has a left adjoint, and *coreflective* if it has a right adjoint.

EXERCISE 1.10.14 (Coequalizers as initial objects). Let  $\mathcal{C}$  be a category.

The first part of the exercise gives a low-tech approach to initial objects (compare Example 1.7.5):

- (1) An object  $c_0 \in \mathcal{C}$  is called an *initial* (or sometimes *universal*) object of  $\mathcal{C}$  if for any object  $c \in \mathcal{C}$ , there exists a unique morphism  $c_0 \rightarrow c$  in  $\mathcal{C}$ . Show that if  $c_0$  and  $c_1$  are initial objects of the category  $\mathcal{C}$ , then there is a unique isomorphism between them (i.e., initial objects are unique up to unique isomorphism).
- (2) Give an example of a category which does not have an initial object.
- (3) What are the initial objects of the categories  $\text{Set}$ ,  $\text{Ab}$ ,  $\text{Ring}$ ,  $\text{Top}$ ?

Next we want to view coequalizers as initial objects in a certain category:

- (4) Let  $f: A \rightarrow B$  be a homomorphism of abelian groups. Consider the following category denoted by  $\mathcal{C}_f$ :

*Objects:* Pairs  $(C, g)$ , where  $C$  is an abelian group,  $g: B \rightarrow C$  is a homomorphism and  $gf = 0$ ;

*Morphisms:* A morphism from  $(C, g)$  to  $(C', g')$  is a homomorphism  $h: C \rightarrow C'$  such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ & \searrow^{g'} & \swarrow_{h} \\ & & C' \end{array}$$

What is the initial object of  $\mathcal{C}_f$ ?

- (5) Using the ideas from the previous paragraph show that coequalizers are initial objects of certain categories and hence unique up to unique isomorphism.

EXERCISE 1.10.15 (Tensor products as initial objects). Let  $R$  be an associative ring,  $M$  a right  $R$ -module and  $N$  a left  $R$ -module. Describe the *tensor product*  $M \otimes_R N$  as an initial object of some category.

Let  $R$  be a commutative ring and let  $A$ ,  $B$  and  $C$  be  $R$ -modules. Show that there is a natural isomorphism of  $R$ -modules

$$\text{Hom}_R(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_R(B, C)).$$

EXERCISE 1.10.16 (No colimits in the homotopy category.). Let  $\mathcal{H}$  denote the category of topological spaces and homotopy classes of maps between them (all are pointed). We will show that colimits do not always exist in  $\mathcal{H}$ .

Consider the following diagram in  $\mathcal{H}$ , where  $2: S^n \rightarrow S^n$  is any map of degree 2:

$$\begin{array}{ccc} S^n & \xrightarrow{2} & S^n \\ \downarrow & & \\ * & & \end{array}$$

Suppose that this diagram has a colimit, and denote the colimit by  $T$ .

- (1) Show that  $\text{Mor}_{\mathcal{H}}(T, X) \cong \{a \in \pi_n(X) \mid 2a = 0\}$  for any pointed space  $X$ .
- (2) Describe a fibration sequence  $A \rightarrow B \rightarrow C$  whose long exact homotopy sequence

$$\cdots \pi_{n+1}(C) \rightarrow \pi_n(A) \rightarrow \pi_n(B) \rightarrow \pi_n(C) \rightarrow \pi_{n-1}(A) \rightarrow \cdots$$

is equal to  $\cdots 0 \rightarrow \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \rightarrow 0 \rightarrow \cdots$ .

- (3) It follows that  $\text{map}(T, A) \rightarrow \text{map}(T, B) \rightarrow \text{map}(T, C)$  is a fibration sequence. What are the homotopy groups of the three spaces?
- (4) Derive a contradiction.

se: cofinality

EXERCISE 1.10.17 (Cofinality). Consider a functor  $F: \mathcal{J} \rightarrow \mathcal{C}$ , with  $\mathcal{C}$  cocomplete. Suppose that  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  is an inclusion of small categories which satisfies that  $\mathcal{I}_{j/}$  is connected (in the sense of Exercise 1.10.10) for every  $j \in \mathcal{J}$ .

Prove that the natural morphism

$$\text{colim}_{\mathcal{I}}(F \circ \phi) \xrightarrow{\cong} \text{colim}_{\mathcal{J}} F$$

is an isomorphism in  $\mathcal{C}$ .

[Functors  $\phi$  with this property are called *cofinal*; we will later see the homotopy invariant strengthening of this property. There is also dual notion call *final* (find this by passing to  $\text{op}!$ )]

[Note that when taking  $\mathcal{J} = \mathbb{N}$ , this notion agrees with the notion of a “cofinal sequence”, as you learned in your first-year calculus class....]

EXERCISE 1.10.18 (Cofinality of a poset of subgroups...). This exercise works out a non-trivial example of a cofinal subcategory, in continuation of Exercise 1.10.17. Let  $\mathcal{C}$  be a subposet of the poset of non-trivial  $p$ -subgroups  $\mathcal{S}_p(G)$  in a finite group  $G$ , and assume that  $\mathcal{C}$  is closed under passage to  $p$ -supergroups.

Prove that the inclusion  $\mathcal{C} \hookrightarrow \mathcal{S}_p(G)$  is cofinal iff  $\mathcal{C}$  contains all non-trivial  $p$ -subgroups  $P$  such that  $\mathcal{S}_p(N_G(P)/P)$  is disconnected or empty.

[Note that  $\mathcal{S}_p(N_G(P)/P)$  is empty iff  $P$  is a Sylow  $p$ -subgroup in  $G$ .]

[Bonus: Can you find a group theoretic criterion for when  $\mathcal{S}_p(G)$  is disconnected?]

**Kan extension exercises:**

EXERCISE 1.10.19. Consider an inclusion  $H < G$  of groups, viewed as a functor  $\mathcal{B}H \rightarrow \mathcal{B}G$ . If  $\mathcal{C}$  is the category of abelian groups, write down a formula for the left and right Kan extension of functors  $\mathcal{B}H \rightarrow \mathcal{C}$ .

[Bonus question: Do you know another, old-school, name for these? ;)]





## CHAPTER 2

### The homotopy theory of simplicial sets

ss-chapter

We will start by introducing simplicial sets. Our goal is to show that we can do homotopy theory in the category of simplicial sets, just like in the category of spaces. The category of simplicial sets has a number of technical advantages over topological spaces, both with respect to formal properties, and in terms of avoiding any continuity considerations.

They also have a direct link to the category of (small) categories via the so-called *nerve* construction. We can hence both view a simplicial set as combinatorial version of a simplicial complex, *and* view it as sort of a generalized category. It is the combination of those two viewpoints that makes them so useful.

**Idea:** A simplicial set can be viewed as a sequence of sets  $X_0, X_1, X_2, \dots$ , where the elements in  $X_n$  are called *simplices of dimension  $n$*  (or *degree  $n$* ), or just  *$n$ -simplices* for short. These will be tied together via various combinatorial relations, like in an ordered simplicial complex. We have face maps  $d_i : X_n \rightarrow X_{n-1}$ , associated to the process of restricting to the  $i^{\text{th}}$  face

(2.0.1) “ $d_1$   $\left( \begin{array}{c} \bullet \\ \nearrow \quad \searrow \\ \bullet \quad \bullet \\ \text{0} \quad \quad \text{2} \end{array} \right) = \bullet \longrightarrow \bullet$ .”

To get the right categorical properties, we also have to allow for certain simplices to be “degenerate”, i.e. obtained from lower-dimensional simplices via degeneracy maps  $s_i : X_n \rightarrow X_{n+1}$ : these simplices often play no substantial role, like identity maps in a category, trivial subgroups in a group etc., and can sometimes be regarded as a sort of “noise”. The remaining “non-degenerate”  $n$ -simplices correspond to the actual  $n$ -cells in the associated CW complex, the *geometric realization* introduced in Section 2.4.

In this combinatorial picture a simplicial set can be be view as a bunch of sets and maps as follows

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_1} \end{array} X_1 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_1} \rightarrow \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_2} \end{array} X_2 \cdots$$

where the maps satisfy certain relations, known as simplicial identities.

### 2.1. The simplex category $\Delta$ and simplicial sets

We now embark on the slick formal definition of simplicial sets (which is due to Dan Kan). First we need the following category:

DEFINITION 2.1.1. The simplex category  $\Delta$  is defined to be the category with objects the non-empty finite totally ordered sets

$$[n] = (0 < 1 < \dots < n).$$

for  $n = 0, 1, \dots$ , and morphisms the order preserving maps between them. In other words  $\Delta$  is the full subcategory of  $\text{Cat}$  with objects the categories  $[0], [1], [2], \dots$

REMARK 2.1.2. The category  $\Delta$  can be described via generators and relations. Namely we have a map  $d^i : [n-1] \rightarrow [n]$  which “skips  $i$ ”, i.e., is the identity for  $j < i$  and sends  $j$  to  $j+1$  for  $j \geq i$ .

Similarly we have  $s^i : [n+1] \rightarrow [n]$  which “repeats  $i$ ”, i.e., is the identity on  $j \leq i$  and sends  $j$  to  $j-1$  for  $j > i$ .

These satisfies some obvious relations:

cosimp-id

$$(2.1.1) \quad \begin{array}{l} d^j d^i = d^i d^{j-1}, i < j \quad s^j s^i = s^i s^{j+1}, i \leq j \\ s^j d^i = d^i s^{j-1}, i < j \quad s^j d^i = d^{i-1} s^j, i > j+1 \quad s^i d^i = s^i d^{i+1} = 1 \end{array}$$

PROPOSITION 2.1.3. *The category  $\Delta$  can be described as being the category with objects  $[0], [1], \dots$ , and morphisms generated by the  $d^i$ 's and  $s^i$ 's, subject to the above relations. In other words, a functor  $F : \Delta \rightarrow \mathcal{C}$  to any category  $\mathcal{C}$  is uniquely determined by a choice of objects  $F([0]), F([1]), \dots$  in  $\mathcal{C}$  and a choice of maps  $F(d^i)$  and  $F(s^i)$  in  $\mathcal{C}$  between them, satisfying the above relations.*

The proof is left as an exercise.

DEFINITION 2.1.4. The category of *simplicial sets*  $\text{sSet}$  is defined to be the functor category  $\Delta^{\text{op}} \rightarrow \text{Set}$ . For  $X$  a simplicial set we write  $X_n = X([n])$  for the set of  $n$ -simplices. We sometimes also write the whole simplicial set as  $X_\bullet$  if it is important to stress the indexing.

For a simplicial set we hence have operations  $d_i = X(d^i) : X_n \rightarrow X_{n-1}$ , (aka *face maps*), and  $s_i = X(s^i) : X_n \rightarrow X_{n+1}$  (aka *degeneracy maps*).

As the  $d_i$  and  $s_i$  generate all morphisms in  $\Delta$ , specifying a *simplicial map*  $f : X_\bullet \rightarrow Y_\bullet$  (i.e. a natural transformation of functors  $\Delta^{\text{op}} \rightarrow \text{Set}$ ) amounts to specifying a sequence of maps of sets  $\{f_n : X_n \rightarrow Y_n\}$  such that

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow d_i & & \downarrow d_i \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ s_i \uparrow & & s_i \uparrow \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array}$$

commute for all  $i$  and  $n$ . The simplicial relations also allow us to describe the objects of  $\text{sSet}$  combinatorially:

prop:simp-id

PROPOSITION 2.1.5 (Combinatorial description of sSet). *A simplicial set  $X_\bullet$  can be described as a collection of sets  $X_0, X_1, \dots, X_n, \dots$ , and maps  $d_i$ 's and  $s_i$ 's, satisfying the relations induced by (2.1.1), which read as follows:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ if } i < j \\ s_i s_j &= s_{j+1} s_i \text{ if } i \leq j \\ d_i s_j &= s_{j-1} d_i \text{ if } i < j \\ d_j s_j &= 1 = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} \text{ if } i > j + 1 \end{aligned}$$

□

It may be memotechnically useful for the reader to say out in words what these relations mean. E.g., the relation “ $d_i d_j = d_{j-1} d_i$  if  $i < j$ ” says that if we first take the  $j^{\text{th}}$  face and then take the  $i^{\text{th}}$  face, then this is the same as first taking the  $i^{\text{th}}$  face and then taking the  $(j-1)^{\text{th}}$  face when  $j$  is larger than  $i$  (as the indexing of what “ $j^{\text{th}}$  face” means has changed). The other relations are similar.

Categorically we can think of the elements in  $X_0$  as objects and the  $X_1$  as arrows, with source and target given by the boundary maps  $d_1$  and  $d_0$ , and  $s_0 : X_0 \rightarrow X_1$  giving the identity morphism.

Geometrically we can think of the map  $d_i$  as the map giving the  $i^{\text{th}}$  face of a simplex, as already drawn in Figure (2.0.1). However, not all simplices really contribute geometrically, as the ones which are in the image of some  $s_i$  are really just repeated copies of things which come from lower dimensions. The simplices in  $X_n$  which in the image of  $s_i : X_{n-1} \rightarrow X_n$  for some  $i$  are called *degenerate* and if not called *non-degenerate*. We will return to them shortly, in Section 2.3, after introducing the nerve functor.

DEFINITION 2.1.6. Define the *simplicial  $n$ -simplex* as the simplicial set

$$\Delta^n = \text{Hom}_{\text{Cat}}(-, [n]) = \text{Hom}_{\Delta}(-, [n])$$

REMARK 2.1.7. Note that in our categorical language from the last chapter the  $\Delta^n$ 's are exactly the representable functors in  $\text{sSet} = \text{Set}^{\Delta^{\text{op}}}$ .

We can already now calculate our first simplicial maps:

lemma:nsimp

LEMMA 2.1.8.

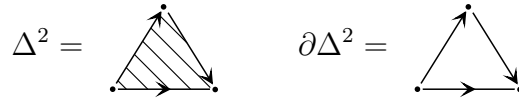
$$\text{Hom}_{\text{sSet}}(\Delta^n, X) \xrightarrow{\cong} X_n$$

PROOF. This follows by the Yoneda lemma, since by definition  $\Delta^n = \text{Hom}_{\Delta}(-, [n])$ . □

The above lemma says that we can think of an  $n$ -simplex in  $X$  as either an element  $x$  in the set  $X_n$  or as a simplicial map  $f : \Delta^n \rightarrow X$ . Applying the face map  $d_i$  to  $x \in X_n$  corresponds restricting  $f$  to the  $i^{\text{th}}$  face by precomposing with the map  $d^i : \Delta^{n-1} \rightarrow \Delta^n$ . In the second notation we are treating  $x$  as an  $n$ -cell in  $X$ , via a map  $\Delta^n \rightarrow X$ . In this way we have a geometric picture in mind, and may pretend that “ $X$ ” is still a space (rather than a simplicial set) and “ $\Delta^n \rightarrow X$ ” is a map of spaces

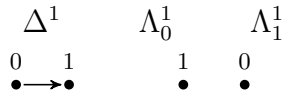
(rather than a map of simplicial sets). It is the first step in our long dictionary between simplicial sets and topological spaces. The dictionary will be so effective that researchers can indeed write entire papers containing results about “spaces”, and only in the end really decide if they meant by the word “space” a topological space or a simplicial set (or perhaps another equivalent  $\infty$ -category?).

EXAMPLE 2.1.9 (Boundary and horns). Define the boundary of the  $n$ -simplex  $\partial\Delta^n$  as the subsimplicial set<sup>1</sup> of  $\Delta^n$  generated by simplices of dimension strictly less than  $n$ . Note that the identity  $(1_{[n]} : [n] \rightarrow [n]) \in \text{Hom}_{\Delta}([n], [n])$  is an  $n$ -simplex in  $\Delta^n$  but not in  $\partial\Delta^n$ , so that  $\partial\Delta^n$  is a proper subsimplicial set of  $\Delta^n$ .

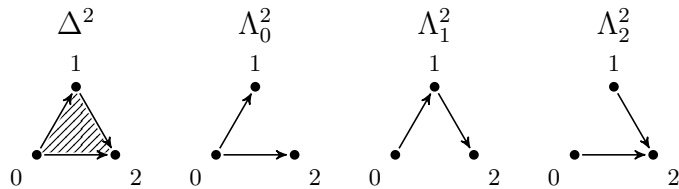


Likewise we define the  $i^{\text{th}}$  horn  $\Lambda_i^n$  as the subobject of  $\Delta^n$  obtained by removing the unique non-degenerate  $n$ -simplex  $\iota$ , as well as the  $(n - 1)$ -simplex  $d_i(\iota)$ : it is thus the subobject generated by the  $(n - 1)$ -simplices  $d_j(\iota)$  for  $j \neq i$  together with all simplices of  $\Delta^n$  of dimension  $\leq n - 2$ . Let us draw some pictures for small  $n$ :

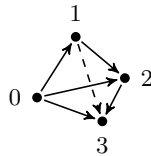
$n = 1 :$



$n = 2 :$

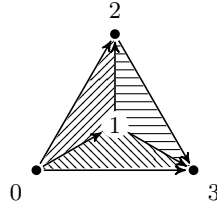


For  $n = 3$ , the simplicial set  $\Delta^3$  is the filled 3-simplex:



<sup>1</sup>A subsimplicial set  $Y_{\bullet}$  of  $X_{\bullet}$  is just a selection of subsets  $Y_n \subseteq X_n$  on which all face and degeneracy maps restrict to give a new simplicial set. We also refer to a subsimplicial set as a subobject in the category of simplicial sets.

and we can e.g. draw the 3-horn  $\Lambda_1^3$  in the plane as follows:



ample:skeleton

EXAMPLE 2.1.10 ( $n^{\text{th}}$  skeleton). Given a simplicial set  $X$  we can consider its  $n^{\text{th}}$  skeleton  $\text{sk}_n X$ , the subobject generated by simplices of dimension at most  $n$ . E.g.,  $\partial\Delta^n = \text{sk}_{n-1}\Delta^n$ . Note also that  $X = \text{colim}_n \text{sk}_n X$ , where the colimit is indexed by the category  $\mathbb{N}$  (it is a *direct limit* in the classical sense, and in this case it can be visualised as an infinite, increasing union).

$$\text{sk}_1 \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, \quad \text{sk}_0 \left( \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \bullet \quad \bullet \end{array}$$

prop:skeleton

PROPOSITION 2.1.11. We have that  $\text{sk}_n(-) = i_! i^*(-)$ , where  $i : \Delta_{\leq n} \rightarrow \Delta$  is the inclusion functor and  $i^* : \text{Fun}(\Delta^{\text{op}}, \text{Set}) \rightarrow \text{Fun}(\Delta_{\leq n}^{\text{op}}, \text{Set})$  and  $i_!$  denotes restriction and left Kan extension respectively.

In particular  $\text{sk}_n(-)$  commutes with colimits.

PROOF. The identification  $\text{sk}_n(-) = i_! i^*(-)$  is left as an exercise (or maybe better, take this description of the definition, and then deduce the description in Example 2.1.10).

As  $i^*$  also has a right adjoint (the right Kan extension, by Proposition 1.9.5) and  $i_!$  is a left adjoint, we have that  $\text{sk}_n(-)$  preserves colimits by Proposition 1.8.3.  $\square$

EXERCISE 2.1.12. Define the coskeleton  $\text{cosk}_n$  dually to  $\text{sk}_n$ . Describe it completely.

PROPOSITION 2.1.13. The category  $\text{sSet}$  is complete and cocomplete and limits and colimits are calculated “levelwise”, i.e. by taking the corresponding limits and colimits in  $\text{Set}$  in each degree  $n \geq 0$ .

PROOF. This follows by the definition of  $\text{sSet}$  as a functor category, by Proposition 1.7.7.  $\square$

REMARK 2.1.14. More generally we define simplicial objects in any category  $\mathcal{C}$  as the functor category  $\mathcal{C}^{\Delta^{\text{op}}}$ . Cosimplicial objects in  $\mathcal{C}$  are similarly defined to be functors  $\Delta \rightarrow \mathcal{C}$ .

REMARK 2.1.15. Note that  $\text{Hom}_{\Delta}(-, -)$  defines a functor  $\Delta^{\text{op}} \times \Delta \rightarrow \text{Set}$ . We can hence view the functor  $(\Delta^\bullet)_\bullet$  which to  $([n], [m])$  assigns the set  $(\Delta^m)_n = \text{Hom}([n], [m])$  of  $n$ -simplices in the simplicial set  $\Delta^m$  as either a cosimplicial simplicial set, or a simplicial cosimplicial set. This dual variance is the key to many formulas.

Before we move on to the nerve functor, let us note that the simplices in  $X$  can also naturally be viewed as a category, the *category of simplices* of  $X$ .

DEFINITION 2.1.16. Define the *category of simplices* of a simplicial set  $X$  as  $\text{el}(X)$ , the category of elements, as defined in Definition 1.9.6. In other words the objects are  $x_n \in X_n$  for some  $n$ . and a morphism  $x_n \rightarrow x_m$  consists of a morphism  $(\theta : [m] \rightarrow [n])$  in  $\Delta$  such that  $\theta^*(x_n) = x_m$ , where  $\theta^* : X([n]) \rightarrow X([m])$  is the induced map. This can again be described as the overcategory  $\iota \downarrow X$ , the overcategory of  $X$  with respect to the Yoneda embedding  $\iota : \Delta^{\text{op}} \rightarrow \text{Set}^{\Delta^{\text{op}}}$ .

The category  $\text{el}(X)$  is also sometimes called the *simplex category* or *subdivision category* of  $X$ , and occasionally denoted  $\text{sd}(X)$  or  $dX$ .

We now translate the abstract Theorem 1.9.7 to describe  $X$  as colimit of its simplices.

PROPOSITION 2.1.17.

$$\text{colim}_{\sigma_n \in \text{el} X} \Delta^n \xrightarrow{\cong} X$$

PROOF. Follows by applying Theorem 1.9.7 to  $X \in \text{sSet} = \text{Set}^{\Delta^{\text{op}}}$ , as  $\Delta^n = \text{Hom}_{\Delta}(-, [n])$  by definition.  $\square$

## 2.2. The nerve functor $N$

Let  $\text{Cat}$  denote the category of small categories, with objects small categories, and morphisms functors.<sup>2</sup> We now want to introduce the nerve functor

$$N : \text{Cat} \rightarrow \text{sSet}$$

and its left adjoint  $h$ .

By definition, it gives us a way to associate a simplicial set (which we think of as a geometric object) to any category. Furthermore it gives us an easy way of constructing interesting examples of simplicial sets. And simplicial sets that come from categories are easier to manipulate, essentially as it is in general easier to write up functors and natural transformations than verifying simplicial identities.

DEFINITION 2.2.1. The nerve functor

$$N : \text{Cat} \rightarrow \text{sSet}$$

associates to a small category  $\mathcal{C}$  the *nerve*  $N\mathcal{C} : \Delta^{\text{op}} \rightarrow \text{Set}$ , defined as the functor  $\text{Hom}_{\text{Cat}}(-, \mathcal{C}) : \Delta^{\text{op}} \rightarrow \text{Set}$  sending

$$[n] \mapsto (N\mathcal{C})_n = \text{Hom}_{\text{Cat}}([n], \mathcal{C})$$

REMARK 2.2.2. We have already seen an instance of the nerve functor, in our definition of  $\Delta^n$ : we can indeed write

$$\Delta^n = N[n].$$

<sup>2</sup>It is even a 2-category, with 2-morphisms natural transformations, but we will ignore this for now.

REMARK 2.2.3. Let us spell out what the above definition means concretely: a functor  $[n] \rightarrow \mathcal{C}$  can be represented as a sequence  $(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n)$  of objects and morphisms in  $\mathcal{C}$ . For  $i \neq 0, n$  we have

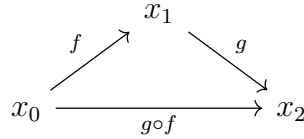
$$d_i(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n) = x_0 \xrightarrow{f_1} \cdots \rightarrow x_{i-1} \xrightarrow{f_{i+1} \circ f_i} x_{i+1} \rightarrow \cdots \rightarrow x_n$$

whereas  $d_0(x_0 \rightarrow \cdots \rightarrow x_n) = (x_1 \rightarrow \cdots \rightarrow x_n)$  and  $d_n(x_0 \rightarrow \cdots \rightarrow x_n) = (x_0 \rightarrow \cdots \rightarrow x_{n-1})$ .

Similarly

$$s_i(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n) = (x_0 \rightarrow \cdots \rightarrow x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{\text{id}} x_i \xrightarrow{f_{i+1}} x_{i+1} \rightarrow \cdots \rightarrow x_n).$$

For example the faces of the 2-simplex  $\sigma = (x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2)$  look as expected:



we have  $d_0\sigma = g$  (really,  $(x_1 \xrightarrow{g} x_2)$ ),  $d_1\sigma = g \circ f$ , and  $d_2\sigma = f$ .

fullyfaithful

PROPOSITION 2.2.4. *The nerve functor  $N : \text{Cat} \rightarrow \text{sSet}$  is fully faithful, i.e., it induces a bijection on morphism sets*

$$\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(N\mathcal{C}, N\mathcal{D}).$$

PROOF. It is clear that  $\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Hom}_{\text{sSet}}(N\mathcal{C}, N\mathcal{D})$  is injective, as a functor is specified by what it does on objects and morphisms, and this information is captured by the behavior of the corresponding simplicial map between nerves in degrees 0 and 1.

We now verify surjectivity: Suppose  $\phi : N\mathcal{C} \rightarrow N\mathcal{D}$  is a morphism of simplicial sets. We want to show that  $\phi = NF$  for a functor  $F$ . We first verify that  $\phi$  restricted to  $(N\mathcal{C})_{\leq 1}$  indeed defines a functor  $F$ . By the definition of a category and a functor (Definitions 1.1.1 and 1.1.6), for this we need to check that it respects composition: Given morphisms  $f_1 : x_0 \rightarrow x_1$  and  $f_2 : x_1 \rightarrow x_2$  in  $\text{Mor}(\mathcal{C}) = (N\mathcal{C})_1$  with  $d_0(f_1) = x_1 = d_1(f_2)$ , we have a unique 2-simplex  $\sigma = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2)$  in  $N\mathcal{C}$ . Write  $\phi(\sigma) = (y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2) \in (N\mathcal{D})_2$ . Then  $F(f_1) = F(d_2\sigma) = d_2\phi(\sigma) = g_1$ . Likewise  $F(f_2) = F(d_0\sigma) = d_0\phi(\sigma) = g_2$  and  $F(f_2 \circ f_1) = F(d_1\sigma) = d_1\phi(\sigma) = g_2 \circ g_1 = F(f_2) \circ F(f_1)$ . So, when restricted to  $(N\mathcal{C})_{\leq 1}$ ,  $\phi$  indeed comes from a (unique) functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . But then  $\phi = NF$ , as the map  $N\mathcal{C}_n \rightarrow N\mathcal{C}_1 \times_{N\mathcal{C}_0} N\mathcal{C}_1 \cdots \times_{N\mathcal{C}_0} N\mathcal{C}_1$  is injective (in fact bijective).  $\square$

We now want to see that the inclusion  $N : \text{Cat} \rightarrow \text{sSet}$  has a left adjoint given by considering the free category on the set of objects  $X_0$  with morphisms generated by the 1-simplices  $X_1$  under formal composition  $\star$ , subject to the following relation: whenever  $f, g \in X_1$  and there exist a 2-simplex  $\sigma \in X_2$  with  $d_2\sigma = f$  and  $d_0\sigma = g$ , then we identify the formal composition  $g \star f$  with the 1-simplex  $d_1\sigma$ .<sup>3</sup> Written down more formally:

<sup>3</sup>This means that if for given  $f, g$  there are several such  $\sigma$ 's, we identify all morphisms  $d_1\sigma$  with each other.

PROPOSITION 2.2.5. *The nerve functor  $N : \text{Cat} \rightarrow \text{sSet}$  has a left adjoint  $h : \text{sSet} \rightarrow \text{Cat}$ .*

and in particular  $N$  preserves limits.

The functor  $h$  associates to  $X$  the category with objects  $X_0$ , and morphism set  $\tilde{X}_1$  defined by letting the morphisms from  $x \rightarrow y$  be freely generated strings  $f_n \star \cdots \star f_1$  with  $f_i \in X_1$  satisfying that  $d_1 f_1 = x$ ,  $d_0 f_n = y$ , and  $d_0 f_i = d_1 f_{i+1}$  for all  $i$ , subject to the equivalence relation  $\sim$  generated by imposing that  $d_1 \sigma \sim d_0 \sigma \star d_2 \sigma$  for all 2-simplices  $\sigma$ :

$$\begin{array}{ccc} & x_1 & \\ d_2 \sigma \nearrow & & \searrow d_0 \sigma \\ x_0 & \xrightarrow{d_1 \sigma} & x_2 \end{array}$$

Hence, as a set, we have

$$\tilde{X}_1 = \left( \coprod_{n \geq 1} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \right) / \sim,$$

where the  $n^{\text{th}}$  set in the disjoint union is a fiber product of  $n$  copies of  $X_1$ . The identity morphisms in this category are given by the  $\sim$ -classes  $1_x$  of  $s_0 x \in X_1$ , for  $x \in X_0$ .

In particular  $h$  only depends on the 2-skeleton of  $X$ , i.e.,  $h(X) = h(\text{sk}_2(X))$ .

PROOF. It is clear that the above definition defines a category  $hX$  with identity maps  $1_x = s_0 x$  and composition  $\circ = \star$ . To show that  $1_y$  is a left identity for all  $y \in X_0$ , fix a 1-simplex  $f : x \rightarrow y$  and note that the 2-simplex  $s_1 f$  ensures commutativity of the following triangle in  $hX$ :

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow s_0 y \\ x & \xrightarrow{f} & y, \\ & s_1 f & \end{array}$$

so  $f = 1_y \circ f$ ; and similarly for right identity, using  $s_0 f$ . It is also clear that  $\circ$  is associative, since  $(h \circ g) \circ f$  by definition is the class of the string of morphisms  $h \star g \star h$ , and so is  $h \circ (g \circ f)$ . As the assignment is obviously functorial, we conclude that we have a well-defined functor  $h : \text{sSet} \rightarrow \text{Cat}$ .

We want to check that this is indeed left adjoint, i.e., that

$$\text{Hom}_{\text{sSet}}(X, \mathcal{NC}) \xrightarrow{\cong} \text{Hom}(hX, \mathcal{C})$$

Each simplex  $\Delta^n \rightarrow \mathcal{NC}$  is determined by its restriction to  $\text{sk}_1 \Delta^n \rightarrow \mathcal{NC}$ , and a map  $\text{sk}_1 \Delta^n \rightarrow \mathcal{NC}$  extends to  $\Delta^n$  iff it extends to  $\text{sk}_2 \Delta^n \rightarrow \mathcal{NC}$  i.e.,

$$\text{Hom}_{\text{sSet}}(\Delta^n, \mathcal{NC}) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(\text{sk}_2 \Delta^n, \mathcal{NC}) \hookrightarrow \text{Hom}_{\text{sSet}}(\text{sk}_1 \Delta^n, \mathcal{NC}).$$

As any simplicial set is a colimit of its simplices by density Proposition 2.1.17, and  $\text{sk}_i$  preserves colimits (Proposition 2.1.11), we more generally have

$$\text{Hom}_{\text{sSet}}(X, \mathcal{NC}) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(\text{sk}_2 X, \mathcal{NC}) \hookrightarrow \text{Hom}_{\text{sSet}}(\text{sk}_1 X, \mathcal{NC}).$$

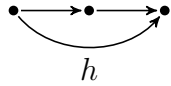


Furthermore the condition that simplicial map  $F_{\leq 1} : \text{sk}_1 X \rightarrow N\mathcal{C}$  extends to  $F_{\leq 2} : \text{sk}_2 X \rightarrow N\mathcal{C}$  is that for each 2-simplex  $\sigma$  in  $X_2$  we have  $F(d_1\sigma) = F(d_0\sigma) \circ F(d_2\sigma)$ . But this is exactly the condition imposed by setting  $d_0\sigma \circ d_2\sigma \sim d_1\sigma$ , and requiring that  $F$  be a functor. Hence maps of simplicial sets  $X \rightarrow N\mathcal{C}$  are in one-to-one correspondence with functors  $\text{h}X \rightarrow \mathcal{C}$  as wanted.  $\square$

example:tauone

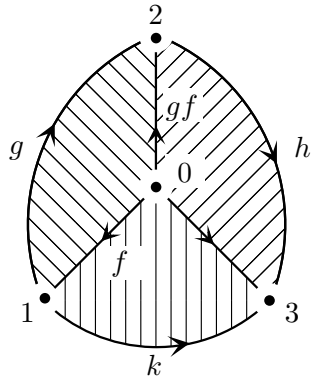
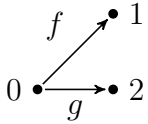
EXAMPLE 2.2.6. Let us describe  $\text{h}$  on  $\partial\Delta^n$  and  $\Lambda_i^n$  both to get some feeling for the construction and as it will be useful for us later.

- (1)  $\text{h}(\Delta^n) = [n]$ .
- (2)  $\text{h}(\partial\Delta^n) = [n]$  for  $n \geq 3$ .
- (3)  $\text{h}(\partial\Delta^2) = 0 \xrightarrow{f} 1 \xrightarrow{g} 2$  and no relations among morphisms. (“=  $\partial\Delta^2$  viewed as a category”).

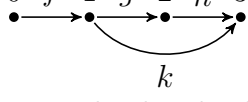


one-innerhorn

- (4)  $\text{h}(\Lambda_i^n) = [n]$  for  $n \geq 4$ .
- (5)  $\text{h}(\Lambda_i^n) = [n]$  for  $0 < i < n$ .
- (6)  $\text{h}(\Lambda_0^2) = \begin{matrix} & & \bullet & 1 \\ & f & \nearrow & \\ \bullet & & & \\ & g & \rightarrow & \bullet & 2 \end{matrix}$  (“=  $\Lambda_0^2$  viewed as a category”), and likewise  $\text{h}(\Lambda_2^2)$  is “ $\Lambda_2^2$  viewed as a category”.
- (7)  $\text{h}\Lambda_0^3$  is described by the picture



where, as the “back simplex” is missing, we are not imposing that the composite  $hg$  equals  $k$ . In other words it has generators  $0 \xrightarrow{f} 1 \xrightarrow{g} 2 \xrightarrow{h} 3$  and one relation



$hgf = kf$ : the (filled) triangles in the diagram imply that both morphisms  $hgf$  and  $kf$  are identified in  $\text{h}\Lambda_0^3$  with the morphism given by the 1-simplex of  $\Lambda_0^3$  having the 0-simplices 0 and 3 as endpoints. There is a dual description of  $\text{h}\Lambda_3^3$ .

One can directly observe:

sproductsprop

PROPOSITION 2.2.7. *The functor  $\text{h} : \text{sSet} \rightarrow \text{Cat}$  preserves finite products.*

### 2.3. Degenerate and non-degenerate simplices in sSet

DEFINITION 2.3.1. An  $n$ -simplex  $x \in X_n$  is called *degenerate* if it is of the form  $\theta^*(y)$  for some  $[n] \twoheadrightarrow [m]$ ,  $m < n$ , and  $y \in X_m$ . Otherwise it is called *non-degenerate*.

LEMMA 2.3.2. Each  $n$ -simplex  $x$  can be written uniquely as  $\theta^*(y)$  for a unique non-degenerate  $y$  and an epimorphism  $\theta: [n] \twoheadrightarrow [m]$ .

The element  $y$  can be described as follows: Let  $i_1 < \dots < i_r$  be the set of indices  $0 \leq i \leq n$ , such that  $x \in s_i X_{n-1}$ . Then

$$y = d_{i_1} \cdots d_{i_r} x.$$

PROOF. We prove the lemma by induction on the number  $r$  of indices  $0 \leq i \leq n$  such that  $x \in s_i X_{n-1}$ . If  $r = 0$ , i.e. if  $x$  is non-degenerate, then  $x$  is not of the form  $\theta^*(y)$  for a surjective, but not bijective map  $\theta$ : it follows that the only choice of  $\theta$  is the identity of  $[n]$ , and the only choice of  $y$  is  $x$  itself (which is non-degenerate).

Suppose now that  $r \geq 1$  and argue by induction: as in the statement, let  $i_1 < \dots < i_r$  be indices  $i$  such that  $x \in s_i X_{n-1}$ , and let  $x' \in X_{n-1}$  be such that  $s_{i_r} x' = x$ . The simplicial identities imply  $x' = d_{i_r} s_{i_r} x' = d_{i_r} x$ , so that  $x'$  is unique.

Suppose now that  $0 \leq j \leq n-1$  is any index such that  $x' \in s_j X_{n-2}$ . Then there are two possibilities:

- if  $j \geq i_r$ , then we have  $x' = s_j z$  for some  $z$  and by the simplicial identities we have  $x = s_{i_r} s_j z = s_{j+1} s_{i_r} z$ , implying that our list of indices  $i_1 < \dots < i_r$  contains  $j+1$ , which is strictly bigger than  $i_r$ ; this is a contradiction, hence we cannot have  $j \geq i_r$ ;
- if  $j < i_r$ , then we have  $x' = s_j z$  for some  $z$  and  $x = s_{i_r} s_j z = s_j s_{i_r-1} z$ , hence  $j$  is one of  $i_1, \dots, i_{r-1}$ ; viceversa, for  $j$  being one of  $i_1, \dots, i_{r-1}$  we can set  $z = d_j x'$  and check that  $s_j z = s_j d_j x' = s_j d_j d_{i_r} x = d_{i_r} x = x'$ , using the simplicial identity  $s_j d_j d_{i_r} = d_{i_r}$ .

The above considerations show that the indices  $0 \leq j \leq n-1$  such that  $x' \in s_j X_{n-2}$  are precisely  $i_1, \dots, i_{r-1}$ , and in particular are  $r-1$  many.

By inductive hypothesis, let  $y'$  and  $\eta: [n-1] \twoheadrightarrow [m]$  be such that  $y'$  is non-degenerate and  $x' = \eta^*(y')$ ; then  $x = (\eta \circ s^{i_r})^*(y')$ , where  $s^{i_r}: [n] \twoheadrightarrow [n-1]$  is the surjective map repeating  $i_r$ . Setting  $y = y'$  and  $\theta = \eta \circ s^{i_r}$ , this shows existence of  $y$  and  $\theta$  for  $x$ .

For uniqueness, suppose now that  $y \in X_m$  and  $\theta: [n] \twoheadrightarrow [m]$  are such that  $y$  is non-degenerate and  $\theta^*(y) = x$ , and write  $\theta = s^{j_1} \circ \dots \circ s^{j_k}$  for suitable  $0 \leq j_1 < \dots < j_k \leq m$ : such a decomposition is unique, and clearly  $k = n - m$ ; since we assume  $r \geq 1$ ,  $x$  is degenerate and hence  $k \geq 1$ . Then  $x = s_{j_k} \dots s_{j_1} y$ , and by the simplicial identities we have  $y = d_{j_1} \dots d_{j_k} s_{j_k} \dots s_{j_1} = d_{j_1} \dots d_{j_k} x$ . Since  $x = s_{i_r} x'$  for some  $x'$  (namely  $x' = d_{i_r} x$ ), we have  $y = d_{j_1} \dots d_{j_k} s_{i_r} x'$ . We cannot have  $i_r > j_k$ , otherwise the simplicial identities would imply  $y' = d_{j_1} \dots d_{j_k} s_{i_r} x' = s_{i_r-k} d_{j_1} \dots d_{j_k} x'$ , hence  $y$  would be degenerate; on the other hand the equality  $x = s_{j_k} \dots s_{j_1} y$  implies that  $j_k$  is one of the indices  $i_1, \dots, i_r$ , in particular  $j_k \leq i_r$ : it follows that  $j_k = i_r$ .

This means that  $\eta := s^{j_1} \dots s^{j_{k-1}}: [n-1] \twoheadrightarrow [m]$  and  $y$  satisfy  $\eta^*(y) = x' = d_{i_r} x$ . By inductive hypothesis (uniqueness of the solution for  $x'$ ) we have  $y = d_{i_1} \dots d_{i_{r-1}} x'$  and  $\eta = s^{i_1} \dots s^{i_{r-1}}$ , and this proves uniqueness of the solution for  $x$ .  $\square$

For illustration and future reference, let us spell out what this means for  $NC$ :

LEMMA 2.3.3. *Consider  $NC$  for some small category  $\mathcal{C}$ .*

*The non-degenerate 0-simplices are the set of objects of  $\mathcal{C}$ .*

*The non-degenerate 1-simplices are the set of non-identity morphisms.*

*The non-degenerate  $n$ -simplices are the  $n$ -fold compositions of non-identity morphisms.  $\square$*

EXAMPLE 2.3.4. Let us work out the non-degenerate simplices in  $\Delta^1 \times \Delta^1$ . We have already seen that  $\Delta^1 \times \Delta^1 = N([1] \times [1])$ , as  $N$  commutes with products. We can easily describe the category  $[1] \times [1]$ : There are 4 objects, 5 non-identity morphisms, and 2 possible compositions, not involving identities. Hence we see that the corresponding simplicial set looks as follows

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (1, 1) \\ \uparrow & \nearrow & \uparrow \\ (0, 0) & \longrightarrow & (1, 0) \end{array}$$

with 4 non-degenerate 0-simplices 5 non-degenerate 1-simplices, and 2 non-degenerate 2-simplices. Note in particular that the non-degenerate 2-simplices correspond to  $(0 \leq 0 \leq 1) \times (0 \leq 1 \leq 1)$  and  $(0 \leq 1 \leq 1) \times (0 \leq 0 \leq 1)$ . In particular, both of them are the product of a pair of degenerate 2-simplices in  $\Delta^1$  and  $\Delta^1$ , but the resulting product 2-simplex is non-degenerate because the two factors are degenerate “for different reasons” (are in the image of different degeneracy maps).<sup>4</sup>

shuffle-ex

EXERCISE 2.3.5. Show that the non-degenerate  $(n+m)$ -simplices in  $\Delta^n \times \Delta^m = N([n] \times [m])$  are in one-to-one correspondence with  $(n, m)$ -shuffles. In particular there are  $\binom{n+m}{n}$  of them.

More precisely, we can view the 0-simplices as the vertices in a grid of size  $(n+1) \times (m+1)$ . A non-degenerate  $k$ -simplex is then a choice of  $(k+1)$  vertices in this grid in non-decreasing order, from bottom-left to top right. The  $(n+m)$ -simplices in particular correspond to paths from  $(0, 0)$  to  $(n, m)$  where we at each step we increase one of the two coordinates by one.

E.g., for  $m = 1$ , there are  $n+1$  non-degenerate  $(n+1)$ -simplices in  $\Delta^n \times \Delta^1$ , corresponding to the sequences

$$\begin{aligned} (0, 0) &< (0, 1) < (1, 1) < (2, 1) < \cdots < (n, 1) \\ (0, 0) &< (1, 0) < (1, 1) < (2, 1) < \cdots < (n, 1) \\ & \vdots \\ (0, 0) &< (1, 0) < \cdots < (n, 0) < (n, 1) \end{aligned}$$

(See also [GJ99, Prop I.4.2].)

<sup>4</sup>This example shows the utility of degenerate simplices and degeneracies in the theory of simplicial sets: it makes the product of simplicial sets agree with the product of the topological spaces associated along geometric realisation.

We can now describe how to obtain an arbitrary simplicial set by “gluing” simplices along attaching maps, as we do for CW complexes by gluing cells:

**cell-attachment**

PROPOSITION 2.3.6 (Cell attachments). *Any simplicial set  $X$  is isomorphic to the colimit of its skeleta along the canonical map*

$$\operatorname{colim}_{n \geq 0} \operatorname{sk}_n X \xrightarrow{\cong} X,$$

and for  $n \geq 0$ ,  $\operatorname{sk}_n X$  is obtained from  $\operatorname{sk}_{n-1} X$  via a pushout.<sup>5</sup>

$$\begin{array}{ccc} \coprod_{\sigma \in X_n^{\text{nd}}} \partial \Delta^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in X_n^{\text{nd}}} \Delta^n & \longrightarrow & \operatorname{sk}_n X. \end{array}$$

Here  $X_n^{\text{nd}} \subset X_n$  denotes the subset of non-degenerate  $n$ -simplices in  $X$ .

In particular any simplicial set  $X$  can be constructed via (potentially infinitely many) simplex attachments, i.e., pushouts of the form

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Z. \end{array}$$

PROOF. To see that  $\operatorname{colim}_{n \geq 0} \operatorname{sk}_n X \xrightarrow{\cong} X$ , we make a Yoneda argument: for any simplicial set  $Y$  we have

$$\begin{aligned} \operatorname{Hom}_{\mathbf{sSet}} \left( \operatorname{colim}_n \operatorname{sk}_n X, Y \right) &\cong \lim_n \operatorname{Hom}_{\mathbf{sSet}} (\operatorname{sk}_n X, Y) \cong \\ &\lim_n \operatorname{Hom}_{\mathbf{Set}^{\Delta_{\leq n}}} (\operatorname{res}_{\Delta_{\leq n}} X, \operatorname{res}_{\Delta_{\leq n}} Y) \cong \operatorname{Hom}_{\mathbf{sSet}} (X, Y). \end{aligned}$$

For the first pushout diagram we just need to verify that in all degrees  $k$ , this is a push-out in  $\mathbf{Set}$ . This is clear in degree  $k < n$  as the vertical maps are bijections. It is also a pushout for  $k = n$ , as  $(\operatorname{sk}_n X)_n \setminus (\operatorname{sk}_{n-1} X)_n$  consists exactly of the non-degenerate  $n$ -simplices, which equals  $\coprod_{\sigma \in X_n^{\text{nd}}} (\Delta^n)_n \setminus (\partial \Delta^n)_n$ .

It is also a pushout in degrees  $k > n$  as each simplex in  $(\operatorname{sk}_n X)_k \setminus (\operatorname{sk}_{n-1} X)_k$  or in  $(\Delta^n)_k \setminus (\partial \Delta^n)_k$  is degenerate, and by Lemma 2.3.2 such simplices are uniquely obtained from non-degenerate ones in degree  $\leq n$ .

The statement about attachments is just a restatement of the previous ones.  $\square$

## 2.4. The functors geometric realization $|\cdot|$ and singular functor $\operatorname{Sing}$ .

DEFINITION 2.4.1. Denote the *topological*  $n$ -simplex as

$$\Delta_{\text{top}}^n = \left\{ \lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n \in \mathbb{R}^{n+1} \mid \lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1 \right\}.$$

<sup>5</sup>By convention  $\operatorname{sk}_{-1} X$  is the empty simplicial set, as well as  $\partial \Delta^0$ .

**secrealsing**

cosimp-delta

DEFINITION 2.4.2. Note that we have a functor  $\Delta \rightarrow \text{Top}$  on objects given by  $[n] \mapsto \Delta_{top}^n$ , and on morphisms given by associating to  $\theta : [n] \rightarrow [m]$  the continuous map  $\theta_* : \Delta_{top}^n \rightarrow \Delta_{top}^m$  given by

$$\theta_* \left( \sum_{i=0}^n \lambda_i e_i \right) = \sum_{i=0}^n \lambda_i e_{\theta(i)}$$

In other words the above functor defines a cosimplicial object in  $\text{Top}$  (aka a cosimplicial space). This cosimplicial space is usually denoted  $\Delta_{top}^\bullet$ .

EXERCISE 2.4.3. Draw this in low dimensions, and check that it is indeed a functor.

We want a procedure to associate a simplicial set to a topological space. This is done via the following construction, which is a “non-linear” version of the singular complex used to define singular homology.

DEFINITION 2.4.4. Define the singular set of a topological space  $Y$  as the simplicial set

$$\text{Sing}_\bullet(Y) = ([n] \mapsto \text{Hom}_{\text{Top}}(\Delta_{top}^n, Y))$$

i.e., composing the functor  $\Delta \rightarrow \text{Top}$  from Definition 2.4.2 with the contravariant functor  $\text{Hom}_{\text{Top}}(-, Y)$ , so  $[n] \mapsto \Delta_{top}^n \mapsto \text{Hom}_{\text{Top}}(\Delta_{top}^n, Y)$ . That is, for  $\theta : [n] \rightarrow [m]$  we have an induced map  $\theta^* : \text{Sing}_m(Y) \rightarrow \text{Sing}_n(Y)$ . We also write

$$\text{Sing}_\bullet(Y) = \text{Hom}_{\text{Top}}(\Delta_{top}^\bullet, Y)$$

for short.<sup>6</sup> We consider  $\text{Sing}_\bullet(-) = \text{Hom}_{\text{Top}}(\Delta_{top}^\bullet, -)$  as a functor  $\text{Top} \rightarrow \text{sSet}$ .

REMARK 2.4.5. Note in particular that  $\text{Sing}_0(Y)$  is the set of points in  $Y$ ,  $\text{Sing}_1(Y)$  is the set of paths in  $Y$ , etc.

REMARK 2.4.6. Note that for a space  $Y$  we have  $C_n^{sing}(Y; \mathbb{Z}) = \mathbb{Z} \text{Sing}_n(Y)$ , and the differential in the singular chain complex is given by  $d = \sum_i (-1)^i d_i$ ; this shows how  $\text{Sing}_\bullet(Y)$  is a non-linear version of singular chain complex defining singular homology. We will see that  $\text{Sing}_\bullet(Y)$  describes the entire weak homotopy type of  $Y$ , and show how to obtain invariants such as homotopy groups of  $Y$  directly from  $\text{Sing}_\bullet(Y)$ .

In the other direction we want to associate a topological space to any simplicial set. As we want  $\Delta_{top}^n$  to correspond to  $\Delta^n$  and the functor to preserve colimits, then the definition is forced by Proposition 2.1.17

DEFINITION 2.4.7. Define the geometric realization functor as

$$|X| = \text{colim}_{\sigma_n \in \text{el } X} \Delta_{top}^n,$$

where  $\Delta_{top}^n$  is the topological  $n$ -simplex. Note that this indeed defines a functor  $|\cdot| : \text{sSet} \rightarrow \text{Top}$ , as a map of simplicial sets  $f : X \rightarrow Y$  induces a functor  $\text{el}(f) : \text{el}(X) \rightarrow \text{el}(Y)$ .

<sup>6</sup>If we consider  $Y$  as a constant cosimplicial space, then we can even write  $\text{Hom}_{\text{Top}\Delta}(\Delta_{top}^\bullet, Y)$ .

In the previous definition we have to clarify the following: what *functor*  $\text{el}(X) \rightarrow \text{Top}$  are we taking the colimit of? Given an object  $\sigma \in \text{el} X$ , it arises as an element  $\sigma$  in one of the sets  $X_n$ , i.e.  $\sigma$  is an  $n$ -simplex for some  $n \geq 0$  (and that is what the notation  $\sigma_n \in \text{el}(X)$ , precisely the index  $n$  in the name of the object, is supposed to suggest): we send  $\sigma \mapsto \Delta_{top}^n$ . Given a morphism  $\sigma \rightarrow \tau$  in  $\text{el}(X)$ , this arises as a map  $\theta: [n] \rightarrow [m]$  in  $\Delta$  such that  $\sigma \in X_n$ ,  $\tau \in X_m$  and  $\theta^*(\tau) = \sigma$ : we send this morphism to the map  $\theta_*: \Delta_{top}^n \rightarrow \Delta_{top}^m$ .

We start by checking that the canonical map  $\Delta_{top}^n \rightarrow |\Delta^n|$  is a homeomorphism, after which we will retire the notation  $\Delta_{top}^n$  and write  $|\Delta^n|$  instead.

deltans-agree

PROPOSITION 2.4.8. *The structure map  $\Delta_{top}^n \rightarrow |\Delta^n|$  corresponding to the  $n$ -simplex  $1_n: [n] \rightarrow [n]$  in  $\Delta^n$ , is a homeomorphism. In other words, the geometric realization of  $\Delta^n$  is canonically homeomorphic to the topological  $n$ -simplex  $\Delta_{top}^n$ .*

PROOF. It is easy to check that there is an inverse  $|\Delta^n| = \text{colim}_{\sigma_i \in \text{el} \Delta^n} \Delta_{top}^i \rightarrow \Delta_{top}^n$  defined as the obvious inclusion map on non-degenerate simplices, and for the non-degenerate simplices first collapsing onto corresponding non-degenerate simplex (cf. Lemma 2.3.2) followed by the inclusion. But we may also use our old favorite, the Yoneda lemma, and check that the representing functors agree:

$$\begin{aligned} \text{Hom}_{\text{Top}}(|\Delta^n|, Y) &\cong \text{Hom}_{\text{Top}}\left(\text{colim}_{\sigma_i \in \text{el}(\Delta^n)} \Delta_{top}^i, Y\right) \\ &\cong \lim_{\sigma_i \in \text{el}(\Delta^n)} \text{Hom}_{\text{Top}}(\Delta_{top}^i, Y) \\ &\cong \lim_{\sigma_i \in \text{el}(\Delta^n)} \text{Hom}_{\text{sSet}}(\Delta^i, \text{Sing}_\bullet(Y)) \\ &\cong \text{Hom}_{\text{sSet}}\left(\text{colim}_{\sigma_i \in \text{el}(\Delta^n)} \Delta^i, \text{Sing}_\bullet(Y)\right) \\ &\cong \text{Hom}_{\text{sSet}}(\Delta^n, \text{Sing}_\bullet(Y)) \\ &\cong \text{Sing}_n(Y) \\ &\cong \text{Hom}_{\text{Top}}(\Delta_{top}^n, Y) \end{aligned}$$

□

rem:ss-coend

REMARK 2.4.9. For  $X_\bullet \in \text{sSet}$ , the geometric realization  $|X_\bullet|$  can also be described more concretely as follows:

$$|X| = \left( \coprod_{n \geq 0} X_n \times |\Delta^n| \right) / \sim$$

where  $\sim$  is the equivalence relation generated by the identifications  $(\theta^*(x), \lambda) \sim (x, \theta_*(\lambda))$ , for varying  $\theta: [n] \rightarrow [m]$ ,  $\lambda \in \Delta_{top}^n$  and  $x \in X_m$ .

Using the notion of a categorical notion of a coend, which we will introduce in Section 5.3.1, this can more succinctly be written as

$$|X| = X_\bullet \otimes_\Delta |\Delta^\bullet|.$$

real-sing-adj

PROPOSITION 2.4.10. *The geometric realization functor  $|\cdot| : \text{sSet} \rightarrow \text{Top}$  is left adjoint to the singular functor  $\text{Sing}_\bullet : \text{Top} \rightarrow \text{sSet}$ .*

$$\text{sSet} \begin{array}{c} \xrightarrow{|\cdot|} \\ \xleftarrow[\text{Sing}_\bullet]{\perp} \end{array} \text{Top},$$

*i.e.*

$$\text{Hom}_{\text{Top}}(|X_\bullet|, Y) \xrightarrow[\simeq]{\phi} \text{Hom}_{\text{sSet}}(X_\bullet, \text{Sing}_\bullet(Y))$$

*natural in both variables.*

*In particular  $|\cdot|$  commutes with colimits, and  $\text{Sing}_\bullet$  commutes with limits.*

REMARK 2.4.11. In the coend language of Remark 2.4.9 the adjunction above becomes a ‘‘Hom-tensor’’ adjunction

$$\text{Hom}_{\text{Top}}(X_\bullet \otimes_\Delta |\Delta^\bullet|, Y) \simeq \text{Hom}_{\text{sSet}}(X_\bullet, \text{Hom}_{\text{Top}}(|\Delta^\bullet|, Y))$$

PROOF OF PROPOSITION 2.4.10. Note that for  $\Delta^n$  the statement becomes

id-for-deltan

$$(2.4.1) \quad \text{Hom}_{\text{Top}}(|\Delta^n|, Y) \cong \text{Hom}_{\text{sSet}}(\Delta^n, \text{Sing}_\bullet(Y)) \cong \text{Sing}_n(Y)$$

where the last rewriting is via the Yoneda lemma. However this amounts to the identification of  $|\Delta^n|$ , defined via the realization functor with the topological  $n$ -simplex, which we verified in Prop 2.4.8. For an arbitrary simplicial set, by definition  $|X| = \text{colim}_{\sigma_n \in \text{el } X} |\Delta^n|$  and the result follows by commuting limits:

$$\begin{aligned} \text{Hom}_{\text{Top}}(|X|, Y) &\cong \text{Hom}_{\text{Top}}\left(\text{colim}_{\sigma_n \in \text{el}(X)} |\Delta^n|, Y\right) \\ &\cong \lim_{\sigma_n \in \text{el}(X)} \text{Hom}_{\text{Top}}(|\Delta^n|, Y) \\ &\cong \lim_{\sigma_n \in \text{el}(X)} \text{Hom}_{\text{sSet}}(\Delta^n, \text{Sing}_\bullet(Y)) \\ &\cong \text{Hom}_{\text{sSet}}\left(\text{colim}_{\sigma_n \in \text{el}(X)} \Delta^n, \text{Sing}_\bullet(Y)\right) \\ &\cong \text{Hom}_{\text{sSet}}(X, \text{Sing}_\bullet(Y)) \end{aligned}$$

□

REMARK 2.4.12. It is worth checking in detail that the maps back and forth in Proposition 2.4.10 are the ‘‘obvious’’ ones: The map  $\phi$  sends  $f : |X_\bullet| \rightarrow Y$ , to the map  $\phi(f)_n : X_n \rightarrow \text{Sing}_n(X)$ , which assigns to  $x \in X_n$  the map that sends  $\lambda \in |\Delta^n|$  to  $(\phi(f)_n(x))(\lambda) = f((x, \lambda)) \in Y$ . The inverse  $\psi$  takes  $g : X_\bullet \rightarrow \text{Sing}_\bullet(Y)$  to  $\psi(g) : |X_\bullet| \rightarrow Y$  given by  $\psi(g)(x, \lambda) = g(x)(\lambda) \in Y$ .

In other words, for each  $n$  we use the adjunction in spaces

$$\text{Hom}_{\text{Top}}(X_n \times |\Delta^n|, Y) \cong \text{Hom}_{\text{Top}}(X_n, \text{map}(|\Delta^n|, Y)),$$

and note that a collection of such maps on the right glue together to form a map  $f : |X_\bullet| \rightarrow Y$  if and only if  $\{\phi(f)_n : X_n \rightarrow \text{Hom}_{\text{Top}}(|\Delta^n|, Y)\}_n$  defines a morphism of simplicial sets.

le: CW-structure

PROPOSITION 2.4.13. *The geometric realisation  $|X|$  of a simplicial set  $X$  has a natural structure of CW complex, whose  $n$ -cells correspond to the non-degenerate  $n$ -simplices in  $X$ .*

PROOF. As  $|-|$  is a left adjoint it commutes with colimits. In particular

$$|X| \cong \operatorname{colim}_n |\operatorname{sk}_n X|$$

Furthermore Proposition 2.3.6, combined again with the fact that  $|-|$  commutes with colimits, implies that we have a pushout square:

$$\begin{array}{ccc} \coprod_{\sigma \in X_n^{\text{nd}}} |\partial \Delta^n| & \longrightarrow & |\operatorname{sk}_{n-1} X| \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in X_n^{\text{nd}}} |\Delta^n| & \longrightarrow & |\operatorname{sk}_n X|. \end{array}$$

This puts a CW structure on  $|X|$ . □

le: RPinfinity

EXAMPLE 2.4.14. Consider  $\mathcal{B}\mathbb{Z}/2$ , i.e., the category with one object and two morphisms 1 and  $g$ , where  $g^2 = 1$ . For all  $n$  there is exactly one non-degenerate  $n$ -simplex which we can write  $(g, g, \dots, g)$ ,  $n$  times. The boundary maps  $d_0$  and  $d_n$  send the non-degenerate  $n$ -simplex to the non-degenerate  $(n-1)$ -simplex, whereas  $d_i$  for  $0 < i < n$  maps it to a degenerate simplex.

In the geometric realization, the 2-skeleton is obtained by gluing a 2-simplex to  $S^1$  along the map that wraps two of the edges (the first and the third) the same way around  $S^1$ , and sends the other edge (the second) constantly to the point representing the unique 0-simplex. A similar but more involved description holds for the attaching maps of the higher simplices. One checks that  $\mathcal{B}\mathbb{Z}/2$  is homotopy equivalent to  $\mathbf{RP}^\infty$ : see also Exercise 2.12.1.

REMARK 2.4.15. The composite functor  $B(-) = |N(-)| : \operatorname{Cat} \xrightarrow{N} \operatorname{sSet} \xrightarrow{|-|} \operatorname{Top}$  is also called the *classifying space* functor, in particular when evaluated on categories of the form  $\mathcal{B}G$ . (The name stems from the fact that, when  $G$  is a group, the space  $B(\mathcal{B}G)$ , also denoted  $BG$ , classifies principal  $G$ -bundles; more about this later!). Sometimes this notation becomes a bit heavy, and the classifying space functor is therefore abbreviated in the literature as just  $|-| : \operatorname{Cat} \rightarrow \operatorname{Top}$ , by an abuse of notation. In yet other parts of the literature the word “space” is taken to mean a simplicial set, and it is then natural to use the notation  $|-|$  for the functor  $\operatorname{Cat} \xrightarrow{N} \operatorname{sSet}$ , here denoted  $N$ . For the even more advanced, distinguishing at all between a category and its image in  $\operatorname{sSet}$  under the fully faithful embedding  $N : \operatorname{Cat} \hookrightarrow \operatorname{sSet}$  is rather cumbersome, and one can simply get rid of both  $N$  and  $|-|$  altogether, by only considering simplicial sets (some of which are (nerves of) categories). The confluence of categories and topology has reached such a pinnacle that one does not even notationally distinguish between the two!!



### 2.5. Geometric realization preserves products

It is also true that geometric realization preserves products, suitably interpreted, though this is not formal.<sup>7</sup> We will now discuss this. The key point is that it holds for products of simplices:

prod-simplices

LEMMA 2.5.1.

$$|\Delta^n \times \Delta^m| \xrightarrow{\cong} |\Delta^n| \times |\Delta^m|$$

IDEA OF PROOF. This essentially follows from our description of the non-degenerate simplices in  $N([n] \times [m])$  in Exercise 2.3.5. This provides a triangulation of  $|\Delta^n \times \Delta^m|$  with top cells corresponding to the  $\binom{n+m}{n}$   $(n, m)$ -shuffles. This is identical to the standard triangulation of  $|\Delta^n| \times |\Delta^m|$ .

[Picture: We imagine an  $n$  times  $m$  grid and a path from  $(0, 0)$  to  $(n, m)$  takes  $m + n$  steps going up and right. We now have to decide where to place the  $n$  steps in which we go right, by selecting  $n$  moments in a sequence of  $n + m$  (e.g., the sublist  $1, 2, 3, 4, \dots, n$  would mean that we first go all the way right, and then all the way up.)

(Note again that when  $m = 1$  this is just the “prism” division of  $|\Delta^n| \times |\Delta^1|$  as a union of  $(n + 1)$  many  $(n + 1)$ -simplices, which occurs when proving homotopy invariance of homology.)

The details are worked out (by you!) are an exercise; see also [GZ67, p. III.3.4].  $\square$

real-pres-prod

THEOREM 2.5.2 (Realization preserves products, v.1). *Let  $X$  and  $Y$  be simplicial sets, and assume that  $Y$  is finite, in the sense that there are only finitely many non-degenerate simplices in  $Y$ , considering all dimensions of simplices at the same time.*<sup>8</sup> Then

$$|X \times Y| \xrightarrow{\cong} |X| \times |Y|$$

PROOF. First note that the statement holds for  $X$  and  $Y$  of the form  $\Delta^n$  and  $\Delta^m$ , by Lemma 2.5.1. Also note that  $X \cong \operatorname{colim}_{\operatorname{el}(X)} \Delta^n$  by the density theorem, that  $|-|$  commutes with colimits being a left adjoint, and that by definition  $|X| \cong \operatorname{colim}_{\sigma_n \in \operatorname{el} X} |\Delta^n|$ . We now assume that  $Y$  is a *finite* simplicial set: as a consequence of Proposition 2.3.6,  $|Y|$  is a finite CW complex, and in particular it is a compact

<sup>7</sup>By not “formal” here we mean that there is no simple proof such as “products are limits, and geometric realisation is a right adjoint, hence it preserves products”, since geometric realisation is a left adjoint but not a right adjoint. An ad hoc argument has to be found!

<sup>8</sup>This condition is *not the same* as asking  $Y_n$  to be finite for each  $n$  (we would say then that  $Y$  is *locally finite*).

space. We can then calculate:

$$\begin{aligned}
|X \times Y| &\cong \left| \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \operatorname{colim}_{\tau_m \in \operatorname{el}(Y)} \Delta^n \times \Delta^m \right| \\
&\cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \operatorname{colim}_{\tau_m \in \operatorname{el}(Y)} |\Delta^n \times \Delta^m| \\
&\cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \operatorname{colim}_{\tau_m \in \operatorname{el}(Y)} (|\Delta^n| \times |\Delta^m|) \\
&\cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} (|\Delta^n| \times |Y|) \\
&\cong \left( \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} |\Delta^n| \right) \times |Y| \\
&\cong |X| \times |Y|.
\end{aligned}$$

In the fourth and fifth line we use that  $|\Delta^n|$  and  $|Y|$  are compact, as otherwise there is only a set-theoretic bijection and the two topologies do not necessarily agree. **[ also use colimits commute with products in ss, as true in sets, which is again because  $(-)\times Y$  is left adjoint. ref to earlier? Example 1.3.11(2) ]**  $\square$

Theorem 2.5.2 can be weakened to  $Y$  only having finitely many cells in each dimension, but is not true without it. This is due to the fact, which the reader probably has encountered in earlier courses on elementary homotopy theory, say [Hat02], that the product in Top of CW complexes will not carry the standard topology on CW complexes, which has the property that a set is open if and only if its restriction to every compact set is open, i.e., the topology. However using Proposition 1.3.13 we get.

**THEOREM 2.5.3** (Realization preserves products, v.2). *Let  $X, Y \in \mathbf{sSet}$ , then  $|X \times Y| \xrightarrow{\cong} |X| \times_k |Y|$ , if the product on the left is taken in the category of compactly generated spaces  $k\mathbf{Top}$ .*

**PROOF.** The argument in Theorem 2.5.2 shows that the map is continuous bijection, but where the left-hand side is topologized by the topology where a set is open if its restriction to every compact is open, which is the same as the topology on  $|X| \times_k |Y|$ , as explained in Proposition 1.3.13. **[ write more formally? ]**  $\square$

## 2.6. Mapping spaces and simplicial homotopy

Recall that  $\Delta^n = N([n])$ , and  $d^i$  is the map  $[n-1] \rightarrow [n]$  that skips  $i$ , so that e.g.,  $N(d^1) : N([0]) \rightarrow N([1])$  geometrically realises to the embedding of a point  $\{0\} = |\Delta^0|$  in the interval  $I = [0, 1] \cong |\Delta^1|$  at 0. There is hence an obvious candidate definition for simplicial homotopy:

**DEFINITION 2.6.1.** Let  $X$  and  $Y$  be simplicial sets. We say that  $f : X \rightarrow Y$  is homotopic to  $g : X \rightarrow Y$  if there exists a map  $H : X \times \Delta^1 \rightarrow Y$  such that

$$X \cong X \times \Delta^0 \xrightarrow{1 \times d^1} X \times \Delta^1 \xrightarrow{H} Y$$

equals  $f$  and

$$X \cong X \times \Delta^0 \xrightarrow{1 \times d^0} X \times \Delta^1 \xrightarrow{H} Y$$

equals  $g$ .

For general simplicial sets  $X$  this definition however has the alarming feature that simplicial homotopy is *not* an equivalence relation in general, due to the “direction” of the homotopy! (I.e., it may not be symmetric.)

EXAMPLE 2.6.2. By Proposition 2.2.4,  $\text{Hom}(\Delta^1, \Delta^1) = \text{Hom}([1], [1])$  consists of 3 maps: the identity map, the constant map 0, and the constant map 1. In particular, while the map  $Nd^1 : \Delta^0 \rightarrow \Delta^1$  is homotopic to  $Nd^0$  via the identity map of  $\Delta^1$ ,  $Nd^0$  is *not* homotopic to  $Nd^1$ , as there is no map  $\Delta^1$  to  $\Delta^1$  flipping the endpoints.

We will describe in the next sections how to rectify this problem by restricting our attention to “nice” simplicial sets, called Kan complexes (or  $\infty$ -groupoids), where such issues cannot occur. However this notion of “niceness” means having sufficiently many simplices to solve natural extension problems. The payback is that “nice” simplicial sets, in this sense, have to be rather large in general, just like an injective resolution of a module or chain complex in homological algebra, is in general much larger than the thing we started out with.

Before moving on to this, let us introduce the internal hom object in simplicial sets:

DEFINITION 2.6.3. For simplicial sets  $X, Y \in \text{sSet}$  define a new simplicial set  $\text{map}(X, Y)_\bullet$ , the *simplicial mapping space*, by setting, for  $n \geq 0$ ,  $\text{map}(X, Y)_n = \text{Hom}_{\text{sSet}}(X \times \Delta^n, Y)$ , with the simplicial structure induced from the cosimplicial structure on  $\Delta^\bullet$ , i.e.,

$$\text{map}(X, Y)_\bullet = \text{Hom}_{\text{sSet}}(X \times \Delta^\bullet, Y).$$

REMARK 2.6.4. Note that we have in particular

$$\text{map}(X, Y)_0 = \{ \text{simplicial maps } X \rightarrow Y \};$$

$$\text{map}(X, Y)_1 = \{ \text{simplicial homotopies } X \times \Delta^1 \rightarrow Y \}.$$

Simplicially we are hence encoding the mapping space  $\text{map}(X, Y)_\bullet$  a bit differently than topologically. We let the points be *all* maps, and then, instead of introducing a topology, we introduce higher order simplices in order to say which maps are homotopic to which, which homotopies are homotopic to which, etc.

Let us check that this makes  $\text{map}(Y, -)$  into a right adjoint to  $(-) \times Y$ :

PROPOSITION 2.6.5. *We have an adjunction isomorphism*

$$\text{Hom}_{\text{sSet}}(X \times Y, Z) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(X, \text{map}(Y, Z))$$

*natural in  $X, Y$ , and  $Z$ .*

PROOF. Naturality in  $X, Y$  and  $Z$  is evident. Both the right-hand side and the left-hand side send colimits in the first variable  $X \in \text{sSet}$  to limits in the category  $\text{Set}$ ; hence it is enough to check the bijection between hom-sets with  $X = \Delta^n$ .

But in this case the lefthand side reads  $\text{Hom}_{\text{sSet}}(\Delta^n \times Y, Z)$  and the right-hand side reads  $\text{Hom}_{\text{sSet}}(\Delta^n, \text{map}(Y, Z)) = \text{map}(Y, Z)_n$ , so this is by definition.  $\square$

EXERCISE 2.6.6. Let  $\mathcal{C}, \mathcal{D}$  be small categories and  $\text{Fun}(\mathcal{C}, \mathcal{D})$  the functor category (which is again a small category). Show that

$$N(\text{Fun}(\mathcal{C}, \mathcal{D})) \cong \text{map}(N\mathcal{C}, N\mathcal{D}).$$

In fancy words,  $\text{Cat}$  is enriched in (small) categories,  $\text{sSet}$  is enriched in simplicial sets, and  $N$  is a functor enriched over  $N$ .

[HINT: Check that  $\text{Hom}_{\text{sSet}}(-, \text{Fun}(\mathcal{C}, \mathcal{D}))$  and  $\text{Hom}_{\text{sSet}}(-, \text{map}(N\mathcal{C}, N\mathcal{D}))$  agree using all our adjunctions, and appeal to Yoneda. You may need that  $h$  commutes with finite products (Proposition 2.2.7).]

## 2.7. Horn filling conditions and Kan complexes

In this section we will introduce the different horn filling conditions. Let us first introduce them, before we try to discuss what they “mean”.

DEFINITION 2.7.1 (Kan complex,  $\infty$ -category, etc.).

- (1)  $X$  is said to satisfy the *horn filling condition* if for any  $n \geq 1$  and  $0 \leq k \leq n$ , and for any  $\Lambda_k^n \rightarrow X$ , there exists a map  $\Delta^n \rightarrow X$  making the following diagram commute:<sup>9</sup>

$$(2.7.1) \quad \begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \exists & \\ \Delta^n & & \end{array}$$

- (2)  $X$  is said to satisfy the *inner horn filling condition* if for any  $n \geq 2$  and  $0 < k < n$ , and for any  $\Lambda_k^n \rightarrow X$ , (i.e., only for “inner horns”) there exists a map  $\Delta^n \rightarrow X$  making the diagram (2.7.1) commute.

Simplicial sets that satisfy the horn filling condition are called *Kan complexes* (or  $\infty$ -groupoids). If they just satisfy the inner horn filling condition are called *weak Kan complexes*, *quasi-categories*, or  *$\infty$ -categories*.

We say that a simplicial set  $X$  satisfies the *unique horn filling condition* if the filler in Definition 2.7.1(1) is unique when  $n \geq 2$ . We define the *unique inner horn filling condition* similarly.

If we think of what the horn filling conditions would say *after* taking geometric realizations, they would be ways of saying that the map  $|X| \rightarrow *$  is a Serre fibration, letting  $n$  vary and for each  $n$  picking just one  $k$  arbitrarily. Of course, this property is satisfied by any topological space  $Y$  replacing  $|X|$ . The two extreme values  $k = 0, n$  would not play a special role. And the lift would also be far from unique. The Kan property is a formulation of this property at the level of simplicial sets, and Kan complexes turn out to model topological spaces — this is a fundamental theorem, see Section 2.10. For now, let us just record that the above argument, combined with adjointness, implies the following proposition.

<sup>9</sup>Note that for  $n = 1$  the horn filling condition is always satisfied.

sing-is-kan

PROPOSITION 2.7.2.  $\text{Sing}_\bullet(Y)$  is a Kan complex for all  $Y \in \text{Top}$ .

PROOF. Filling in the dotted arrow in the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{Sing}_\bullet(Y) \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

is by adjunction equivalent to filling in the dotted arrow in the following diagram in spaces

$$\begin{array}{ccc} |\Lambda_k^n| & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \\ |\Delta^n| & & \end{array}$$

which is possible as  $|\Lambda_k^n|$  is a (deformation) retract of  $|\Delta^n|$ . □

For general simplicial sets, the extension conditions are far from automatic, and it turns out that the extension condition for  $k = 0, n$  play a special role, in particular if we think of simplicial sets as generalized nerves of categories. Namely for a nerve of category one can check that the filler for  $\Lambda_1^2 \rightarrow \Delta^2$  always exists and is unique, as we can uniquely compose morphisms, whereas having fillers for  $\Lambda_0^2 \rightarrow \Delta^2$  and  $\Lambda_2^2 \rightarrow \Delta^2$  translates into having left and right inverses for morphisms, respectively. (Try to check this! ... or see below). We will work out the general versions of these statements below. The name  $\infty$ -category comes from the fact that the fillers still allow us to compose morphisms, though composition will now only be well-defined up to what turns out to be a contractible space of choices. Besides other applications,  $\infty$ -categories turn out to model topologically enriched categories, i.e., categories in which each hom-set  $\text{Hom}(x, y)$  is endowed with a topology (and composition of morphisms is continuous); see also Remark 2.8.2.

**2.7.1. Horn filling conditions for nerves.** In this subsection we will see how to identify  $\text{Cat}$  inside  $\text{sSet}$  as the subcategory of simplicial sets with the unique inner horn filling condition, and also see that  $\mathcal{NC}$  will satisfy the general horn filling condition if and only if  $\mathcal{C}$  is a groupoid, i.e., a category in which every morphism is invertible.

PROPOSITION 2.7.3 (Unique inner horn filling condition). *A simplicial set  $X$  is in the essential image of  $N : \text{Cat} \rightarrow \text{sSet}$  iff  $X$  satisfies the unique inner horn filling condition. Hence  $\text{Cat}$  identifies with the full subcategory of  $\text{sSet}$  with objects simplicial sets satisfying the unique inner horn filling condition.*

PROOF. We start by checking that any simplicial set of the form  $\mathcal{NC}$  has the unique inner horn filling condition: For  $n = 2$  the condition says that for  $f : x \rightarrow y$

and  $g : y \rightarrow z$  there exists a unique 2-simplex filling in the diagram

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & & z \end{array}$$

which of course holds true, as the unique 2 simplex is  $(x \xrightarrow{f} y \xrightarrow{g} z)$ . The argument for higher simplices is similar, as the  $n$ -simplex  $(x_0 \xrightarrow{f_1} x_1 \rightarrow \cdots \xrightarrow{f_n} x_n)$  is uniquely determined by its edges  $f_1, \dots, f_n$ . Or, said in a more fancy-pants way: By adjunction, the lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathcal{NC} \\ \downarrow & \dashrightarrow & \\ \Delta^n & & \end{array}$$

for  $n \geq 2$  and  $0 < i < n$  is equivalent to the lifting problem

$$\begin{array}{ccc} [n] = \mathbf{h} \Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & \dashrightarrow & \\ [n] = \mathbf{h} \Delta^n & & \end{array}$$

which of course has a unique solution, where we have used  $[n] = \mathbf{h} \Lambda_i^n = \mathbf{h} \Delta^n$  by Example 2.2.6.

Conversely, suppose that  $X$  is a simplicial set with the unique inner horn filling condition. We need to define a category  $\mathcal{C}$  such that  $\mathcal{NC} \cong X$ . For this we can take  $\mathcal{C} = \mathbf{h}X$ .

Let us just for concreteness spell out what  $\mathcal{C}$  is in this case. Consider the diagram

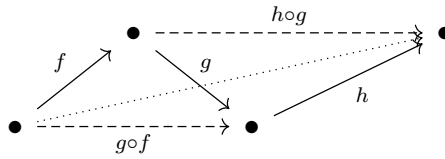
$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{(f,g)} & X \\ \downarrow & \dashrightarrow & \\ \Delta^2 & & \end{array}$$

and define the composition  $g \circ f$  as  $d_1$  of the filler

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & \cdots \xrightarrow{g \circ f} & z \end{array}$$

which by our assumption exists and is unique. This assignment indeed defines a category  $\mathcal{C}$ , with  $X_1$  as morphisms.  $s_0 x \in X_1$  provides identity maps  $1_x$  for all  $x \in X_0$  and associativity follows from the uniqueness of dotted filler in the following

diagram:

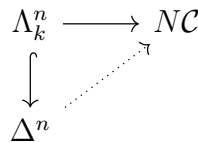


That  $X \cong NC$  again follows by the uniqueness of the filler map. □

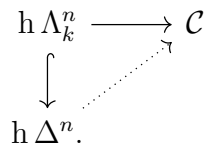
then-is-NC-kan

PROPOSITION 2.7.4 (Horn filling condition for nerves). *The nerve  $NC$  satisfies the horn filling condition if and only if  $\mathcal{C}$  is a groupoid.*

PROOF. Note by adjointness that a lift in the diagram

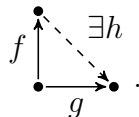


is equivalent to a lift in



Furthermore recall by Example 2.2.6 that  $h(\Lambda_i^n) = [n]$  for almost all  $n \geq 2$  and  $0 \leq i \leq n$ , making the lifting problem trivial, except the four outer horns  $h(\Lambda_0^2)$ ,  $h(\Lambda_2^2)$ ,  $h(\Lambda_0^3)$ ,  $h(\Lambda_3^3)$ .

By Example 2.2.6, the lifting problem with  $h(\Lambda_0^2)$  is solvable if for any two morphisms  $f : x \rightarrow y$  and  $g : x \rightarrow z$  in  $\mathcal{C}$  there exists  $h : y \rightarrow z$  with  $h \circ f = g$



This holds if and only if every morphism  $f : x \rightarrow y$  in  $\mathcal{C}$  has a left inverse (this is clearly sufficient, and setting  $g = 1_x$  for some object  $x$  shows it is also necessary).

Similarly the lifting problem with  $h(\Lambda_2^2)$  is solvable if and only if every morphism  $f : x \rightarrow y$  in  $\mathcal{C}$  has a right inverse. Hence an extension exist for all diagrams of type  $h(\Lambda_0^2)$  and  $h(\Lambda_2^2)$  if and only if every morphism in  $\mathcal{C}$  is invertible, and in this case the extension is unique. This already forces  $\mathcal{C}$  to be a groupoid.

To verify that extensions always exist and are unique for  $n \geq 2$  in the groupoid case, let us check what the extension condition says for  $h(\Lambda_0^3)$ , a category we already described in Example 2.2.6 (you checked the details of that example, right?): In this case, the requirement on  $\mathcal{C}$  is that if we are given morphisms  $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$  and  $y \xrightarrow{k} w$  with  $h \circ g \circ f = k \circ f$ , then  $h \circ g = k$ . In other words the condition is equivalent to that  $\mathcal{C}$  satisfies right cancellation, and this is always true in a groupoid.

The extension is obviously unique as  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$  is a subcategory of  $h(\Lambda_0^3)$ .



Similarly the lifting problem for  $h(\Lambda_3^3)$  is equivalent to left cancellation, and the extension is unique for the same reason as before.  $\square$

REMARK 2.7.5. The above remark explains why a Kan complex, i.e., “space”, is sometimes referred to as an  $\infty$ -groupoid: A groupoid is a category such that also the outer horns admit fillers. Analogously an  $\infty$ -groupoid is an  $\infty$ -category such that the outer horns admit fillers.

### 2.7.2. The fundamental groupoid: the functor $h$ on Kan complexes.

PROPOSITION 2.7.6. *Suppose that  $X$  is a Kan complex, then  $hX$  (a priori just a category) is a groupoid, denoted  $\pi(X)$ , and called the fundamental groupoid of  $X$ . Concretely we have*

$$\mathrm{Hom}_{\pi(X)}(x, y) \cong \{f \in X_1 \mid d_1 f = x, d_0 f = y\} / \overset{h}{\sim}$$

where  $\overset{h}{\sim}$  is an equivalence relation defined by  $f \overset{h}{\sim} g$  iff there exists  $\sigma \in X_2$  with  $\partial(\sigma) := (d_0\sigma, d_1\sigma, d_2\sigma) = (s_0y, g, f)$ . Composition is given by  $g \circ f = d_1\sigma$ , where  $\sigma \in X_2$  is any simplex satisfying  $d_2\sigma = f$  and  $d_0\sigma = g$  (such  $\sigma$  exist by the Kan condition):

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & \xrightarrow{d_1\sigma} & z. \end{array}$$

PROOF. Let  $f \in X_1$ . Then  $f$  is invertible in  $hX$ , since by the Kan condition on  $\Lambda_0^2$  there exists  $\sigma \in X_2$ , so that  $d_0\sigma \circ f = 1$ , so  $f$  has a left inverse in  $hX$ , and similarly for a right inverse.

Now let us check that  $\pi(X)$  has the given description. As we can always find  $\sigma$  as in the diagram above, any morphism in  $hX$  can be represented by a single element in  $X_1$ .

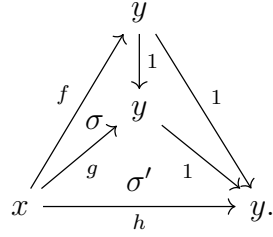
We need to check that  $\overset{h}{\sim}$  agrees with the  $\sim$  from the description of  $h$ . Let us start by verifying that  $\overset{h}{\sim}$  is itself an equivalence relation. Obviously  $f \overset{h}{\sim} f$  via  $s_1f$ . Also  $f \overset{h}{\sim} g$  implies  $g \overset{h}{\sim} f$  as we can find a 3-simplex  $\tau$  filling in the following  $\Lambda_3^3$

$$\begin{array}{ccc} & y & \\ & \downarrow 1 & \\ f \nearrow & y & \searrow 1 \\ \sigma \nearrow & & \nwarrow 1 \\ g \nearrow & & \nwarrow 1 \\ x & \xrightarrow{g} & y \end{array}$$

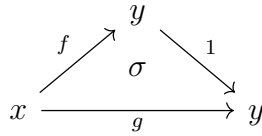
and  $g \overset{h}{\sim} f$  is witnessed by  $d_3\tau$ .



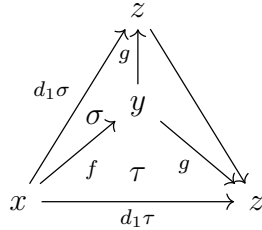
Likewise it is transitive: if  $f \stackrel{h}{\sim} g$  and  $g \stackrel{h}{\sim} h$ , we can fill in the following  $\Lambda_2^3$  with a 3-simplex  $\tau$  and use  $d_2\tau$  to witness  $f \stackrel{h}{\sim} h$ :



We now verify that  $\sim$  agrees with  $\stackrel{h}{\sim}$ . If  $f \stackrel{h}{\sim} g$ , then there exists a 2-simplex  $\sigma$  in the following diagram.



so  $f \sim 1_y \circ f \sim g$ . On the other hand if  $\sigma$  and  $\tau$  are two 2-simplices such that  $d_2\sigma = d_2\tau = f$  and  $d_0\sigma = d_0\tau = g$ , then we can fill in the following  $\Lambda_1^3$



with a 3-simplex, whose  $d_1$ -face shows that  $d_1\sigma \stackrel{h}{\sim} d_1\tau$ . And since  $d_1\sigma \sim d_1\tau$  generates  $\sim$ , we conclude that  $\sim$  and  $\stackrel{h}{\sim}$  have to be the same equivalence relation.  $\square$

**COROLLARY 2.7.7.** *Suppose  $X$  is a Kan complex, then  $\pi(X)$  is a category with set of components  $\pi_0(\pi(X))$  given by  $X_0$  modulo the equivalence relation that  $x \sim y$  if there exists  $(f : x \rightarrow y) \in X_1$ . Furthermore  $\text{Aut}_{\pi(X)}(x) = \{(f : x \rightarrow x) \in X_1\} / \sim$  where  $f \sim g$  iff there exists  $\sigma \in X_2$  with  $\partial(\sigma) = (s_0y, g, f)$ .  $\square$*

**COROLLARY 2.7.8.** *Suppose  $X$  is a Kan complex, then  $\pi(X) \xrightarrow{\cong} \pi(|X|)$  is an equivalence of categories, where the last notation means the topological fundamental groupoid.*

**PROOF.** The functor is given on objects by sending a 0-simplex to the corresponding point in the geometric realisation, and on morphisms by sending the class of a 1-simplex to the class of the corresponding path in  $|X|$ . It is essentially surjective, as any point in  $|X|$  is equivalent to a point coming from  $X_0$  by the description of the CW structure.

Likewise surjective on morphisms: Any path from  $x$  to  $y$  in  $|X|$  can be approximated by a simplicial path, i.e. a composition of edges and inverses of edges; using the Kan property we can always replace an edge run in the wrong direction by two

edges run in the correct direction. So any path from  $x$  to  $y$  in  $|X|$  can be approximated by a concatenation of paths corresponding to 1-simplices in  $X$ . Using the inner horn filling property of  $X$  and the fact that a 2-simplex  $\sigma$  in  $X$  can be used to give a homotopy (relative to its endpoints) between the paths  $|d_0\sigma| \star |d_2\sigma|$  and  $|d_1\sigma|$ , we can further approximate our path in  $|X|$  by one coming from a single 1-simplex in  $X$ .

To show faithfulness, if two 1-simplices  $f, g \in X_1$  are such that  $\partial(f) = \partial(g) = (y, x)$  and moreover  $|f| \sim |g|$  rel.  $x, y$ , then we can write a homotopy  $H: [0, 1] \times [0, 1] \rightarrow |X|$  between  $|f|$  and  $|g|$ , i.e. a continuous map which is already simplicial on  $\partial([0, 1] \times [0, 1])$ , in that it sends the edges of the square to  $|f|, 1_y, g$  and  $1_x$  respectively. We can replace  $H$  by a simplicial approximation that agrees with  $H$  on  $\partial([0, 1] \times [0, 1])$ , by suitably subdividing  $[0, 1] \times [0, 1]$  (but without the need to subdivide  $\partial[0, 1] \times [0, 1]$ ). We can then use the 2-simplices of the new  $H$  to witness the equality  $f \sim g$  in  $\mathbf{h}X$ : here it is crucial that whenever a 2-simplex is oriented “wrongly”, we can use the Kan condition to change  $H$  so that it is oriented correctly.  $\square$

### 2.8. Kan complexes and simplicial homotopy

The following proposition should be compared with the fact from homological algebra that if  $R$  is a commutative ring, and  $M, N$  are  $R$ -modules with  $M$  projective (or flat) and  $N$  injective, then the  $R$ -module  $\text{Hom}_R(M, N)$  is again injective.

map-is-kan

PROPOSITION 2.8.1. *If  $Y$  is a Kan complex then so is  $\text{map}(X, Y)$ .*

PROOF. We need to solve the following lifting problem:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{map}(X, Y) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

which by the adjunction of  $(-)\times X$  and  $\text{map}(X, -)$  is equivalent to

$$\begin{array}{ccc} \Lambda_k^n \times X & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \Delta^n \times X & & \end{array}$$

Now recall that by Proposition 2.3.6,  $X$  can be constructed by iterative simplex attachments.

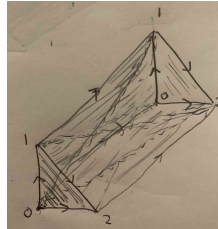
Suppose that  $X$  is obtained from  $X'$  by attaching an  $m$ -simplex, and we have constructed the map on  $\Delta^n \times X'$ . Then we have a diagram of simplicial sets

$$\begin{array}{ccc} (\Lambda_k^n \times X) \cup (\Delta^n \times X') & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \Delta^n \times X & & \end{array}$$

However, extending along this map is equivalent to extending along the inclusion  $(\Lambda_k^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m) \hookrightarrow \Delta^n \times \Delta^m$ . (In  $\Lambda_k^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m$  the interior of

$\Delta^n \times \Delta^m$  and one  $\Delta^{n-1} \times \Delta^m$  faces is missing.) This we can do, by extending over the  $\binom{n+m}{n}$  many  $(m+n)$ -simplices, one at a time, using the horn filling condition for  $X$ . This should be geometrically fairly obvious, but we will not provide the combinatorial details of this here.

Just consider for example the following picture for  $n = 2, k = 0$  and  $m = 1$ :



Here the outwards facing side and interior is removed, and we want to fill in the three 3-simplices  $(0, 0) < (0, 1) < (1, 1) < (2, 1)$ ,  $(0, 0) < (1, 0) < (1, 1) < (2, 1)$  and  $(0, 0) < (1, 0) < (2, 0) < (2, 1)$  (remember the shuffle picture from Exercise 2.3.5). The first 3-simplex is filled in with the hornfilling condition, and similarly we can fill the 2-simplex  $(0, 0) < (1, 0) < (2, 1)$ ; we can then fill in the remaining 3-simplices using the horn filling condition.

The argument in general is obtained by passing to a colimit to attach all  $m$ -simplices, then moving on to the  $m+1$ -simplices etc. See also [GJ99, Prop I.4.2].  $\square$

rem:infytcats

REMARK 2.8.2 ( $\infty$ -categories). It turns out that weak Kan complexes, i.e., simplicial sets  $X$  with the weak inner horn filling condition (also sometimes called quasi-categories), model “ $\infty$ -categories”. The elements in  $X_0$  are the objects (viewed as “homotopy types”). And for two points  $x, y \in X_0$  we can associate a space of maps between them “ $\text{Hom}_X(x, y)$ ”, given as the fiber over  $(x, y)$  of the map  $\text{map}(\Delta^1, X) \rightarrow X \times X$  given by evaluation at endpoints. It turns out to be a Kan complex [apparently by Proposition 2.4.1.8 of HTT].

We will combine the results of the previous subsections by applying  $h$  to  $\text{map}(X, Y)$  to see that simplicial homotopy defines an equivalence relation when mapping into Kan complexes. This fact could of course also have been established directly (Exercise 2.8.8), but we do it this way to illustrate our “higher” viewpoint, where properties of morphisms can be reinterpreted as properties of an object, the mapping space.

grpds-map

COROLLARY 2.8.3. *Suppose  $X$  is an arbitrary simplicial set, and  $Y$  a Kan complex. Then  $\pi(\text{map}(X, Y)) \cong h \text{map}(X, Y)$  is a groupoid with objects maps  $f : X \rightarrow Y$ , and*

$$\text{Hom}_{\pi(\text{map}(X, Y))}(f, g) = \{H : X \times I \rightarrow Y \mid H(-, 0) = f, H(-, 1) = g\} / \sim,$$

where  $H \sim H'$  if there exists a map  $G : X \times \Delta^2 \rightarrow Y$  which restricts to  $(s_0g, H', H)$  on the boundary pieces  $(X \times d_0\Delta^2, X \times d_1\Delta^2, X \times d_2\Delta^2)$ . Elements of  $\text{Hom}_{\pi(\text{map}(X, Y))}(f, g)$  are thus “homotopy classes of homotopies between  $f$  and  $g$ ”. In particular homotopy between maps into a Kan complex is an equivalence relation, corresponding to connected components of the groupoid  $\pi(\text{map}(X, Y))$ .

PROOF. By Proposition 2.8.1  $\text{map}(X, Y)$  is a Kan complex, and the description can now be obtained by writing out Proposition 2.7.6 in that case.  $\square$

DEFINITION 2.8.4. Suppose  $Y$  is a Kan complex, then we denote by  $[X, Y]$  the (free) homotopy classes of maps  $X \rightarrow Y$  (being homotopic is an equivalence relation on the set of maps  $X \rightarrow Y$  by Corollary 2.8.3).

If  $X$  and  $Y$  are pointed simplicial sets, with  $Y$  still Kan, then we denote by  $[X, Y]_*$  the set of pointed homotopy classes of pointed maps, i.e., the set of equivalence classes of pointed maps  $X \rightarrow Y$  under the equivalence relation  $f \sim g$  if there is a homotopy  $H: X \times \Delta^1 \rightarrow Y$  restricting to  $f$  on  $X \times d_1 \Delta^1$ , to  $g$  on  $X \times d_0 \Delta^1$  and to the constant map to  $*$  on  $* \times \Delta^1$ : here  $*$  denotes the subsimplicial set of  $X$ , respectively of  $Y$ , generated by the 0-simplex giving the basepoint (along degeneracies).

EXERCISE 2.8.5. Check that pointed homotopy indeed defines an equivalence relation when the target simplicial set  $Y$  is Kan.

adj-homotopy

COROLLARY 2.8.6. *The adjunction between  $|-|$  and  $\text{Sing}_\bullet$  passes to homotopy categories, i.e. for any simplicial set  $X$  and any topological space  $Y$  we have a natural bijection*

$$[|X|, Y] \cong [X, \text{Sing}_\bullet Y];$$

and if  $X$  and  $Y$  are pointed we also have a natural bijection

$$[|X|, Y]_* \cong [X, \text{Sing}_\bullet Y]_*.$$

PROOF. By Corollary 2.8.3, simplicial homotopy is an equivalence relation when mapping into Kan complexes, and by Proposition 2.7.2  $\text{Sing}_\bullet(Y)$  is Kan. By the  $|-|$ - $\text{Sing}_\bullet(-)$  adjunction, maps  $X \times \Delta^1 \rightarrow \text{Sing}_\bullet(Y)$  are in bijection with maps  $|X \times \Delta^1| \rightarrow Y$ . Since  $|X \times \Delta^1| \cong |X| \times |\Delta^1|$ , by Theorem 2.5.2, the two equivalence relations on maps agree. We leave it as an exercise to verify that this adjunction restricts to the pointed category.  $\square$

ex:simhdual

EXERCISE 2.8.7. Check that the equivalence relation  $\overset{h}{\sim}$  on the set of 1-simplices of a Kan complex can also be described as  $f \overset{h}{\sim} g$  if there exists 2-simplex  $\sigma$  filling in the diagram

$$\begin{array}{ccc} & x & \\ 1_x = s_0(x) \nearrow & & \searrow f \\ x & \xrightarrow{g} & y \end{array}$$

exerc:sh-eq-rel

EXERCISE 2.8.8. Establish directly (without referring to the fundamental groupoids of mapping spaces) that simplicial homotopy is an equivalence relation when mapping into Kan complexes.

EXERCISE 2.8.9. Verify that  $|-|$  and  $\text{Sing}_\bullet(-)$  provide an adjunction between pointed simplicial sets and pointed topological spaces, and verify that this adjunction passes to homotopy classes. (That is, fill in the step left to the reader in Proposition 2.8.6.)

### 2.9. Simplicial homotopy groups and fibrations

In this section we will define simplicial homotopy groups of Kan complexes and establish their basic properties. As homotopy groups require basepoints, we shall mainly work in the category of pointed simplicial sets, which we denote by  $\mathbf{sSet}_*$ .

#### 2.9.1. Simplicial homotopy groups.

DEFINITION 2.9.1. Define the simplicial  $n$ -sphere as  $S^n = \Delta^n / \partial\Delta^n$  for  $n > 0$ , and set  $S^0 = \partial\Delta^1$ .

REMARK 2.9.2. Note that  $\Delta^n / \partial\Delta^n$  has one non-degenerate 0-simplex and one non-degenerate  $n$ -simplex. In particular there are exactly two maps  $\Delta^n / \partial\Delta^n \rightarrow \Delta^n / \partial\Delta^n$ , the trivial map and the identity. This shows the importance of replacing complexes with Kan complexes to get enough maps: the naive simplicial definition of  $\pi_n(S^n)$  as  $\{f: \Delta^n / \partial\Delta^n \rightarrow \Delta^n / \partial\Delta^n\} / \sim$  would lead to a set with (at most) two elements, instead of “ $\mathbb{Z}$ ” many as we expect from the topological, classical notion.

REMARK 2.9.3. By the Yoneda lemma

$$\mathrm{Hom}_{\mathbf{sSet}_*}(S^n, X) \xrightarrow{\cong} \{x \in X_n \mid \partial x := (d_0x, d_1x, \dots, d_nx) = (*, *, \dots, *)\},$$

where by abuse of notation we write  $*$  for the  $(n - 1)$ -simplex  $s_0^{n-1}(*)$  (a practice we will continue to simplify notation!).

DEFINITION 2.9.4 (Simplicial homotopy groups). Let  $X$  be a Kan complex with basepoint  $*$ . Define

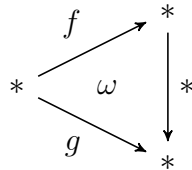
$$\pi_n(X, *) = [S^n, X]_* \quad \text{for } n \geq 0.$$

Let us start by verifying that this gives the right thing for  $n = 0, 1$ . The following lemma gives an alternative way of viewing  $\pi_0$  and  $\pi_1$  in terms of the fundamental groupoid.

groups-groupoid

LEMMA 2.9.5. *Let  $X$  be a Kan complex. Then*

- $\pi_0(X) \cong \{\text{isomorphism classes of objects in } \pi(X)\} = X_0 / \sim$ , where  $x \sim y$  iff there exists  $(f : x \rightarrow y) \in X_1$  (in particular this is an equivalence relation).
- $\pi_1(X, *) \cong \mathrm{Hom}_{\pi(X)}(*) = \{(f : * \rightarrow *) \in X_1\} / \sim$ , where  $f \sim g$  iff there exists  $\omega \in X_2$  with  $\partial\omega = (*, g, f)$  (in particular this is an equivalence relation).



PROOF. We start with  $\pi_0(X)$ . The proof can be summarized by the following picture:

$$\begin{array}{ccc} * & \xrightarrow{*} & * \\ & H & \\ x & \xrightarrow{f} & y \end{array}$$

In slightly more detail: By definition  $S^0$  is isomorphic to the coproduct  $d_1\Delta^1 \sqcup d_0\Delta^1$  of two 0-simplices, where  $*$  =  $d_1\Delta^1$  is the basepoint. A map  $\tilde{x} : S^0 \rightarrow X$  is hence equivalent to a choice of  $x \in X_0$  where to send  $d_0\Delta^1$ , i.e. an object in  $\pi(X)$ . A pointed homotopy  $H : S^0 \times \Delta^1 \rightarrow X$  is a map

$$(* \sqcup d_0\Delta^1) \times \Delta^1 \cong * \times \Delta^1 \sqcup d_0\Delta^1 \times \Delta^1 \rightarrow X$$

restricting to the constant map to  $*$   $\in X$  on  $* \times \Delta^1$ : such a map is uniquely determined by its behavior on  $d_0\Delta^1 \times \Delta^1 \cong \Delta^1$ , corresponding to an  $f \in X_1$ . In other words: Saying that  $H$  witnesses that  $\tilde{x}$  and  $\tilde{y}$  are equivalent in  $\pi_0(X)$ , with  $\tilde{x}$  and  $\tilde{y}$  corresponding to  $x$  and  $y$  in  $X_0$ , respectively, means that  $d_1f = x$  and  $d_0f = y$  and that  $f : x \rightarrow y$  witnesses that  $x$  and  $y$  are equivalent in  $\pi(X)$ .

We now turn to  $\pi_1(X, *)$ . Note that giving a pointed map  $S^1 = \Delta^1/\partial\Delta^1 \rightarrow X$  means specifying a simplex  $(f : * \rightarrow *) \in X_1$ . Furthermore, a pointed simplicial homotopy  $H : S^1 \times \Delta^1 \rightarrow X$  from  $f$  to  $g$  is uniquely determined by a 1-simplex  $h \in X_1$  and two 2-simplices  $\sigma, \tau \in X_2$  satisfying  $\partial\sigma = (*, h, f)$  and  $\partial\tau = (g, h, *)$ , as pictured below:

$$\begin{array}{ccc} * & \xrightarrow{f} & * \\ * \downarrow & \searrow h & \sigma \downarrow \\ * & \xrightarrow{g} & * \end{array}$$

Now given  $\omega \in X_2$  with  $\partial\omega = (*, g, f)$  we can construct a simplicial homotopy  $H$  as follows

$$\begin{array}{ccc} * & \xrightarrow{f} & * \\ * \downarrow & \searrow g & \omega \downarrow \\ * & \xrightarrow{g} & * \end{array}$$

Conversely, given  $h, \sigma$  and  $\tau$ , we can fill in the following horn  $\Lambda_3^3$ :

$$\begin{array}{ccc} & * & \\ & \downarrow & \\ f \nearrow & * & \searrow g \\ \sigma \nearrow & * & \searrow g \\ h \nearrow & * & \searrow g \\ * & \xrightarrow{\tau} & * \end{array}$$

which shows  $f \sim g$ , using Exercise 2.8.7. □

topicsimplices

LEMMA 2.9.6.  $f, g \in \pi_n(X, *)$  are homotopic if and only if there exists  $\omega : \Delta^{n+1} \rightarrow X$  such that  $\omega \circ d^n = g$  and  $\omega \circ d^{n+1} = f$ , and all other faces of  $\omega$  are constantly  $*$  (i.e.,  $\omega \in X_{n+1}$  such that  $\partial\omega = (*, \dots, *, g, f)$ ).

PROOF. It is clear by definition that elements in  $\pi_n(X, *)$  are represented by  $n$ -simplices whose boundary faces are constantly  $*$ .

The statement about the equivalence relations is not just the definition: A simplicial homotopy is a map  $\Delta^n \times \Delta^1 \rightarrow X$  sending  $\partial\Delta^n \times \Delta^1$  to the basepoint  $*$  in  $X$ . The datum of such a homotopy is equivalent to the datum of  $n + 1$  many  $(n + 1)$ -simplices  $\sigma_0, \dots, \sigma_n \in X_{n+1}$  such that one face of  $\sigma_0$  is  $f$ , one face of  $\sigma_n$  is  $g$ , and the

other faces are partially required to be equal in pairs, and partially required to be constantly  $*$ .

We have a simplicial collapse map  $\Delta^n \times \Delta^1 \rightarrow \Delta^{n+1}$ , which shows that an  $\omega$  as above induces a simplicial homotopy.

Conversely, if we have a simplicial homotopy, then we see that  $f$  is equivalent to  $g$  in the sense above by going through the  $n+1$  levels of the prism. More precisely, there are  $n$ -simplices  $f = f_{n+1}, f_n, \dots, f_1, f_0 = g$  such that  $\partial\sigma_i = (*, \dots, *, f_i, f_{i+1}, *, \dots, *)$ , where  $f_i$  occurs as the  $i^{\text{th}}$  face.

It is a general principle that if  $\sigma$  is a  $(n+1)$ -simplex with  $\partial\sigma = (*, \dots, *, a, b, *, \dots, *)$ , for some  $n$ -simplices  $a, b$  occurring as  $i^{\text{th}}$  and  $(i+1)^{\text{st}}$  faces, with  $i \leq n$ , then we can form a horn with faces  $(*, \dots, *, \sigma, s_i a, ?, s_{i+2} b, *, \dots, *)$ , whose  $(i+2)^{\text{nd}}$  face, denoted  $? = \tilde{\sigma}$ , satisfies  $\partial\tilde{\sigma} = (*, \dots, *, a, b, *, \dots, *)$ , where now  $a, b$  occur as  $(i+1)^{\text{st}}$  and  $(i+2)^{\text{nd}}$  faces. To justify this principle, use also Lemma 2.9.9 in the following. Using this principle many times we find simplices  $\tilde{\sigma}_i$  with  $\partial\tilde{\sigma}_i = (*, \dots, *, f_i, f_{i+1})$ .

It is another general principle that if  $a, b, c$  are  $n$ -simplices with  $\partial a = \partial b = \partial c = (*, \dots, *)$ , then as soon as we have two of the following three types of  $(n+1)$ -simplices:

- a simplex  $\alpha$  with  $\partial\alpha = (*, \dots, *, a, b)$ ;
- a simplex  $\beta$  with  $\partial\alpha = (*, \dots, *, b, c)$ ;
- a simplex  $\gamma$  with  $\partial\alpha = (*, \dots, *, a, c)$ ,

then we can construct the third by filling a horn whose faces are  $(*, \dots, *, \alpha, \gamma, \beta)$  (where one of  $\alpha, \gamma, \beta$  replaced by a  $?$ ), and filling this horn gives an example of the missing type of the three. Using this principle many times we can construct a simplex  $\omega$  with  $\partial\omega = (*, \dots, *, g, f)$ . Exercise: check that this indeed works (and that I have not made mistakes with the indices). **[ todo: Add details. Draw this for  $\Delta^1 \times \Delta^1$ . ]**  $\square$

PROPOSITION 2.9.7. *For a pointed topological space  $(Y, *)$ , let  $* \in \text{Sing}_\bullet(Y)_0 = \text{Hom}_{\text{Top}}(|\Delta^0|, Y)$  be given by the composite  $|\Delta^0| \rightarrow * \hookrightarrow Y$ . Then*

$$\pi_n(\text{Sing}_\bullet(Y), *) = [\Delta^n / \partial\Delta^n, \text{Sing}_\bullet(Y)]_* \cong [|\Delta^n / \partial\Delta^n|, Y]_* = \pi_n(Y, *),$$

PROOF. This follows from the adjunction of Corollary 2.8.6, as  $|\Delta^n / \partial\Delta^n| \cong S^n$ .  $\square$

PROPOSITION 2.9.8. *Let  $\mathcal{G}$  be a groupoid,  $* \in \mathcal{G}$ . Then*

$$\pi_n(N\mathcal{G}, *) = 0 \text{ for } n \geq 2$$

*and in particular if  $\mathcal{G} = \mathcal{B}G$  for a group  $G$ , then  $N\mathcal{G}$  is a  $K(G, 1)$ .*

PROOF. Note that  $N\mathcal{G}$  is Kan by Proposition 2.7.4, so the homotopy groups are defined. Furthermore if  $x \in (N\mathcal{G})_n$ ,  $n \geq 2$  and  $\partial x = (s_0^{n-1}(*), \dots, s_0^{n-1}(*))$ , then  $x = s_0^n(*)$ , because  $x$  is determined by its 1-skeleton, as this is true for the nerve of any category.  $\square$

It may be useful at this point to restate the horn filling condition in concrete simplicial terms:

em:specifyhorn

LEMMA 2.9.9. *Specifying a simplicial map  $f : \Lambda_k^n \rightarrow X$  is equivalent to specifying elements  $x_0, \dots, \hat{x}_k, \dots, x_n \in X_{n-1}$  such that  $d_i x_j = d_{j-1} x_i$  for  $0 \leq i < j \leq n$ , with  $i, j \neq k$ . Extending  $f$  to  $\Delta^n$  is equivalent to finding  $y \in X_n$  such that  $d_i y = x_i$  for  $i \neq k$ .*

PROOF. Recall that the simplicial identities say also that  $d_i d_j = d_{j-1} d_i$  if  $i < j$ ; these identities correspond to how the  $(n-1)$ -faces of an  $n$ -simplex identify along the  $(n-2)$ -faces, cf. Proposition 2.1.5. Correspondingly, specifying a map on  $\Lambda_k^n$  exactly amounts to specifying  $x_i$  for  $i \neq k$ , subject to these constraints for  $i, j \neq k$ . And extending a map to all of  $\Delta^n$  means finding  $x \in X_n$  such that  $d_i x = x_i$  for all  $i \neq k$ . As a general principle, if  $X$  is finite then a simplicial map  $X \rightarrow Y$  can be defined by declaring its behavior only on those simplices of  $X$  that are not in the image of any face or degeneracy map (we might call these the “free simplices” of  $X$ ), under all constraints ensuring that the above choices actually glue to a map out of  $X$  (It is a very good exercise to check all the details of this).  $\square$

PROPOSITION 2.9.10 (Group structure). *Let  $n \geq 1$ . The following operation  $\star$  makes  $\pi_n(X, *)$  into a group, which is abelian if  $n \geq 2$  (in this case we write  $+$  for  $\star$ ). Given  $x, y \in X_n$  with  $\partial x = \partial y = *$ , let  $\omega$  be any lift in*

$$\begin{array}{ccc} \Lambda_n^{n+1} & \xrightarrow{(*, \dots, *, x, -, y)} & X \\ \downarrow & \nearrow \exists \omega & \\ \Delta^{n+1} & & \end{array},$$

i.e. let  $\omega \in X_{n+1}$  be such that  $\partial \omega = (*, \dots, *, x, d_n \omega, y)$ , and set

$$[x] \star [y] = [d_n \omega].$$

PROOF.  $\partial(d_n \omega)$  is determined by the behavior of the map  $\Lambda_n^{n+1} \rightarrow X$  on a part of the  $(n-1)$ -skeleton of  $\Lambda_n^{n+1}$ , and in particular  $\partial(d_n \omega) = (*, \dots, *)$ , so  $d_n \omega \in X_n$  represents a homotopy class. Associativity of  $\star$  and existence of inverses is similar to verifying that  $\pi(X)$  is a groupoid. The proof that  $\pi_n(X, *)$  is abelian when  $n \geq 2$  uses an Eckmann-Hilton argument [elaborate].  $\square$

OBSERVATION 2.9.11. We saw in Lemma 2.9.5 that  $\pi_1(X, *) \cong \text{Hom}_{\pi(X)}(*, *)$ . We can now add that the group operation on  $\pi_1(X, *) \cong \text{Hom}_{\pi(X)}(*, *)$  coincides with the composition on  $\text{Hom}_{\pi(X)}(*, *)$ :

$$(x, -, y) \text{ gives } \omega \in X_2 \text{ with } \begin{array}{ccc} & \bullet & \\ y \nearrow & \omega & \searrow x \\ \bullet & \dashrightarrow & \bullet \\ & d_1 \omega & \end{array} \text{ and } [x] \star [y] = [d_1 \omega].$$

In the same way  $\omega$  witnesses that  $[x] \circ [y] = [d_1 \omega]$  in  $\pi(X)$ . [reformulate?]

**2.9.2. Kan fibrations.** Extending the notion of a Kan complex, we can define a Kan fibration:



**defn:kanfib**

DEFINITION 2.9.12 (Kan fibration). A simplicial map  $p: X \rightarrow Y$  is a *Kan fibration* if it has the following lifting property ( $\forall n > 0, 0 \leq k \leq n$ )

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f} & X \\
 \downarrow & \nearrow \exists \lambda & \downarrow p \\
 \Delta^n & \xrightarrow{g} & Y
 \end{array}$$

i.e. given  $f, g$  making the square commute, there exists a  $\lambda$  making the two triangles commute.

REMARK 2.9.13. Note that a simplicial set  $X$  is Kan if and only if the map  $X \rightarrow *$  is a Kan fibration.

PROPOSITION 2.9.14. Let  $f: Z \rightarrow W$  be a map in  $\text{Top}$ . Then  $\text{Sing}_\bullet(f)$  is a Kan fibration if and only if  $f$  is a Serre fibration.

PROOF. By adjunction, for a continuous map of spaces  $f: Z \rightarrow W$ , the lifting problem in  $\text{sSet}$

$$\begin{array}{ccc}
 \Lambda_k^n & \longrightarrow & \text{Sing}_\bullet(Z) \\
 \downarrow & \nearrow & \downarrow \text{Sing}_\bullet(f) \\
 \Delta^n & \longrightarrow & \text{Sing}_\bullet(W)
 \end{array}$$

is equivalent to the lifting problem in  $\text{Top}$

$$\begin{array}{ccc}
 |\Lambda_k^n| & \longrightarrow & Z \\
 \downarrow & \nearrow & \downarrow f \\
 |\Delta^n| & \longrightarrow & W
 \end{array}$$

□

The following is a vast generalization of Proposition 2.8.1:

**sm7**

THEOREM 2.9.15 (The notorious SM7). If  $p: X \rightarrow Y$  is a (Kan) fibration of simplicial sets and  $i: A \hookrightarrow B$  is an inclusion of simplicial sets, then

$$\text{map}(B, X) \xrightarrow{(i^*, p_*)} \text{map}(A, X) \times_{\text{map}(A, Y)} \text{map}(B, Y)$$

is a (Kan) fibration. In particular,  $\text{map}(B, X) \rightarrow \text{map}(A, X)$  is a fibration for  $X$  Kan, and  $\text{map}(A, X) \rightarrow \text{map}(A, Y)$  is a fibration if  $p$  is a fibration.

The result can be strengthened: Below in Definition 2.10.4, we define the notion of a weak equivalence of simplicial sets. With this notion, the map  $(i^*, p_*)$  is a weak equivalence if  $\iota$  and  $p$  are.

ABOUT THE PROOF OF THEOREM 2.9.15. Note that it is a generalization of Proposition 2.8.1. We will not give the proof here: it can be found in [GJ99, Prop. I.5.2], following an approach of Gabriel–Zisman [GZ67].

It involves the same ingredients as the baby case Proposition 2.8.1, which we also did not prove in details—the key input is constructing a lifting cell-by-cell, which again involves keeping track of the “prism” division of products of simplices into simplices. A good way of keeping track of this is via the theory of “anodyne extensions”. (“Anodyne” might sound like a scary word, but it actually means roughly “inoffensive”; they are inclusions such as  $\Lambda_i^n \rightarrow \Delta^n$ ,  $(\Lambda_i^n \times \Delta^1 \cup \Delta^n \times \partial\Delta^1) \rightarrow \Delta^n \times \Delta^1$  etc., obtained by iterately attaching a new simplex along a “contractible” part of its boundary (in a precise sense), and you prove that you can lift over anodyne extensions.)  $\square$

The (non-formal) fact that geometric realization commutes with products in fact has the following generalization:

**THEOREM 2.9.16** (Quillen). *If  $p: X \rightarrow Y$  is a Kan fibration, then  $|p|: |X| \rightarrow |Y|$  is a Serre fibration.*

ABOUT THE PROOF. The result is certainly not a formality, and is a bit surprising since  $|-|$  is a left adjoint rather than a right adjoint, in the same way that it was surprising that  $|-|$  commuted with products. We will skip the proof, which requires some work. The original reference is [Qui68]. The statement and proof is also given in [GJ99, Thm. I.10.10].  $\square$

**REMARK 2.9.17.** Note that the converse of Theorem 2.9.16 is false: The map  $|X| \rightarrow *$  is always a Serre fibration, even if  $X$  is not Kan.

**PROPOSITION 2.9.18.** *The composition of Kan fibrations is a Kan fibration, and the pullback of a Kan fibration is a Kan fibration.*

**PROOF.** This follows directly from the lifting property, and the universal property of the pullback.  $\square$

**DEFINITION 2.9.19.** For a Kan complex  $X$ , define the path space and loop space via the pullback squares

$$\begin{array}{ccc} \Omega X & \longrightarrow & \text{map}(\Delta^1, X) \\ \downarrow & & \downarrow \text{ev}_0 \times \text{ev}_1 \\ * & \longrightarrow & X \times X \end{array} \quad \text{and} \quad \begin{array}{ccc} PX & \longrightarrow & \text{map}(\Delta^1, X) \\ \downarrow & & \downarrow \text{ev}_0 \times \text{ev}_1 \\ X & \xrightarrow{* \times \text{id}_X} & X \times X. \end{array}$$

Note in particular:

**PROPOSITION 2.9.20.**  *$PX$  and  $\Omega X$  are Kan complexes, and  $PX \rightarrow X$  is a Kan fibration.*

**PROOF.** Theorem 2.9.15 guarantees that the right-hand vertical maps in the two squares in Definition 2.9.19 are Kan fibrations, and then so are the left-hand ones by the pull-back property, Proposition 2.9.18.  $\square$

**2.9.3. The long exact sequence in homotopy groups.**

**THEOREM 2.9.21.** *Let  $p: X \rightarrow Y$  be a Kan fibration between Kan complexes and let  $F = p^{-1}(*) \xrightarrow{i} X$  be the fiber of a vertex  $* \in Y_0$ . Assume a lift  $* \in F_0 \subset X_0$  of  $* \in Y_0$  is given. Then  $F$  is a Kan complex and there is a long exact sequence of homotopy groups*

$$\begin{aligned} \cdots \longrightarrow \pi_n(F, *) &\xrightarrow{i_*} \pi_n(X, *) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, *) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_1(Y, *) &\xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y). \end{aligned}$$

Moreover, there is an action of the group  $\pi_1(Y, *)$  on the set  $\pi_0(F)$  such that  $i_*[x] = i_*[y] \in \pi_0(X)$  if and only if  $[x], [y] \in \pi_0(F)$  are in the same orbit.

The entire statement is natural in the Kan fibration: a commutative diagram of simplicial sets

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ F' & \longrightarrow & X' & \longrightarrow & Y' \end{array}$$

with  $p$  and  $p'$  being Kan fibrations (and compatible choices of basepoints in  $Y$  and in  $Y'$  as well as in  $F := p^{-1}(*) \subset X$  and  $F' := (p')^{-1}(*) \subset X'$ ) gives rise to a commutative diagram of sets (for  $n \geq 0$ )/of groups (for  $n \geq 1$ )

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & \pi_n(F, *) & \longrightarrow & \pi_n(X, *) & \xrightarrow{p_*} & \pi_n(Y, *) & \xrightarrow{\partial} & \pi_{n-1}(F, *) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\partial} & \pi_n(F', *) & \longrightarrow & \pi_n(X', *) & \xrightarrow{p'_*} & \pi_n(Y', *) & \xrightarrow{\partial} & \pi_{n-1}(F', *) & \longrightarrow & \cdots \end{array}$$

**PROOF.** The boundary map  $\partial: \pi_n(Y, *) \rightarrow \pi_{n-1}(F, *)$  is defined as follows: Let  $y \in Y_n$  with  $\partial y = *$  be given. Let  $\lambda \in X_n$  be a solution of the lifting problem

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{(-, *, \dots, *)} & X \\ \downarrow & \nearrow \exists \lambda & \downarrow p \\ \Delta^n & \xrightarrow{y} & Y, \end{array}$$

i.e.  $\partial \lambda = (d_0 \lambda, *, \dots, *)$ ,  $p(\lambda) = y$ . So  $p(d_0 \lambda) = d_0 p(\lambda) = d_0 y = *$  and therefore  $d_0 \lambda \in F$ . Moreover,  $\partial(d_0 \lambda) = *$ . So  $d_0 \lambda$  represents a class in  $\pi_{n-1}(F, *)$ . Define

$$\partial[y] = [d_0 \lambda].$$

One checks that  $\partial$  is well-defined and a homomorphism for  $n > 1$  (exercise!). For the exactness, we check it at

$$\pi_n(X, *) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, *).$$

For  $x \in X_n$ ,  $\partial x = *$ , we have

$$\partial p_*[x] = \partial[p(x)] = [d_0x] = [*] = *,$$

since we can choose  $\lambda = x$ . Conversely, suppose that  $0 = \partial[y] = [d_0\lambda]$ . Then by Lemma 2.9.6, there exists  $\omega \in F_n$  such that  $\partial\omega = (*, \dots, *, d_0\lambda)$ . The lifting problem

$$\begin{array}{ccc} \Lambda_n^{n+1} & \xrightarrow{(\omega, *, \dots, *, -, \lambda)} & X \\ \downarrow & \nearrow \exists \Theta & \\ \Delta^{n+1} & & \end{array}$$

gives  $\Theta \in X_{n+1}$  with  $\partial\Theta = (\omega, *, \dots, *, \eta, \lambda)$  for some  $\eta \in X_n$  with  $\partial\eta = *$ . Since  $p\omega = *$  (indeed  $\omega \in F$ ), it follows  $\partial p\Theta = p\partial\Theta = (*, \dots, *, p\eta, y)$ . So  $p\Theta$  exhibits  $[y] = [p\eta] = p_*[\eta]$ . The rest of the proof is left as exercise.  $\square$

secsSetvsTop

## 2.10. The equivalence between sSet and Top

thm:isopn

THEOREM 2.10.1. *For any Kan complex  $X$ , the canonical map*

$$X \xrightarrow{\eta_X} \text{Sing}_\bullet |X|$$

*induces an isomorphism  $\pi_n(X, *) \xrightarrow{\cong} \pi_n(\text{Sing}_\bullet |X|, *) = \pi_n(|X|, *)$  in homotopy groups.*

PROOF. The proof goes by induction on  $n$ , using the path-loop fibration  $\Omega X \rightarrow PX \rightarrow X$ , where  $PX$  and  $\Omega X$  are defined through the pullback squares

$$\begin{array}{ccc} PX & \longrightarrow & \text{map}(\Delta^1, X) \\ \downarrow & & \downarrow \text{ev}_0 \\ * & \longrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega X & \longrightarrow & PX \\ \downarrow & & \downarrow \text{ev}_1 \\ * & \longrightarrow & X. \end{array}$$

We do the induction step, using two facts:

- (1)  $PX \rightarrow X$  is indeed a fibration (Proposition 2.9.20).
- (2)  $PX$  has trivial homotopy groups.

By Theorem 2.9.16,  $|\Omega X| \rightarrow |PX| \rightarrow |X|$  is a (Serre) fibration in Top and hence,  $\text{Sing}_\bullet(|\Omega X|) \rightarrow \text{Sing}_\bullet(|PX|) \rightarrow \text{Sing}_\bullet(|X|)$  is a (Kan) fibration in sSet. The commutative diagram of Kan fibrations

$$\begin{array}{ccccc} \Omega X & \longrightarrow & PX & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \text{Sing}_\bullet(|\Omega X|) & \longrightarrow & \text{Sing}_\bullet(|PX|) & \longrightarrow & \text{Sing}_\bullet(|X|) \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccc}
 \pi_n(PX) & \longrightarrow & \pi_n(X) & \xrightarrow{\partial} & \pi_{n-1}(\Omega X) & \longrightarrow & \pi_{n-1}(PX) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \pi_n(\text{Sing}_\bullet |PX|) & \longrightarrow & \pi_n(\text{Sing}_\bullet(|X|)) & \xrightarrow{\partial} & \pi_{n-1}(\text{Sing}_\bullet(|\Omega X|)) & \longrightarrow & \pi_{n-1}(\text{Sing}_\bullet |PX|).
 \end{array}$$

By the second fact (the fact the  $PX$  has trivial homotopy groups), it follows that also  $\text{Sing}_\bullet |PX|$  is contractible. So the boundary maps  $\partial$  in the diagram are isomorphisms and since by induction hypothesis,  $\pi_{n-1}(\Omega X) \rightarrow \pi_{n-1}(\text{Sing}_\bullet(|\Omega X|))$  is an isomorphism, we conclude the induction step.  $\square$

**cor:weakhom**

COROLLARY 2.10.2. *For any topological space  $Y$ , the canonical map*

$$|\text{Sing}_\bullet(Y)| \xrightarrow{\varepsilon_Y} Y$$

*is a weak homotopy equivalence.*

PROOF. Applying  $\pi_n$  to the commutative diagram

$$\begin{array}{ccc}
 \text{Sing}_\bullet(Y) & \xrightarrow{\eta_{\text{Sing}_\bullet(Y)}} & \text{Sing}_\bullet | \text{Sing}_\bullet(Y) | \\
 & \searrow 1 & \downarrow \text{Sing}_\bullet(\varepsilon_U) \\
 & & \text{Sing}_\bullet(Y)
 \end{array}$$

yields a commutative diagram

$$\begin{array}{ccc}
 \pi_n(\text{Sing}_\bullet(Y)) & \xrightarrow{\cong} & \pi_n(\text{Sing}_\bullet | \text{Sing}_\bullet(Y) |) = \pi_n(| \text{Sing}_\bullet(Y) |) \\
 & \searrow 1 & \downarrow \\
 & & \pi_n(\text{Sing}_\bullet(Y)) = \pi_n(Y)
 \end{array} ,$$

where the horizontal morphism is an isomorphism Theorem 2.10.1. So also the map  $\pi_n(| \text{Sing}_\bullet(Y) |) \rightarrow \pi_n(Y)$  induced by the counit of the adjunction is an isomorphism.  $\square$

**[ would still like to see that weak homotopy equivalence of simplicial sets agrees with “strong” homotopy equivalence defined via the path object on Kan complexes ]**

REMARK 2.10.3.  $|\text{Sing}_\bullet(U)| \xrightarrow{\varepsilon_U} U$  is a functorial CW approximation of  $U$ , i.e. the functor  $|\text{Sing}_\bullet(-)|: \text{Top} \rightarrow \text{Top}$  produces for every topological space a CW approximation.

Recall that for any adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D} ,$$

the functors  $F, G$  are equivalences of categories if and only if  $X \xrightarrow{\eta_X} GF(X)$  and  $FG(Y) \xrightarrow{\varepsilon_Y} Y$  are isomorphisms for all  $X \in \mathcal{C}$  and  $Y \in \mathcal{D}$ .

In the case of the adjunction

$$\begin{array}{ccc} & | - | & \\ \text{sSet} & \xrightarrow{\quad} & \text{Top} , \\ & \text{Sing}_\bullet & \end{array}$$

the morphisms  $X \xrightarrow{\eta_X} \text{Sing}_\bullet |X|$  and  $|\text{Sing}_\bullet(Y)| \xrightarrow{\varepsilon_Y} Y$  are not necessarily isomorphisms, but they are weak homotopy equivalences, and therefore they induce an equivalence of homotopy categories

$$\begin{array}{ccc} & | - | & \\ \text{Ho}(\text{sSet}) & \xrightarrow{\quad} & \text{Ho}(\text{Top}) , \\ & \text{Sing}_\bullet & \end{array}$$

where  $\text{Ho}(\mathcal{C})$  is  $\mathcal{C}$  with weak equivalences formally inverted (this will be discussed in more detail in Section 4.1).

defweequiv

DEFINITION 2.10.4. A simplicial map between any simplicial sets  $f: X \rightarrow Y$  is a *weak equivalence* if  $|f|: |X| \rightarrow |Y|$  is a weak homotopy equivalence.

We conclude that for any simplicial set  $X$ , the morphism  $X \xrightarrow{\eta_X} \text{Sing}_\bullet |X|$  is a weak equivalence by the commutative diagram

$$\begin{array}{ccc} |X| & \xrightarrow{|\eta_X|} & |\text{Sing}_\bullet |X|| \\ & \searrow 1 & \downarrow \varepsilon_{|X|} \\ & & |X| \end{array}$$

and Corollary 2.10.2. Similarly we have the following:

PROPOSITION 2.10.5. For  $X$  and  $Y$  Kan,  $f: X \rightarrow Y$  is a weak equivalence if and only if  $f_*: \pi_0(X) \rightarrow \pi_0(Y)$  and  $f_*: \pi_n(X, *) \rightarrow \pi_n(Y, f(*))$  are isomorphisms for all  $n \geq 1$  and all  $* \in X_0$ .

remmodelcat

REMARK 2.10.6 (A small digression on model categories). The above comparison between simplicial sets and topological spaces can be formulated within the framework of model categories. A *model category* is a category that comes equipped with three classes of morphisms, namely *cofibrations*, *fibrations* and *weak equivalences* subject to a list of axioms, see Chapter 2 of [GJ99] or also [Hov99]. One should think of cofibrations (fibrations) as a nice class of injections (surjections) although this is nothing more than an intuition. A weak equivalence should be thought of as morphism that preserves all essential information, but is typically not an isomorphism (not an *invertible* morphism). A cofibration (fibration) that is also a weak equivalence is called a *trivial cofibration (fibration)* (instead of the word ‘trivial’, the word

‘acyclic’ is also in use). Bicompleteness is part of the definition of a model category, in particular there is always an initial object  $\emptyset$  and a terminal object  $\star$ . An object  $X$  is called *cofibrant* if the unique map  $\emptyset \rightarrow X$  is a cofibration; it is called *fibrant* if the unique map  $X \rightarrow \star$  is a fibration. In any model category, the classes of weak equivalences and fibrations determine the class of cofibrations; dually, the class of fibrations can be recovered from knowledge of the classes of weak equivalences and cofibrations.

One can show that simplicial sets form a model category in which the cofibrations are monomorphisms (concretely, simplicial maps that are injective in each degree), fibrations are Kan fibrations and weak equivalences are exactly the ones from Definition 2.10.4. As a consequence, all objects in sSet with this model structure are cofibrant, but only the Kan complexes are fibrant.

The category of topological spaces with its usual weak equivalences and Serre fibrations as fibrations is a model category. The cofibrations (that are determined by this choice of weak equivalences and fibrations) are the cofibrations encountered in AlgTop2, and defined in terms of the homotopy extension property.

A good notion for the comparison of model categories  $\mathcal{C}$  and  $\mathcal{D}$  is a *Quillen adjunction*, i.e. an adjoint pair  $L \dashv R$

$$\mathcal{C} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} \mathcal{D}$$

such that  $L$  preserves cofibrations and trivial cofibrations (equivalently, we can ask that  $R$  preserves fibrations and trivial fibrations). A Quillen adjunction  $L \dashv R$  is called *Quillen equivalence* if for any cofibrant object  $X$  in  $\mathcal{C}$  and any fibrant object  $Y$  in  $\mathcal{D}$  a morphism  $X \rightarrow RY$  is a weak equivalence if and only if the adjoint morphism  $LX \rightarrow Y$  is a weak equivalence (an equivalent definition asks the so-called *derived unit* and *derived counit* to be weak equivalences).

From the results above, one concludes that the adjunction

$$\text{sSet} \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow{\text{Sing}_\bullet} \end{array} \text{Top}$$

is a Quillen equivalence for the model structures on sSet and Top that we have just described. This adjunction induces an equivalence of homotopy categories

$$\text{Ho}(\text{sSet}) \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow{\text{Sing}_\bullet} \end{array} \text{Ho}(\text{Top})$$

between the homotopy categories of sSet and Top that are obtained by localization at weak equivalences. As a word of warning: It is not true that any Quillen adjunction will descend to the homotopy categories. This is true for  $| - | \dashv \text{Sing}_\bullet$  because both

functors preserve weak equivalences. If this is not the case, we need to *derive*, using the techniques developed in Section 4.2.

secclassfib

### 2.11. The classification of fibrations

In this section, we discuss the simplicial approach to the classification of fibrations or fiber bundles. In other lectures, you might have encountered the problem of classifying all complex (or real) vector bundles of rank  $n$  over a given space  $X$ . If  $X$  satisfies some basic requirements, more precisely paracompactness in order to allow partition of unity arguments, then homotopy theory offers the following solution to this problem.

For instance, if we want to classify complex vector bundles of rank  $n$ , we can consider the group of linear automorphisms of the fiber, here  $\mathbb{C}^n$ . This gives us the Lie group  $\mathrm{GL}(n, \mathbb{C})$  which is homotopy equivalent to the Lie group  $\mathrm{U}(n)$  of  $\mathbb{C}$ -linear *isometries* of  $\mathbb{C}^n$ , with respect to the standard hermitian metric; the inclusion  $\mathrm{U}(n) \xrightarrow{\simeq} \mathrm{GL}(n, \mathbb{C})$  is a homotopy equivalence, and in fact classifying  $\mathbb{C}^n$ -vector bundles over  $X$  is as difficult as classifying *hermitian* vector bundles, i.e. vector bundles all of whose fibres are endowed with a hermitian metric (positive definite, symmetric and sesquilinear form): this follows from the observation that the *space* of hermitian structures that one can put on a vector bundle has a natural “convex” structure, and is hence contractible.

One may then observe that instead of classifying hermitian complex vector bundles of rank  $n$  over  $X$ , we may equivalently classify principal  $\mathrm{U}(n)$ -bundles over  $X$ . From the topological group  $\mathrm{U}(n)$ , as for any other topological group, we can construct a certain space  $\mathrm{BU}(n)$ , the *classifying space* of  $\mathrm{U}(n)$ . One can then prove that the isomorphism classes of principal  $\mathrm{U}(n)$ -bundles over  $X$  (and hence the isomorphism classes of complex vector bundles of rank  $n$  over  $X$ ) are in bijection with the set  $[X, \mathrm{BU}(n)]$  of homotopy classes of maps from  $X$  to  $\mathrm{BU}(n)$  (a very detailed treatment is given in Section 14 of [Die08]).

If for example we are interested in complex line bundles over  $X$ , we need to compute  $[X, \mathrm{BU}(1)]$ ; and we have seen in AlgTop2 that  $\mathrm{BU}(1)$  is actually a  $K(\mathbb{Z}, 2)$ -space. Therefore, line bundles over  $X$  up to isomorphism are in bijection with  $H^2(X; \mathbb{Z})$ .

We will formulate a simplicial version of the solution to the problem of classifying fibrations. It is a little different in flavor (there will be no technical issues coming from point-set topology). Moreover, a lot of the concepts developed along the way will be of independent interest. For example, the notion of a minimal fibration introduced below is key to the proof given for Theorem 2.9.16 in [GJ99].

Our goal is to classify, up to homotopy equivalence, the Kan fibrations over any simplicial set  $X$  whose fiber is homotopy equivalent to a given Kan complex  $F$ . To this end, we need a more rigid version of a Kan fibration, a so-called *minimal fibration*. It will have the property that if two  $n$ -simplices in the total space have equal boundary and equal projections onto the base space, and are homotopic relative boundary by a homotopy that projects to the constant homotopy, they are already equal. The precise definition is as follows:



DEFINITION 2.11.1. A Kan fibration  $p : X \rightarrow Y$  is called a *minimal fibration* if for every commutative diagram

$$\begin{array}{ccccc} \partial\Delta^n \times \Delta^1 & \longrightarrow & \Delta^n \times \Delta^1 & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow h & & \downarrow \\ \partial\Delta^n & \longrightarrow & X & \xrightarrow{p} & Y \end{array}$$

the two compositions  $h \circ d^0, h \circ d^1 : \Delta^n \rightarrow \Delta^n \times \Delta^1 \rightarrow X$  are equal.

minimalpullback

REMARK 2.11.2. The property of being a minimal fibration is stable under pullbacks.

approxminimal

PROPOSITION 2.11.3. *For any Kan fibration  $p : X \rightarrow Y$ , there is a strong fiber-wise deformation retract  $q : Z \rightarrow Y$  which is a minimal fibration, i.e. there is a simplicial subset  $Z \subset X$  such that  $q := p|_Z : Z \rightarrow Y$  is a minimal fibration and a fiber-wise homotopy  $H : X \times \Delta^1 \rightarrow X$ , i.e. the following properties are satisfied:*

- $p \circ H$  is equal to the composition of the projection  $X \times \Delta^1 \rightarrow X$  with  $p$ ;
- $H(-, 0) = \text{id}_X$ ;
- $H$  restricts to the projection  $Z \times \Delta^1 \rightarrow Z$  on  $Z \times \Delta^1$ ;
- $H(-, 1)$  has image in  $Z \subset X$ .

PROOF. This is Proposition 10.3 in [GJ99]. The idea is to construct  $Z$  skeleton by skeleton. As the 0-skeleton of  $Z$ , one takes a system of representatives for the fiber-wise homotopy classes of vertices of  $X$ . Then one continues in a similar way to the higher skeleta.  $\square$

REMARK 2.11.4. A Kan complex  $X$  is called minimal if  $X \rightarrow *$  is a minimal fibration. The above result tells us that we can replace any Kan complex by a homotopy equivalent minimal Kan (sub)complex. Moreover, using stability of being minimal under pullback, we obtain that if  $p : X \rightarrow Y$  is a minimal fibration, then the fibres of  $p$  over vertices of  $Y$  are minimal Kan complexes.

minimalfibiso

LEMMA 2.11.5. *Let  $p : X \rightarrow Y$  and  $q : X' \rightarrow Y$  be minimal fibrations and let  $f : X \rightarrow X'$  be a map of fibrations over  $Y$ , i.e.  $q \circ f = p$ . If  $f$  is fiber-wise a homotopy equivalence, then  $f$  is an isomorphism.*

PROOF. This is Lemma 10.4 in [GJ99]. The proof can be reduced to the case  $Y = *$ , i.e. to the case of a single fiber. Then the statement follows in a relatively straightforward way from the minimality.  $\square$

Proposition 2.11.3 and Lemma 2.11.5 tell us that, instead of investigating the homotopy classes of Kan fibrations over  $X$  with fiber homotopy equivalent to  $F$ , we can equivalently consider isomorphism classes of minimal fibrations over  $X$  whose fiber is isomorphic to  $F^{\min}$  (a minimal replacement of  $F$ ).

pyinvpullback

LEMMA 2.11.6. *Let  $p : X \rightarrow Y$  be a Kan fibration and let  $f_0, f_1 : A \rightarrow Y$  be maps such that there exists a homotopy  $A \times \Delta^1 \rightarrow Y$  from  $f_0$  to  $f_1$ . Consider the pullback*

$$\begin{array}{ccc}
 P_i & \longrightarrow & X \\
 \downarrow & & \downarrow p \\
 A & \xrightarrow{f_i} & Y
 \end{array}$$

for  $i = 0, 1$ . Then there is a fiber-wise homotopy equivalence  $P_0 \rightarrow P_1$ , i.e. a homotopy equivalence  $P_0 \rightarrow P_1$  over  $A$ . If  $p$  is minimal, there is even an isomorphism  $P_0 \rightarrow P_1$  over  $A$ .

PROOF. The first part is Lemma 10.6 in [GJ99]. The addendum for minimal fibrations follows from Lemma 2.11.5.  $\square$

Next we will establish the local triviality of minimal fibrations.

DEFINITION 2.11.7. A map  $p : E \rightarrow B$  of simplicial sets is called a *fiber bundle* with fiber  $F$  if it is surjective and if for any  $b : \Delta^n \rightarrow B$  the pullback  $E \times_B \Delta^n \rightarrow \Delta^n$  of  $p$  along  $b$  is isomorphic over  $\Delta^n$  to the projection  $F \times \Delta^n \rightarrow \Delta^n$ . If  $F$  is additionally a Kan complex, we call the fiber bundle a *Kan fiber bundle*.

PROPOSITION 2.11.8. Any Kan fiber bundle is a Kan fibration, and any minimal fibration with connected base is a Kan fiber bundle.

PROOF. This is Lemma 11.9 and 11.10 in [May67].  $\square$

In the above recollection of the classification of vector bundles (in the topological setting), it was a key step to pass from bundles with a certain fiber to principal bundles for the automorphism group of the fiber. We will pursue essentially the same strategy in the simplicial setting.

DEFINITION 2.11.9. Let  $G$  be a simplicial group, i.e. a functor  $\Delta^{\text{op}} \rightarrow \text{Grp}$ : in particular, we have a sequence of groups  $G_n = G([n])$  for  $n \geq 0$ . A *principal  $G$ -bundle over  $B$*  is a map of simplicial sets  $p : E \rightarrow B$  equipped with a *free* right action of  $G$  on  $E$  (i.e. compatible, free right actions  $E_n \times G_n \rightarrow E_n$  of each group  $G_n$  on the corresponding set  $E_n$ ) such that  $p : E \rightarrow B$  is equivariant with respect to the trivial  $G$ -action on  $B$ , and such that the induced map  $E/G \rightarrow B$  is an isomorphism, where  $E/G$  is the simplicial set with  $E_n/G_n$  as set of  $n$ -simplices, and with induced face and degeneracy maps.

principalpullback

REMARK 2.11.10. A pullback of a principal fibration is again a principal fibration.

REMARK 2.11.11. A morphism of principal  $G$ -bundles over a fixed base is a simplicial map over that base that respects the  $G$ -action. One can prove that such a morphism is always an isomorphism, i.e. principal  $G$ -bundles over a fixed base form a groupoid.

PROPOSITION 2.11.12. Any principal  $G$ -bundle  $E \rightarrow B$  is a Kan fiber bundle with fiber  $G$ . In particular,  $E \rightarrow B$  is a Kan fibration.

PROOF. The map  $E \rightarrow B$  is surjective, and its fiber, namely  $G$ , is a Kan complex because every simplicial group is a Kan complex (a lemma you might want to prove).

In order to prove that for  $b : \Delta^n \rightarrow B$  the pullback  $E \times_B \Delta^n \rightarrow \Delta^n$  is trivial, we can assume  $B = \Delta^n$  and  $b = \text{id}_{\Delta^n}$  by Remark 2.11.10. By surjectivity we can find a lift  $\tilde{b} : \Delta^n \rightarrow E$  of  $b$ . Now one can verify that  $G \times \Delta^n \xrightarrow{G \times \tilde{b}} G \times E \xrightarrow{\text{action}} E$  is an isomorphism.  $\square$

DEFINITION 2.11.13. For a Kan complex  $F$ , we define the *automorphism group*  $\text{Aut}(F)$  of  $F$  as the subsimplicial set of  $\text{map}(F, F)$  of those simplices  $\sigma : F \times \Delta^n \rightarrow F$  such that

$$\sigma' : F \times \Delta^n \rightarrow F \times \Delta^n, \quad (f, x) \mapsto (\sigma(f, x), x)$$

is an isomorphism (over  $\Delta^n$ ).  $\text{Aut}(F)$  is a simplicial group whose multiplication is in particular as follows: given  $\sigma, \tau \in \text{Aut}(F)_n$ , i.e. maps  $F \times \Delta^n \rightarrow F$ , we set  $\sigma \circ \tau : (f, x) \mapsto \sigma(\tau(f, x), x)$ .

The simplicial set  $\text{Aut}(F)$  is a simplicial group (and, as any simplicial group, a Kan complex). We have a left  $\text{Aut}(F)$ -action on  $F$  by

$$\text{Aut}_n(F) \times F_n \rightarrow F_n, \quad (\sigma, (f : \Delta^n \rightarrow F)) \mapsto (\Delta^n \xrightarrow{f \times \text{id}} F \times \Delta^n \xrightarrow{\sigma} F).$$

Given simplicial sets  $X$  and  $Y$  with right and left  $G$ -action, respectively, for any simplicial group  $G$ , we have a left action on  $X \times Y$  given by  $g.(x, y) = (x.g^{-1}, g.y)$  for  $g \in G_n$ ,  $x \in X_n$  and  $y \in Y_n$ . We denote the orbits by  $X \times_G Y := (X \times Y)/G$ . This will be relevant for us in the following situation: Using the left  $\text{Aut}(F)$ -action on  $F$ , we can form  $E \times_{\text{Aut}(F)} F$  for any  $\text{Aut}(F)$ -principal bundle  $E \rightarrow B$ ; then the map  $E \times_{\text{Aut}(F)} F \rightarrow E \times_{\text{Aut}(F)} * \cong B$  is an  $F$ -fibre bundle. In fact we have the following:

THEOREM 2.11.14. *For any simplicial set  $B$  and any Kan complex  $F$ , the assignment sending a principal  $\text{Aut}(F)$ -bundle  $E \rightarrow B$  to the fiber bundle  $E \times_{\text{Aut}(F)} F \rightarrow B$  establishes an equivalence of groupoids*

$$\left\{ \begin{array}{c} \text{groupoid of} \\ \text{principal } \text{Aut}(F)\text{-bundles} \\ \text{over } B \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{groupoid of} \\ \text{fiber bundles over } B \text{ with fiber } F \\ \text{and their isomorphisms} \end{array} \right\}.$$

PROOF. This is (in a slightly different language) Proposition 20.7 in [May67]. The proof is technical, but again the idea is to reduce to  $B = \Delta^n$ . Then all principal bundles and fiber bundles are trivial, hence the above equivalence reduces to a statement about the automorphism groups of the trivial  $\text{Aut}(F)$ -principal bundle and the trivial bundle with fiber  $F$  over  $\Delta^n$ : Indeed, the group of automorphisms for the trivial principal  $\text{Aut}(F)$ -bundle over  $\Delta^n$  is  $\text{Aut}_n(F)$ ; in fact, more generally the automorphism group of the trivial principal  $G$ -bundle over  $\Delta^n$  can be identified with the group of maps  $\Delta^n \rightarrow G$  and hence with  $G_n$ . On the other hand, the group of automorphisms of the trivial bundle  $F \times \Delta^n$  is really  $\text{Aut}_n(F)$  by definition. This implies the assertion in this special case.  $\square$

For a given simplicial group  $G$  we will need in the sequel a contractible Kan complex  $EG$  with free right  $G$ -action. This makes the quotient map  $EG \rightarrow BG := EG/G$  into a principal  $G$ -bundle, and the hypotheses on  $EG$  imply that  $BG$  is also a Kan complex. We call  $EG \rightarrow BG$  the *universal principal  $G$ -bundle*. One can build

a concrete model for  $EG$  with  $n$ -simplices given by the set  $EG_n = G_n \times \cdots \times G_n$ , but we will not rely too much on this model. The contractibility of this model of  $EG$  can be seen by exhibiting extra degeneracies in the sense of [GJ99], page 200 (there will be an exercise on that).

The properties that  $EG$  has by definition characterize it uniquely up to equivariant homotopy. This follows from the following statement:

uniquemaptoEG

LEMMA 2.11.15. *For any simplicial group  $G$  and any simplicial set  $E$  with free  $G$ -action, there is a map  $E \rightarrow EG$  unique up to equivariant homotopy.*

We will not give a proof here, but just mention that if one establishes a simplicial model structure on simplicial sets with (right)  $G$ -action, this is a direct consequence of (the analogue of) the (SM7) axiom. More precisely, one can see that the trivial Kan fibration  $EG \rightarrow *$  induces a trivial Kan fibration  $\text{map}_G(E, EG) \rightarrow \text{map}_G(E, *) = *$  between the mapping spaces of  $G$ -equivariant maps (this needs that  $E$  has a free  $G$ -action, i.e. it is cofibrant as a  $G$ -simplicial set). This tells us that the Kan complex  $\text{map}_G(E, EG)$  is contractible and gives us Lemma 2.11.15.

THEOREM 2.11.16. *For any simplicial set  $X$  and any simplicial group  $G$ , the assignment sending a homotopy class of maps  $f : X \rightarrow BG$  to the pullback  $f^*EG$  of the universal  $G$ -bundle  $EG \rightarrow BG$  along  $f$  establishes a bijection of sets*

$$[X, BG] \xrightarrow{\cong} \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles over } X \end{array} \right\}.$$

PROOF. The well-definedness of the map of set is Lemma 3.4 in [GJ99] (the needed statement is similar to Lemma 2.11.6). We sketch the construction of the inverse map of sets: Let a principal  $G$ -bundle  $P \rightarrow X$  be given. Since the  $G$ -action on  $P$  is free, there is by Lemma 2.11.15 a  $G$ -equivariant map  $P \rightarrow EG$  that is unique up to  $G$ -equivariant homotopy. Since this map is  $G$ -equivariant, it descends to a map  $X = P/G \rightarrow BG = EG/G$ : this gives us the inverse assignment from isomorphism classes of principal  $G$ -bundles over  $X$  to  $[X, BG]$ .  $\square$

In summary, we have achieved the following result:

THEOREM 2.11.17. *For any simplicial set  $X$  and any Kan complex  $F$  with minimal replacement  $F^{\min}$ , there is a bijection*

$$[X, B \text{Aut}(F^{\min})] \xrightarrow{\cong} \left\{ \begin{array}{l} \text{homotopy equivalence classes of} \\ \text{Kan fibrations over } X \text{ with fiber} \\ \text{homotopy equivalent to } F \end{array} \right\}.$$

## 2.12. More exercises

**The definition of simplicial sets, the nerve functor, and (non-)degenerate simplices**

se:RPinfinity

EXERCISE 2.12.1. In continuation of Example 2.4.14, verify that  $|N\mathbb{Z}/2| \simeq \mathbf{RP}^\infty$ .

EXERCISE 2.12.2. Show that  $|N\mathbb{Z}| \simeq S^1$ .

**Simplicial homotopy**

EXERCISE 2.12.3. Show that for a group  $G$  and an element  $g \in G$ , the simplicial map  $NG \rightarrow NG$  given by conjugation by  $g$ , given explicitly on  $n$ -simplices by

$$(* \xrightarrow{g_1} * \xrightarrow{g_2} \dots \xrightarrow{g_n} *) \mapsto (* \xrightarrow{gg_1g^{-1}} * \xrightarrow{gg_2g^{-1}} \dots \xrightarrow{gg_ng^{-1}} *)$$

is homotopic to the identity.

EXERCISE 2.12.4. For  $G, H$  groups calculate the Kan complex  $\text{map}(NG, NH)$ . What is the set of components? What is the homotopy type of each component?



## CHAPTER 3

### The homotopy theory of chain complexes and the Dold–Kan correspondence

The Dold–Kan correspondence relates the category of simplicial abelian groups to the category of non-negatively graded chain complexes  $\text{Ch}_{\geq 0}(\mathbb{Z})$  of abelian groups. More precisely, it shows that they are equivalent and have equivalent homotopy theories. The statement holds for any abelian category (instead of abelian groups) and suggests a way of ‘doing homological algebra’ in a non-abelian category by studying simplicial objects in that category.

A good reference for this section is Weibel’s ‘An introduction to homological algebra’ [Wei94], Sections 8.3 and 8.4, see also [GJ99], III.2. We also recommend the online resource ‘Kerodon’.

#### 3.1. (Normalized) chains on simplicial abelian groups

To a simplicial abelian group, we can associate several chain complexes.

chainsimpabgr

DEFINITION 3.1.1. For a simplicial abelian group  $A : \Delta^{\text{op}} \rightarrow \text{Ab}$ , we define the complex  $C_*(A)$  of  $A$  by  $C_n(A) := A_n$  for  $n \geq 0$ . The differential  $\partial_n : C_n(A) = A_n \rightarrow C_{n-1}(A) = A_{n-1}$  is given by  $\partial_n = \sum_i (-1)^i d_i$ .

The simplicial identities ensure that this differential actually squares to zero. If  $X$  is a topological space, then we have performed this computation for the simplicial set  $\mathbb{Z}\text{Sing}_\bullet X$  (the simplicial abelian group obtained by taking degree-wise the free abelian group on  $\text{Sing}_\bullet X$ ) in previous lectures. With our new terminology, we see now: The singular homology of  $X$  was (secretly) defined as the homology of the complex associated to the simplicial abelian group  $\mathbb{Z}\text{Sing}_\bullet X$ ;

eqnsingspace

$$(3.1.1) \quad H_*(X; \mathbb{Z}) = H_*(\mathbb{Z}\text{Sing}_\bullet X) .$$

It is natural to extend this to all simplicial sets:

DEFINITION 3.1.2. For any simplicial set  $X$ , we define the *homology of  $X$  (with integer coefficients)* by

$$H_*(X; \mathbb{Z}) := H_*(\mathbb{Z}X) ,$$

where  $\mathbb{Z}(-) : \text{sSet} \rightarrow \text{sAb}$  forms the simplicial abelian group freely generated by a simplicial set (it forms the free abelian group on the given set of simplices degree-wise; in fact,  $\mathbb{Z}(-)$  is the left adjoint to the forgetful functor from simplicial abelian groups to simplicial sets).

lem:C=N+D

LEMMA 3.1.3. *Let  $A$  be a simplicial abelian group.*

(1) *The subgroups*

$$D_n(A) := \sum_i s_i(A_{n-1}) \subseteq A_n$$

for  $n \geq 0$  form a sub-chain complex  $D_*(A) \subset C_*(A)$ .

(2) *The subgroups*

$$N_n(A) = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \rightarrow A_{n-1})$$

form a sub-chain complex  $N_*(A) \subset C_*(A)$ . We call  $N_*(A)$  the normalized chain complex of  $A$ .

(3) *The two previous inclusions of chain complexes combine into an isomorphism of chain complexes*

$$N_*(A) \oplus D_*(A) \cong C_*(A) ,$$

and in particular we have a natural isomorphism

$$N_*(A) \xrightarrow{\cong} C_*(A)/D_*(A) .$$

(4) *The complex  $D_*(A)$  has zero homology,  $H_*(D_*(A)) = 0$ . Hence, the inclusion  $N_*(A) \rightarrow C_*(A)$  is a quasi-isomorphism.*

partofDcx

PROOF. Most of these statements follow directly from the definition, and we refer to Section 8.3 in [Wei94] for details. The most non-trivial part is (4): Let us abbreviate  $D := D_*(A)$  and let us introduce, for  $p \geq 0$ , a sub-chain complex  $F_p D \subset D$  by setting  $(F_0 D)_n := 0$ ,  $(F_p D)_n = D_n$  for  $p \geq n$  and  $(F_p D)_n := \sum_{i=0}^p s_i(A_{n-1}) \subset D_n$  for  $n \geq p$ . Then  $F_* D$  gives us a bounded filtration<sup>1</sup> of the complex  $D$ . The convergent spectral sequence  $E_{pq}^1 = H_{p+q}(F_p D/F_{p-1} D) \Rightarrow H_{p+q}(D)$  tells us that it is sufficient to prove that  $F_p D/F_{p-1} D$  has vanishing homology. In fact,  $F_p D/F_{p-1} D$  in level  $n$  is a quotient of  $s_p A_{n-1}$  (and it is zero if  $p > n$ ). A direct computation shows that  $h_n := (-1)^{p+1} s_p$  induces a chain contraction of  $F_p D/F_{p-1} D$ , i.e. a chain homotopy from the identity of the chain complex  $F_p D/F_{p-1} D$  to the zero chain self-map. This implies the assertion.  $\square$

### 3.2. The Dold–Kan correspondence

We can now state the main result relating simplicial abelian groups and chain complexes of abelian groups.

THEOREM 3.2.1 (Dold–Kan-correspondence). *The normalized chain functor*

$$(3.2.1) \quad N : \text{sAb} \xrightarrow{\cong} \text{Ch}_{\geq 0}(\mathbb{Z})$$

is an equivalence of categories. Its inverse and right adjoint  $C \mapsto K(C)$  is given on objects by “adding degeneracies”, i.e. by setting  $K_n(C) = \bigoplus_{[n] \rightarrow [m]} C_m$ .

Under the Dold–Kan correspondence, simplicial homotopy corresponds to homology, i.e.  $\pi_*(A, 0) \cong H_*(A)$  by a natural isomorphism.

<sup>1</sup>In each chain degree we have finitely many non-trivial filtration quotients.

correspondence

eqnNfunctor



REMARK 3.2.2. Recall that there are  $\binom{n}{m}$  surjective maps  $[n] \twoheadrightarrow [m]$  in  $\mathbf{\Delta}$ . (And  $\binom{n+1}{m+1}$  maps  $[m] \hookrightarrow [n]$ , by the way.)

SKETCH OF THE PROOF. Consider the cosimplicial chain complex  $\mathbf{\Delta} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$  sending  $[n]$  to  $N_*(\mathbb{Z}\Delta^n)$ . Then the functor  $N$  from (3.2.1) can be seen as the left Kan extension of this cosimplicial object along the composite functor  $\mathbf{\Delta} \rightarrow \text{Fun}(\mathbf{\Delta}^{\text{op}}, \text{Set}) \xrightarrow{\mathbb{Z}(-)} \text{Fun}(\mathbf{\Delta}^{\text{op}}, \text{Ab}) = \text{sAb}$ . Now by abstract properties of this extension (this is a rather general statement about an abstract and enriched version of the nerve; we will cover this in the exercises), we have an adjunction  $N \dashv K$ , where  $K$  sends a chain complex  $C$  to the simplicial abelian group

$$[n] \mapsto \text{Hom}_{\text{Ch}_{\geq 0}(\mathbb{Z})}(N_*(\mathbb{Z}\Delta^n), C)$$

which, when spelled out, gives us the description of  $K$  from above (we should say here that the adjunction can also be obtained by hand). For the direct verification that  $KN \cong \text{id}$  and  $NK \cong \text{id}$ , we refer to Section 8.4 of [Wei94].  $\square$

REMARK 3.2.3. The Dold–Kan correspondence can be formulated for *any* abelian category instead of abelian groups; this includes in particular categories of modules over a ring  $R$ . Moreover, there is a dual version featuring *cosimplicial* objects in an abelian category and *cochain* complexes in that abelian category.

EXERCISE 3.2.4 (Exercise 8.4.2 in [Wei94]). Recall that a semisimplicial object in a category  $\mathcal{C}$  is a functor  $\mathbf{\Delta}_+^{\text{op}} \rightarrow \mathcal{C}$ , where  $\mathbf{\Delta}_+$  is the subcategory of the simplex category  $\mathbf{\Delta}$  containing all objects  $[n]$ , but only injective maps  $[n] \hookrightarrow [m]$ . Generating morphisms are the face inclusions  $d^i: [n-1] \rightarrow [n]$ , for  $0 \leq i \leq n$ ; conversely, a semisimplicial object in  $\mathcal{C}$  can be described as a collection  $X_n$  of objects in  $\mathcal{C}$ , together with face maps  $d_i: X_n \rightarrow X_{n-1}$ , for  $0 \leq i \leq n$ , satisfying the semisimplicial identities  $d_j d_i = d_i d_{j+1}$  for  $i \leq j$ .

- (1) Given a chain complex  $(C, d^C) \in \text{Ch}_{\geq 0}(\mathbb{Z})$ , show that the assignment  $[n] \mapsto (\kappa C)_n := C_n$ , together with the face maps  $d_i: (\kappa C)_n \rightarrow (\kappa C)_{n-1}$  given by  $d^C$  for  $i = n$  and by 0 for  $i < n$ , defines a semisimplicial abelian group  $\kappa C$ . Show that this gives a functor  $\kappa: \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{semisAb}$ .
- (2) There is a forgetful functor  $U: \text{sAb} \rightarrow \text{semisAb}$ , induced by the inclusion of categories  $\mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}$ . Show that  $U$  admits a left adjoint  $F: \text{semisAb} \rightarrow \text{sAb}$ , given on objects by sending  $X \in \text{semisAb}$  to the simplicial abelian group  $[n] \mapsto \bigoplus_{[n] \rightarrow [m]} X_m$ .
- (3) Prove that  $K: \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{sAb}$  from the Dold–Kan correspondence is naturally isomorphic to the composite functor  $F \circ \kappa$ .

defweab

DEFINITION 3.2.5. A map  $f: A \rightarrow B$  of simplicial abelian groups is called a *weak equivalence* if the map of the underlying simplicial sets is a weak equivalence.

One can directly observe by means of the Dold–Kan correspondence:

corquasiisodk

COROLLARY 3.2.6. A map  $f: A \rightarrow B$  of simplicial abelian groups is a weak equivalence if and only if  $N_*(f): N_*(A) \rightarrow N_*(B)$  is a quasi-isomorphism.

PROOF. This follows from the natural isomorphisms  $\pi_*(A, 0) \cong H_*(A)$  for any simplicial abelian group  $A$ , and the fact that there are isomorphisms  $\pi_*(A, 0) \cong \pi_*(A, a)$ , for any vertex  $a$  of  $A$  (any element  $a \in A_0$ ) induced by the “addition with  $a$ ” (the map of simplicial sets  $A \rightarrow A$  given on  $n$ -simplices by  $- + s_0^n(a) : A_n \rightarrow A_n$ ).  $\square$

odelstructure2

REMARK 3.2.7. There is a model structure on simplicial abelian groups in which the weak equivalences are the ones from Definition 3.2.5 and fibrations are the maps that are Kan fibrations on the underlying simplicial sets (cofibrations are then determined by the choice of weak equivalences and of fibrations), thereby making the adjunction between  $\mathbb{Z}(-) : \mathbf{sSet} \rightarrow \mathbf{sAb}$  and the forgetful functor  $\mathbf{sAb} \rightarrow \mathbf{sSet}$  into a Quillen adjunction.

There is also a model structure on  $\text{Ch}_{\geq 0}(\mathbb{Z})$  in which weak equivalences are quasi-isomorphisms and fibrations are degree-wise epimorphisms (cofibrations are again determined by that; it turns out that they are exactly degree-wise monomorphisms with degree-wise *projective* cokernel<sup>2</sup>). Then the adjunction  $N_* \dashv K$  becomes a Quillen equivalence.

### 3.3. Monoidality of the Dold–Kan correspondence and the Eilenberg–Zilber Theorem

For  $C, D \in \text{Ch}_{\geq 0}(\mathbb{Z})$ , we can define the monoidal product complex  $C \otimes D$  by  $(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$  (here  $\otimes$  is the monoidal product of abelian groups). The differential is determined by the product rule  $\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|} c \otimes \partial d$ , where  $|c|$  is the degree of  $c$ . This endows  $\text{Ch}_{\geq 0}(\mathbb{Z})$  with a symmetric monoidal structure (actually, this monoidal structure is also closed; the ‘mapping spaces’ needed for that are the so-called *hom complexes*). Similarly,  $\mathbf{sAb}$  is a symmetric monoidal category by considering the level-wise monoidal product. Note that this makes  $\mathbb{Z}(-) : \mathbf{sSet} \rightarrow \mathbf{sAb}$  into a symmetric monoidal functor, i.e. there are natural isomorphisms (both in  $X$  and in  $Y$  in  $\mathbf{sSet}$ )

$$\mathbb{Z}(X \times Y) \cong \mathbb{Z}X \otimes \mathbb{Z}Y.$$

The monoidal structures on  $\text{Ch}_{\geq 0}(\mathbb{Z})$  and  $\mathbf{sAb}$  are defined in a somewhat different way, so it is not obvious to what extent the Dold–Kan correspondence is compatible with these monoidal structures.

In order to investigate this, let  $A$  and  $B$  be simplicial abelian groups. For  $a \in A_p$  and  $b \in B_q$ , define

$$\nabla_{A,B}(a \otimes b) := \sum_{(p,q)\text{-shuffle } (\mu,\nu)} \text{sgn}(\mu,\nu) s_\nu(a) \otimes s_\mu(b),$$

where:

- the sum runs over all  $(p, q)$ -shuffles  $(\mu, \nu) = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$  (a  $(p, q)$ -shuffle is a permutation of the set  $\{1, \dots, p + q\}$  with  $\mu_1 < \dots < \mu_p$  and  $\nu_1 < \dots < \nu_q$ );
- $s_\mu := s_{\mu_p-1} \dots s_{\mu_1-1}$  and  $s_\nu = s_{\nu_q-1} \dots s_{\nu_1-1}$  are iterated degeneracy operators;
- and  $\text{sgn}(\mu, \nu)$  is the sign of the shuffle permutation.

---

<sup>2</sup>For abelian groups/ $\mathbb{Z}$ -modules, “projective” just means *free*.

The formula for  $\nabla$  is given here for unnormalized chains, but it induces a map  $\nabla_{A,B} : N_*(A) \otimes N_*(B) \rightarrow N_*(A \otimes B)$  on normalized chains that is called the *Eilenberg-Zilber map*.

PROPOSITION 3.3.1. *The Eilenberg-Zilber maps make the normalized chain functor  $N_*$  into a lax symmetric monoidal functor, i.e. the Eilenberg-Zilber maps*

$$(3.3.1) \quad \nabla_{A,B} : N_*(A) \otimes N_*(B) \rightarrow N_*(A \otimes B)$$

are compatible with the monoidal structures on  $\mathbf{sAb}$  and  $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$  (i.e. they are natural in  $A$  and  $B$ ) and are also compatible with the symmetric braiding; here the word ‘lax’ indicates the direction of the maps, from  $N_*(-) \otimes N_*(-)$  to  $N_*(- \otimes -)$ , and also reminds us that these maps are not supposed to be isomorphisms.

Moreover,  $N_*$  is also oplax monoidal (but not oplax symmetric monoidal!), i.e. one can define maps

$$\Delta_{A,B} : N_*(A \otimes B) \rightarrow N_*(A) \otimes N_*(B),$$

that are compatible with the monoidal structure (i.e. natural in  $A$  and  $B$ ), the so-called *Alexander-Whitney maps*.

The fact that  $N_*$  is lax monoidal implies that it translates an algebra  $A$  in simplicial abelian groups to a differential graded algebra  $N_*(A)$ . If  $A$  is commutative, so is  $N_*(A)$ : this is because  $N_*$  is also symmetric.

By a concept called *doctrinal adjunctions* this implies automatically that the right adjoint  $K : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \mathbf{sAb}$  inherits a lax monoidal structure and an oplax symmetric monoidal structure. Again,  $K$  will not be lax *symmetric* monoidal, but the symmetry will be satisfied *up to coherent homotopy*, see [Ric03].

The Eilenberg-Zilber and Alexander-Whitney maps satisfy the following:

PROPOSITION 3.3.2. *For simplicial abelian groups  $A$  and  $B$ , the composition*

$$N_*(A) \otimes N_*(B) \xrightarrow{\text{Eilenberg-Zilber}} N_*(A \otimes B) \xrightarrow{\text{Alexander-Whitney}} N_*(A) \otimes N_*(B)$$

is the identity, while the composition

$$N_*(A \otimes B) \xrightarrow{\text{Alexander-Whitney}} N_*(A) \otimes N_*(B) \xrightarrow{\text{Eilenberg-Zilber}} N_*(A \otimes B)$$

is chain homotopic to the identity.

PROPOSITION 3.3.3 (Addendum to the Dold-Kan correspondence). *Let  $A, B$  be simplicial abelian groups and let  $f, g : A \rightarrow B$  be simplicial group homomorphisms. Then there is a canonical correspondence between the sets of:*

- *homotopies of maps of simplicial abelian groups from  $f$  to  $g$ , i.e. maps  $h : A \otimes \mathbb{Z}\Delta^1 \rightarrow B$  restricting to  $f$  and  $g$ , respectively, on the sub-simplicial abelian groups  $A \otimes d^1\mathbb{Z}\Delta^0$  and  $A \otimes d^0\mathbb{Z}\Delta^0$ ;*
- *chain homotopies between the chain maps  $N_*(f), N_*(g) : N_*(A) \rightarrow N_*(B)$ .*

PROOF. Since  $N_*$  is fully faithful, a simplicial homotopy  $h : A \otimes \mathbb{Z}\Delta^1 \rightarrow B$  corresponds to a chain map  $N_*(h) : N_*(A \otimes \mathbb{Z}\Delta^1) \rightarrow N_*(B)$ . By precomposing with the Eilenberg-Zilber map (which is a chain homotopy equivalence) we obtain a chain map  $N_*(h) : N_*(A) \otimes N_*(\mathbb{Z}\Delta^1) \rightarrow N_*(B)$ . After writing out explicitly the

chain complex  $N_*(\mathbb{Z}\Delta^1)$ , we realize that  $N_*(h)$  amounts exactly to a choice of abelian groups  $H_n : N_n(A) \rightarrow N_{n+1}(B)$  with  $\partial H + H\partial = g - f$ , i.e. a chain homotopy from  $f$  to  $g$ .  $\square$

propZwe

PROPOSITION 3.3.4. *The functor  $\mathbb{Z}(-) : \text{sSet} \rightarrow \text{sAb}$  preserves weak equivalences.*

PROOF. For a direct proof, see [GJ99] around page 173. The main idea is as follows: We can use a contracting homotopy  $h : \Delta^n \times \Delta^1 \rightarrow \Delta^n$  of the  $n$ -simplex to one of its vertices in order to obtain a contracting homotopy  $\mathbb{Z}\Delta^n \otimes \mathbb{Z}\Delta^1 \rightarrow \mathbb{Z}\Delta^n$  and then via Proposition 3.3.3 a chain homotopy equivalence between  $N_*(\mathbb{Z}\Delta^n)$  and  $N_*(\mathbb{Z}\Delta^0) = \mathbb{Z}$ . Together with (3.1.1), this tells us that the unit map  $\Delta^n \rightarrow \text{Sing}_\bullet |\Delta^n|$  induces a weak equivalence  $\mathbb{Z}\Delta^n \rightarrow \mathbb{Z}\text{Sing}_\bullet |\Delta^n|$  (because for this we just need that it induces a quasi-isomorphism after taking  $N_*$  by Corollary 3.2.6 — which we have just seen). Using this, one shows inductively that  $\mathbb{Z}\text{sk}_n X \rightarrow \mathbb{Z}\text{Sing}_\bullet |\text{sk}_n X|$  is a weak equivalence and then, finally, that the same is true for  $\mathbb{Z}X \rightarrow \mathbb{Z}\text{Sing}_\bullet |X|$ .

Let us now suppose that  $f : X \rightarrow Y$  is a weak equivalence. Then the square

$$\begin{array}{ccc}
 \mathbb{Z}X & \longrightarrow & \mathbb{Z}\text{Sing}_\bullet |X| \\
 \downarrow f & & \downarrow \mathbb{Z}\text{Sing}_\bullet |f| \\
 \mathbb{Z}Y & \longrightarrow & \mathbb{Z}\text{Sing}_\bullet |Y|
 \end{array}
 \tag{3.3.2}$$

eqnunitZ

commutes, and the horizontal maps are weak equivalences. Since  $H_*(\mathbb{Z}\text{Sing}_\bullet -)$  is the functor taking singular homology of a space and since  $|f|$  is a weak equivalence (both is true by definition), it suffices to see in the *topological* setting that weak equivalences induce homology isomorphisms — and this is a statement that we have proven in AlgTop2.  $\square$

The fact that the horizontal maps in (3.3.2) induce an isomorphism in homology is of course interesting in its own right because it tells us the following:

COROLLARY 3.3.5. *The weak equivalence  $X \rightarrow \text{Sing}_\bullet |X|$  for  $X \in \text{sSet}$  induces an isomorphism  $H_*(X; \mathbb{Z}) \cong H_*(|X|; \mathbb{Z})$ .*

We will now formulate the so-called *generalized Eilenberg-Zilber Theorem* using bisimplicial abelian groups.

DEFINITION 3.3.6. A bisimplicial abelian group is a functor  $A : \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Ab}$ .

Via the Dold–Kan correspondence applied to the simplicial objects in the abelian category of simplicial abelian groups, the category  $\text{ssAb}$  of bisimplicial abelian groups is equivalent to the category  $\text{Ch}_{\geq 0}(\text{Ch}_{\geq 0}(\mathbb{Z}))$  of first quadrant double chain complexes of abelian groups. We denote by  $C(A)$  the unnormalized version of this first quadrant double complex associated to a bisimplicial abelian group; concretely,  $C_{p,q}(A) = A_{p,q}$ . The alternating sums of the face maps coming from the two copies of  $\Delta$  give us the horizontal and vertical differential, respectively.

Any bisimplicial abelian group  $A$  (more generally: any bisimplicial object in *any* category) can be turned into a simplicial one by precomposing with the diagonal functor  $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$ . We call this simplicial object the *diagonal*  $d(A)$  of  $A$ .

**THEOREM 3.3.7** (Generalized Eilenberg–Zilber Theorem of Dold–Puppe). *For any bisimplicial abelian group  $A$ , there is a natural chain homotopy equivalence*

$$C_*(d(A)) \simeq \text{Tot}_\oplus C_*(A)$$

where  $\text{Tot}_\oplus$  denotes the  $\oplus$ -totalization of the double complex  $C_*(A)$ .

**SKETCH OF THE PROOF.** We give the idea and refer to Theorem IV.2.5 in [GJ99] for the details. For  $p, q \geq 0$ , we have the bisimplicial set  $\Delta^{p,q} = \text{Hom}_\Delta(-1, [p]) \times \text{Hom}_\Delta(-2, [q])$ .

Every bisimplicial abelian group can be built from the bisimplicial abelian groups  $\mathbb{Z}\Delta^{p,q}$  via colimits, and both functors  $C_*(d(-)), \text{Tot}_\oplus C_*(-): \text{ssAb} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$  preserve colimits, hence the proof of the statement can be reduced to the case  $A = \mathbb{Z}\Delta^{p,q}$ .

Proposition 3.3.2 gives us for simplicial sets  $X$  and  $Y$  a homotopy equivalence  $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$  — this is the ‘ordinary’ Eilenberg–Zilber Theorem. For  $X = \Delta^p$  and  $Y = \Delta^q$ , this will give us, together with the identification  $d(\Delta^{p,q}) = \Delta^p \times \Delta^q$ , a chain homotopy equivalence

$$C_*(d\Delta^{p,q}) \simeq C_*(\Delta^p) \otimes C_*(\Delta^q).$$

Moreover, one can directly observe an isomorphism of chain complexes  $C_*(\Delta^p) \otimes C_*(\Delta^q) \cong \text{Tot}_\oplus C_*(\Delta^{p,q})$  (which is natural in the two simplicial variables, i.e. as a functor out of  $\mathbf{\Delta} \times \mathbf{\Delta}$ ). This implies the assertion in the special case  $A = \mathbb{Z}\Delta^{p,q}$ .  $\square$

### 3.4. Eilenberg MacLane spaces

For any abelian group  $G$ , we can consider the chain complex whose only non-trivial term is  $G$  in degree  $n$ , where  $n \geq 0$  is fixed; we denote this complex by  $G[n]$ . By applying the functor  $K : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{sAb}$  we get a simplicial abelian group whose underlying simplicial set we denote by  $K(G, n)$ , i.e.  $K(G, n) = UK(G[n])$  for the forgetful functor  $U$  from simplicial abelian groups to simplicial sets. Thanks to the Dold–Kan correspondence, we know that it satisfies  $\pi_n(K(G, n)) \cong G$  and  $\pi_k(K(G, n)) = 0$  for  $k \neq n$  (this is always with respect to the canonical basepoint 0 that any simplicial abelian group has). In other words, the Kan complexes  $K(G, n)$  are simplicial analogues of the Eilenberg–MacLane spaces that we have constructed as topological spaces in AlgTop2. Note that for  $n = 1$ , we can get a  $K(G, 1)$  also for a non-abelian group, but not via the Dold–Kan correspondence; instead, we can take the nerve of the category with one object and automorphism group  $G$ .

**DEFINITION 3.4.1.** For any simplicial set  $X$  and an abelian group  $G$ , we denote by  $C^*(X; G)$  the cochain complex which in cohomological degree  $n \geq 0$  is given by  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[X_n] \rightarrow G)$ , i.e. the abelian group of maps of abelian groups from the free abelian group over the set  $X_n$  to  $G$ . In other words  $C^*(X; G)$  is the hom complex  $\text{Hom}(C_*(X; \mathbb{Z}), G)$ . This allows us to define the *cohomology of  $X$  with coefficients in  $G$*  by  $H^*(X; G) := H^*(C^*(X; G))$ .

The following result is a simplicial version of the fact that Eilenberg–MacLane spaces represent cohomology:

**THEOREM 3.4.2.** *For any simplicial set  $X$ , any abelian group  $G$ , there are natural isomorphisms*

$$H^n(X; G) \cong [X, K(G, n)] , \quad n \geq 0.$$

**PROOF.** Via the (Quillen) adjunction between  $\mathbb{Z}(-)$  and the forgetful functor  $U$  from simplicial abelian groups to simplicial sets (Remark 3.2.7), the homotopy classes  $[X, K(G, n)]$  of maps  $X \rightarrow K(G, n)$  can be identified with homotopy classes of maps  $\mathbb{Z}X \rightarrow K(G[n])$  of simplicial abelian groups; this uses the definition  $K(G, n) := UK(G[n])$ . Using the Quillen equivalence  $N_* \dashv K$ , the homotopy classes of maps  $\mathbb{Z}X \rightarrow K(G[n])$  are in bijection to chain maps  $N_*(X) \rightarrow G[n]$  up to chain homotopy; this follows from Proposition 3.3.3. In summary,

$$[X, K(G, n)] \cong [\mathbb{Z}X, K(G[n])] \cong [N_*(X), G[n]]$$

(it is suppressed here that we take the homotopy classes of maps first in  $\mathbf{sSet}$ , then in  $\mathbf{sAb}$  and finally in  $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ ). As a last step, we realize that  $[N_*(X), G[n]] \cong [C_*(X), G[n]] \cong H^n(X; G)$ .  $\square$

The Dold–Kan correspondence can be used to decompose a simplicial abelian group into a product of Eilenberg–MacLane spaces:

**THEOREM 3.4.3** (Simplicial Dold–Thom theorem). *Suppose  $A$  is a simplicial abelian group. Then as a simplicial set  $A \simeq \prod_i K(\pi_i(A), i)$ , non-canonically.*

**PROOF.** We follow the proof given in [GJ99], page 175: Let  $C$  be a chain complex of abelian groups, and denote by  $Z_n$  and  $B_n$  the subgroups of  $C_n$  given by cycles and boundaries, respectively. We can pick an epimorphism  $r_n : F_n \rightarrow Z_n$ , where  $F_n$  is a free abelian group, and define

$$K_n := \ker \left( F_n \xrightarrow{r_n} Z_n \rightarrow H_n(C) \right) .$$

By definition the restriction of  $r_n$  to  $K_n$  factors through a map  $r'_n : K_n \rightarrow B_n$ . Note that we have an epimorphism  $C_{n+1} \rightarrow B_n$ . Since  $K_n$ , as subgroup of a free abelian group, is also free, we can lift  $r'_n$  to a map  $K_n \rightarrow C_{n+1}$ . We denote by  $F_n C$  the chain complex

$$\cdots \rightarrow 0 \rightarrow K_n \rightarrow F_n \rightarrow 0 \rightarrow \cdots$$

with  $K_n$  in degree  $n + 1$  and  $F_n$  in degree  $n$ . The epimorphism  $F_n C \rightarrow H_n C$  yields a quasi-isomorphism  $q_n : F_n C \rightarrow H_n C[n]$ , where  $H_n C[n]$  is the notation for the abelian group  $H_n C$  seen as chain complex supported in degree  $n$ . The maps  $K_n \rightarrow C_{n+1}$  and  $F_n \rightarrow Z_n \subseteq C_n$  previously constructed give us a map  $p_n : F_n C \rightarrow C$  that induces an isomorphism  $H_n(F_n C) \cong H_n C$ . As a result, we have quasi-isomorphisms

$$\bigoplus_{n \geq 0} H_n C[n] \xleftarrow{\simeq} \bigoplus_{n \geq 0} F_n C \xrightarrow{\simeq} C .$$

Moreover, one can observe that the canonical map  $\bigoplus_{n \geq 0} H_n C[n] \rightarrow \prod_{n \geq 0} H_n C[n]$  from the coproduct to the product of chain complexes is an isomorphism.

Now we concentrate on the case  $C = N_*(A)$ , i.e. to

$$\prod_{n \geq 0} \pi_n(A)[n] \xleftarrow{\simeq} \bigoplus_{n \geq 0} F_n N_*(A) \xrightarrow{\simeq} N_*(A)$$

and apply the weak inverse functor  $K : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{sAb}$  to  $N_*$ . This translates the quasi-isomorphisms above to weak equivalences and yields

$$\prod_{n \geq 0} K(\pi_n(A), n) \xleftarrow{\cong} \bigoplus_{n \geq 0} K(F_n N_*(A)) \xrightarrow{\cong} K N_*(A) \cong A .$$

□

### 3.5. The Hurewicz map

The Hurewicz map that we have discussed in AlgTop1.5 and AlgTop2 relates the homology of a space to its homotopy groups. We will now discuss a simplicial version of this map.

If  $X$  is a simplicial set and  $A \subset X$  a simplicial subset, we can define the *homology of  $X$  relative to  $A$*  by  $H_*(X, A; \mathbb{Z}) := H_*(\mathbb{Z}X/\mathbb{Z}A)$ . For a simplicial set  $X$  with base point  $*$ , we will use the shorthand  $\tilde{\mathbb{Z}}X := \mathbb{Z}X/\mathbb{Z}*$  and refer to  $\tilde{H}_*(X; \mathbb{Z}) := H_*(X, *; \mathbb{Z})$  as the *reduced homology*. By the Dold–Kan correspondence, we have

$$(3.5.1) \quad \pi_*(\mathbb{Z}X/\mathbb{Z}*, 0) \cong \tilde{H}_*(X; \mathbb{Z}) .$$

LEMMA 3.5.1. *For pointed simplicial sets  $X$  and  $Y$ , we have a natural isomorphism of simplicial abelian groups*

$$\tilde{\mathbb{Z}}(X \vee Y) \xrightarrow{\cong} \tilde{\mathbb{Z}}X \times \tilde{\mathbb{Z}}Y .$$

PROOF. The functor  $\tilde{\mathbb{Z}}$  from pointed simplicial sets to simplicial abelian groups is a left adjoint and hence takes coproducts to coproducts. But finite coproducts in simplicial abelian groups  $\text{sAb}$  are the same as finite products (by the additive structure), so the map is an isomorphism. (As a sanity check, we can also observe directly that the map induces an isomorphism on  $\pi_i$  by noting that this is just the isomorphism  $\tilde{H}_i(X \vee Y) \xrightarrow{\cong} \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$  that we have encountered in topology.) □

For any simplicial set  $X$ , we denote by  $h : X \rightarrow \mathbb{Z}X$  the unit of the adjunction  $\mathbb{Z}(-) \dashv U$  (we suppress the forgetful functor  $U : \text{sAb} \rightarrow \text{sSet}$  in the notation here). This map is also called the *Hurewicz map*. It sends an  $n$ -simplex  $\sigma \in X_n$  to the corresponding basis element  $\sigma$  in the free abelian group  $\mathbb{Z}X_n = (\mathbb{Z}X)_n$ .

DEFINITION 3.5.2. Let  $X$  be a connected Kan complex with base point  $*$ . Then for  $n \geq 1$  we define the *Hurewicz morphism* as the group homomorphism

$$h_n : \pi_n(X, *) \xrightarrow{\text{Hurewicz map from above}} \pi_n(\mathbb{Z}X, *) \rightarrow \pi_n(\tilde{\mathbb{Z}}X, 0) \stackrel{(3.5.1)}{\cong} \tilde{H}_*(X; \mathbb{Z}) .$$

One can now formulate and prove the following version of the Hurewicz Theorem(s); for the details we refer to Section III.3 in [GJ99].

PROPOSITION 3.5.3. *Let  $X$  be a connected pointed Kan complex. Then the Hurewicz morphism  $h_1 : \pi_1(X, *) \rightarrow \tilde{H}_1(X; \mathbb{Z})$  induces an isomorphism*

$$\pi_1(X, *) / [\pi_1(X, *), \pi_1(X, *)] \xrightarrow{\cong} \tilde{H}_1(X; \mathbb{Z}) ,$$

where  $\pi_1(X, *) / [\pi_1(X, *), \pi_1(X, *)]$  is the abelianization of the group  $\pi_1(X, *)$ .

THEOREM 3.5.4. *Let  $n \geq 1$  and let  $X$  be a pointed  $n$ -connected Kan complex, i.e.  $\pi_i(X, *) = 0$  for  $0 \leq i \leq n$ . Then the Hurewicz morphism  $h_i : \pi_i(X, *) \rightarrow \tilde{H}_i(X; \mathbb{Z})$  is an isomorphism for  $i = n + 1$  and an epimorphism for  $i = n + 2$ .*

### 3.6. More exercises



## Homotopy categories and derived functors

ch:hocat

In this chapter we will introduce the notion of a category equipped with a class of weak equivalences. Given this data, one wants to define a localization, i.e. a new category in which these weak equivalences are *formally inverted*. It will then be natural to replace limits and colimits, as we know then from ordinary category theory, with their *derived* versions, namely *homotopy limits* and *homotopy colimits* which are compatible with weak equivalences (and hence with the localization at the weak equivalences).

[ ch1 has undergone revisions since this was written. It would be good to restructure material here and connect more directly to ch1.]

localizations

## 4.1. Homotopy categories and localizations

DEFINITION 4.1.1. A *relative category* is a pair  $(\mathcal{C}, \mathcal{W})$  where  $\mathcal{C}$  is a category and  $\mathcal{W}$  is a collection of “weak equivalences” in  $\mathcal{C}$ ; more precisely,  $\mathcal{W}$  is a subcategory of  $\mathcal{C}$  containing all objects and all isomorphisms. A *homotopical category* is a relative category where  $\mathcal{W}$  satisfies the *2-of-3 property*: given two composable morphisms  $f, g$  in  $\mathcal{C}$ , if any two out of  $f, g, gf$  are in  $\mathcal{W}$ , then so is the third.<sup>1</sup> We will often leave the class of weak equivalences implicit in the notation and just talk about a homotopical category  $\mathcal{C}$ .

The next proposition says that we can always invert a set of morphisms in a small category:

gabriel-zisman

PROPOSITION 4.1.2 (Gabriel-Zisman). *Suppose  $(\mathcal{C}, \mathcal{W})$  is a relative category where  $\mathcal{C}$  is small. Then there exists a small category  $\mathcal{C}[\mathcal{W}^{-1}]$  and a functor  $L: \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$  with the universal property that  $L$  is initial among functors  $\mathcal{C} \rightarrow \mathcal{D}$  to a small category  $\mathcal{D}$  that take the morphisms in  $\mathcal{W}$  to isomorphisms in  $\mathcal{D}$ .*

SKETCH OF PROOF. Define a new (small) category  $\mathcal{C}[\mathcal{W}^{-1}]$  with objects the objects of  $\mathcal{C}$  and morphisms given by formal zig-zags of maps in  $\mathcal{C}$ , where maps in  $\mathcal{W}$  are allowed to go the wrong way, modulo the obvious identifications given by composition, removing identities, and cancelling  $a \xrightarrow{w} b \xleftarrow{w} a$  and  $b \xleftarrow{w} a \xrightarrow{w} b$  where  $w$  is in  $\mathcal{W}$ . The construction is spelled out in [GZ67].  $\square$

<sup>1</sup>Sometimes it is better to assume the stronger *2-of-6* property: if we have composable maps  $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$  where  $gf$  and  $hg$  are in  $\mathcal{W}$ , then  $f, g, h$  and  $hgf$  are also in  $\mathcal{W}$ . (Note that this is sort of saying that  $g$  has both a left and a right “inverse”.) This stronger condition is assumed in the definition of homotopical category in [Rie14] or [Dwy+04], but here we will just need 2-of-3.

If  $(\mathcal{C}, \mathcal{W})$  is a homotopical category, we also use the notation  $\mathrm{Ho}(\mathcal{C})$  for  $\mathcal{C}[\mathcal{W}^{-1}]$ , leaving  $\mathcal{W}$  implicit. We call this the *homotopy category* of  $\mathcal{C}$  with respect to  $\mathcal{W}$ .

REMARK 4.1.3. We would like to apply this construction to categories such as  $\mathrm{Top}$  and  $\mathrm{Top}^{\mathcal{I}}$  (where  $\mathcal{I}$  is an indexing category), and such categories are not small. However, there is a set-theoretical issue involved in doing this: as soon as  $\mathcal{C}$  contains a proper class of objects and morphisms (i.e. a class that is possibly not a set), also formal zig-zags and their equivalence classes might form only a proper class. We will generally ignore this issue, but to avoid consternation, let us point out that we can avoid this issue by assuming that inside our set theory there is a sub-class of *small* sets that themselves form a model of set theory; then we can use the large (i.e. not necessarily small) sets instead of worrying about classes. (The precise version of this idea is called a *Grothendieck universe*.) We then use the following terminology:

- a category is the datum of a (potentially large) *set* of objects and a (potentially large) *set* of morphisms, together with source, target, identity and composition maps satisfying the usual requirements;
- a category is *small* if its set of objects and all its Hom-sets are small;
- a category is *locally small* if its Hom-sets are all small but its set of objects is (potentially) large;
- a category is *large* if its set of objects and its Hom-sets are all (potentially) large.

We can then take  $\mathrm{Set}$  to be the (locally small) category of small sets,  $\mathrm{Top}$  to be the (locally small) category of topological spaces whose underlying sets are small, etc.

The construction of  $\mathcal{C}[\mathcal{W}^{-1}]$  now works just as well for locally small but not small homotopical categories  $(\mathcal{C}, \mathcal{W})$ , it just yields in general a large category  $\mathcal{C}[\mathcal{W}^{-1}]$ ; so we can for example define the homotopy category  $\mathrm{Ho}(\mathrm{Top})$  by starting with the category  $\mathrm{Top}$  with objects all topological spaces and then formally inverting the weak homotopy equivalences. Similarly, we can define the (bounded) derived category of  $R$ -modules by taking all bounded chain complexes over  $R$  and formally inverting the quasi-isomorphisms (homology isomorphisms). We can also construct homotopy categories of diagrams:

DEFINITION 4.1.4. Suppose  $\mathcal{I}$  is a small category and  $(\mathcal{C}, \mathcal{W})$  is a (possibly large) homotopical category. Then  $\mathcal{C}^{\mathcal{I}}$  is a homotopical category if we equip it with the *natural weak equivalences*, i.e. the weak equivalences  $\mathcal{W}_{\mathcal{I}}$  are those natural transformations  $\eta: F \rightarrow G$  such that  $\eta_i: F(i) \rightarrow G(i)$  is in  $\mathcal{W}$  for all  $i \in \mathcal{I}$ . Then we can define  $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$  as  $\mathcal{C}^{\mathcal{I}}[\mathcal{W}_{\mathcal{I}}^{-1}]$ .

REMARK 4.1.5. There is a less formal set-theoretical issue with these localizations, however: although the categories we start with ( $\mathrm{Top}$ ,  $\mathrm{Top}^{\mathcal{I}}$ , etc.) are all locally small, a priori the localized categories are just large categories — the Hom-sets are not necessarily small. In practice, to work with these categories it is useful to know that the localizations are also locally small. We will not deal with this issue here, however, as it will not actually affect us. That being said, local smallness of the homotopy category is part of the package you get from a *model structure*, and all the categories we will consider do have model structures, so in any case there is nothing to worry about.

REMARK 4.1.6. We stress right away that the category  $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$  is in general *not* the same as the category  $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ .  $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$  is a much richer category since diagrams are required to strictly commute, whereas  $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$  is generally too weak to be of much interest (see e.g. Example 4.1.19 below). Another problem with  $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$  is that the functor  $\delta : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$  forming constant diagrams in general does not have an adjoint, i.e. limits and colimits do not exist in the homotopy category (see e.g. Example 4.1.22 below). If we instead work with  $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ , then  $\delta$  often *does* have an adjoint, and this gives one definition of the homotopy colimit: the *homotopy colimit* will be the left adjoint of  $\delta : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ .

Inside a homotopical category  $\mathcal{C}$  we can often find full subcategories  $\mathcal{C}'$  of “good” (or “cofibrant”) objects such that  $\mathrm{Ho}\mathcal{C}'$  is equivalent to  $\mathrm{Ho}\mathcal{C}$ , but where  $\mathrm{Ho}\mathcal{C}'$  is simpler to describe, or some functor we are interested in is better-behaved on the objects in  $\mathcal{C}'$ . Making this precise gives the notion of *deformation*:

defn:def

DEFINITION 4.1.7. Let  $\mathcal{C}$  be a homotopical category. A *left deformation* of  $\mathcal{C}$  is a functor  $Q : \mathcal{C} \rightarrow \mathcal{C}$  and a natural transformation  $q : Q \rightarrow \mathrm{id}_{\mathcal{C}}$  such that  $q_c : Qc \rightarrow c$  is a weak equivalence for all  $c \in \mathcal{C}$ . We write  $\mathcal{C}_Q$  for the full subcategory of  $\mathcal{C}$  spanned by the essential image of  $Q$ .

A *right deformation* is defined dually.

REMARK 4.1.8. Note that by the 2-of-3 property  $Q$  takes weak equivalences to weak equivalences.

lem:dehhoec

LEMMA 4.1.9. Suppose  $(Q, q)$  is a left deformation of a homotopical category  $\mathcal{C}$ . If  $\mathcal{C}'$  is a full subcategory containing the image of  $Q$  (for instance  $\mathcal{C}_Q$ ), then the functor  $i : \mathrm{Ho}\mathcal{C}' \rightarrow \mathrm{Ho}\mathcal{C}$  induced by the inclusion is an equivalence of categories.

SKETCH PROOF. Since the natural map  $q_c : Qc \rightarrow c$  is a weak equivalence, and hence an isomorphism in  $\mathrm{Ho}\mathcal{C}$ , the functor  $i$  is essentially surjective. Suppose  $x$  and  $y$  are objects of  $\mathcal{C}'$ , then we need to prove that  $\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}'}(x, y) \rightarrow \mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(ix, iy)$  is a bijection. Using  $q$  we can replace any zig-zag of morphisms from  $x$  to  $y$  in  $\mathcal{C}$  by an equivalent zig-zag in  $\mathcal{C}_Q \subseteq \mathcal{C}'$ , so this map is surjective. Similarly if two zig-zags become equivalent in  $\mathcal{C}$  we can use  $q$  to show they are equivalent also in  $\mathcal{C}_Q$ , giving injectivity.  $\square$

For example, if  $\mathcal{C}$  is  $\mathrm{Top}$  we can take  $Q = |\mathrm{Sing}_{\bullet}(-)|$  and  $\mathcal{C}_Q$  to be the full subcategory of CW complexes. In fact we can do this also for diagrams:

diag-CWapprox

PROPOSITION 4.1.10. Let  $\mathcal{I}$  be a (small) indexing category. Then  $|\mathrm{Sing}_{\bullet}(-)|$  is a left deformation of  $\mathrm{Top}^{\mathcal{I}}$  into  $\mathrm{CW}^{\mathcal{I}}$ , and hence the inclusion  $\mathrm{Ho}(\mathrm{CW}^{\mathcal{I}}) \rightarrow \mathrm{Ho}(\mathrm{Top}^{\mathcal{I}})$  is an equivalence of categories.

PROOF. The counit  $q : |\mathrm{Sing}_{\bullet}(-)| \rightarrow \mathrm{id}(-)$  induces for each  $F \in \mathrm{Top}^{\mathcal{I}}$  and  $i \in \mathcal{I}$  a weak homotopy equivalence  $|\mathrm{Sing}_{\bullet}(F(i))| \rightarrow F(i)$ , so the result follows from Lemma 4.1.9.  $\square$

While Proposition 4.1.10 shows that we can without any real restriction assume that our functors take values in CW complexes, it does not show that such functors

have any special properties, as we elaborate in Remark 4.1.12 below. Only when  $\mathcal{I}$  is a point, does this deformation allow us to identify the homotopy category with the usual construction:

prop:HoTop=hCW

PROPOSITION 4.1.11.  $|\mathrm{Sing}_\bullet(-)|$  induces an equivalence  $\mathrm{Ho}(\mathrm{Top}) \rightarrow \mathrm{hCW}$ , the category with objects CW complexes and morphisms homotopy classes of maps.

PROOF. First note that  $\mathrm{Ho}(\mathrm{CW}) \xrightarrow{\cong} \mathrm{Ho}(\mathrm{Top})$ , as a special case of Lemma 4.1.10, with  $\mathcal{I}$  a point.

It remains to show that  $\mathrm{Ho}(\mathrm{CW})$  is equivalent to  $\mathrm{hCW}$ . By Whitehead's Theorem weak homotopy equivalences between CW complexes are homotopy equivalences, so the functor  $\mathrm{Ho}(\mathrm{CW}) \rightarrow \mathrm{hCW}$  is well-defined. Likewise the functor  $\mathrm{CW} \rightarrow \mathrm{Ho}(\mathrm{CW})$  descends to  $\mathrm{hCW} \rightarrow \mathrm{Ho}(\mathrm{CW})$ : Namely the two inclusion  $i_0 : X \rightarrow X \times I$  and  $i_1 : X \rightarrow X \times I$  are both weak equivalences and right inverses to the projection  $X \times I \rightarrow X$ , and hence they become equal isomorphisms in  $\mathrm{Ho}(\mathrm{CW})$ . If  $f_0, f_1 : X \rightarrow Y$  are homotopy homotopic, then there exists  $F : X \times I \rightarrow Y$  so that  $f_0 = F \circ i_0$  and  $f_1 = F \circ i_1$ . But then  $f_0$  and  $f_1$  are equal in  $\mathrm{Ho}(\mathrm{CW})$ .

We have hence seen that the identity functor on CW induces inverse equivalences of categories between  $\mathrm{Ho}(\mathrm{CW})$  and  $\mathrm{hCW}$ .  $\square$

diagramwarning

REMARK 4.1.12. Note that in  $\mathrm{CW}^{\mathcal{I}}$  we invert maps that are *objectwise* homotopy equivalences. We have not shown that such maps have *natural* homotopy inverse. In particular we have *not* shown in Proposition 4.1.10 an equivalence of  $\mathrm{Ho}(\mathrm{CW}^{\mathcal{I}})$  with  $h(\mathrm{CW}^{\mathcal{I}})$ , where the latter means the homotopy category obtained using the notion of homotopy obtained by crossing with the unit interval. In fact this is *false*, this requires much stronger assumptions on the diagram than just being objectwise given by CW complexes. We will see in Proposition 4.1.15 what this stronger condition is for pushout diagrams, and generalizing this for arbitrary diagrams is the subject of Chapter 5.

REMARK 4.1.13. Note that  $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$ , as defined here, is not a homotopical object in our definition, as  $\mathrm{Ho}(\mathcal{C})$  is just an ordinary 1-category, not a category with extra structure such as a collection of weak equivalences, since we have set the weak equivalences in  $\mathcal{W}$  to be *isomorphisms* in  $\mathcal{C}[\mathcal{W}^{-1}]$ .

This why one sometimes means something more refined by “localization” so that the localized category also has the higher order structure, i.e., the localization “is” the category  $(\mathcal{C}, \mathcal{W})$  up to a suitable notion of equivalence, or one interprets all this  $\infty$ -categorically, where weak equivalences become isomorphisms in the  $\infty$ -categorical sense.

We also mention the following example, well known from homological algebra, without proof.

PROPOSITION 4.1.14. Let  $R$  be a ring, and suppose  $\mathcal{C} = \mathrm{Ch}_R^{\geq 0}$  be the category of non-negatively graded  $R$ -chain complexes, with weak equivalences given by quasi-isomorphisms. Take  $\mathcal{C}'$  to be the full subcategory  $\mathrm{Proj}_R^{\geq 0}$  of chain complexes of projective  $R$ -modules. Then  $\mathrm{Ho} \mathrm{Ch}_R^{\geq 0} \simeq \mathrm{hProj}_R^{\geq 0}$  where the right-hand side means we take chain homotopy equivalences of maps.

Use the dictionary “topological spaces  $\leftrightarrow$  non-negatively graded  $R$ -chain complexes” and “CW complexes  $\leftrightarrow$  non-negatively graded  $R$ -chain complexes of projectives” to compare the previous example with Proposition 4.1.11.

We will make use of deformations for general diagram categories later. For now we will just describe how such replacements can work in some fundamental examples:

propn:pohocat

PROPOSITION 4.1.15. *Let  $\mathcal{I} = (0 \leftarrow 1 \rightarrow 2)$  be the pushout category. Then  $\text{Ho}(\text{Top}^{\mathcal{I}})$  is equivalent to the category with objects  $\mathcal{I}$ -diagrams of CW complexes and CW complex inclusions, and morphisms homotopy classes of maps between diagrams.*

We need a technical lemma that will be useful again later. Recall that we call a map a *cofibration* (or sometimes Hurewicz or h-cofibration) if it has the homotopy extension property (cf. [Hat02, Ch. 0]). An example is an inclusion of a CW subcomplex into a CW complex.

lem:cofibsqinv

LEMMA 4.1.16. *Suppose we are given a commutative square*

$$\begin{array}{ccc} X & \xleftarrow{i} & Y \\ \downarrow f & & \downarrow g \\ X' & \xleftarrow{i'} & Y' \end{array}$$

*of topological spaces, where  $i$  and  $i'$  are cofibrations and  $f$  and  $g$  are homotopy equivalences. Then every homotopy inverse of  $f$  can be extended to a homotopy inverse of  $g$  (meaning more precisely that the homotopies to the identity map on  $Y$  and  $Y'$  respectively can be chosen such as to extend any given homotopies on  $X$  and  $X'$ ).*

PROOF. See [May99, §6.5]. □

PROOF OF PROPOSITION 4.1.15. Any diagram is weakly equivalent to a diagram of this form: The map  $|\text{Sing}_\bullet(X)| \rightarrow X$  allows us to replace our diagram by a diagram of CW complexes. Then turn the two maps into CW inclusions using mapping cylinders. We conclude that for all diagrams  $X$  there exist a diagram  $QX$  of CW complexes and CW inclusions, and a natural weak equivalence  $QX \rightarrow X$ . By Lemma 4.1.9 this means inverting the weak equivalences in  $\text{Top}^{\mathcal{I}}$  is equivalent to inverting them in the subcategory of such diagrams. It remains to show that a weak equivalence  $\phi: F \rightarrow G$  between two of these is a homotopy equivalence. By Whitehead’s Theorem the maps  $F(i) \rightarrow G(i)$  are all homotopy equivalences, so we need to show we can choose a natural homotopy inverse. This follows from Lemma 4.1.16. It then follows by the same argument as for  $\text{Top}$  that inverting homotopy equivalences is equivalent to taking homotopy classes of maps. □

As we mentioned already, the categories  $\text{Ho}(\mathcal{C}^{\mathcal{I}})$  and  $\text{Ho}(\mathcal{C})^{\mathcal{I}}$  are quite different in general. We can now give an explicit example of this:

EXAMPLE 4.1.17. Let  $\delta: \text{Top} \rightarrow \text{Top}^{\mathcal{I}}$  be the constant-diagram functor. With  $\mathcal{I}$  as above and  $n \geq 1$ , we have bijections of sets

$$\begin{aligned} \text{Hom}_{\text{Ho}(\text{Top})^{\mathcal{I}}}([* \leftarrow S^{n-1} \rightarrow *], \delta S^n) &\cong *; \\ \text{Hom}_{\text{Ho}(\text{Top}^{\mathcal{I}})}([* \leftarrow S^{n-1} \rightarrow *], \delta S^n) &\cong \mathbb{Z}. \end{aligned}$$

As an exercise, try to verify this: The first claim is easy to verify. The second claim takes a bit more work (replacing  $[* \leftarrow S^{n-1} \rightarrow *]$  with the weakly equivalent diagram  $[D^n \leftarrow S^{n-1} \rightarrow D^n]$  is a good start), but will follow at once later when we get our model for homotopy colimit.

The next example is also good to keep in mind, but we will not give a full proof here:

freeG-spaces

PROPOSITION 4.1.18. *For  $G$  a (finite) group, let  $\mathcal{B}G$  denote the category with one object and  $G$  as morphisms. Then  $\text{Top}^{\mathcal{B}G}$  is the category of spaces with a  $G$ -action, and  $\text{Ho}(\text{Top}^{\mathcal{B}G})$  is the category where we invert  $G$ -maps that are ordinary weak equivalences. This is equivalent to the category whose objects are CW complexes with free  $G$ -action and morphisms  $G$ -homotopy equivalence classes of  $G$ -maps.*

PROOF. The idea is analogous to the previous proposition. As before, we can functorially replace any  $G$ -space  $X$  by a  $G$ -CW complex, namely  $|\text{Sing}_\bullet(X)|$  (check that it has a (non-free) cellular  $G$ -action!). Now consider the space  $EG$ , the classifying space of the category with objects the elements of  $G$  and exactly one morphism between any two objects. This space is contractible and naturally carries a free  $G$ -action induced by the translation action on the category. Hence the map

$$QX := EG \times |\text{Sing}_\bullet(X)| \rightarrow X$$

is a weak equivalence in  $\text{Top}^{\mathcal{B}G}$  and replaces  $X$  by a CW complex  $QX$  with a free  $G$ -action.

We can now apply Lemma 4.1.9 to the subcategory  $\text{CW}^{\mathcal{B}G, \text{free}} \subset \text{Top}^{\mathcal{B}G}$  of CW-complexes with free  $G$ -action, together with the deformation

$$Q = EG \times |\text{Sing}_\bullet(-)|: \text{Top}^{\mathcal{B}G} \rightarrow \text{CW}^{\mathcal{B}G, \text{free}} \subset \text{Top}^{\mathcal{B}G}.$$

We would now like to see that given a  $G$ -equivariant homotopy equivalence  $f: X \rightarrow Y$  between free  $G$ -CW complexes, there exists a  $G$ -equivariant homotopy inverse  $g: Y \rightarrow X$ , such that the two composites  $g \circ f$  and  $f \circ g$  are  $G$ -equivariantly homotopic to the identities of  $X$  and  $Y$ , respectively. This is easy with a bit of equivariant homotopy theory, but we will not give all details here: we just suggest that a  $G$ -equivariant map  $g: Y \rightarrow X$  can be constructed inductively over skeleta of  $Y$ , and that each time, trying to extend  $g$  over a  $G$ -equivariant *free* cell (which consists of  $|G|$  cells of the CW complex  $Y$ ), we extend  $g$  suitably over one cell and extend  $G$ -equivariantly over the entire  $G$ -equivariant cell in the unique possible way (see e.g. [Bre67]).

Once this is done, the proof proceeds similarly as for Proposition 4.1.11. □

homotopycategory

EXAMPLE 4.1.19. Specifying an object in  $\text{Ho}(\text{Top}^{\mathcal{B}G})$  amounts to giving a space with a group action. Note how this is quite different from  $(\text{Ho}(\text{Top}))^{\mathcal{B}G}$  which is just a topological space  $X$  and a homomorphism  $G \rightarrow [X, X]^{\text{inv}}$  from  $G$  to the group of homotopy classes of self-homotopy equivalences of  $X$ . For example, if  $X = S^n$ ,  $[X, X]^{\text{inv}} = \mathbb{Z}/2$ . This example tells us that, generally, specifying a homomorphism  $G \rightarrow [X, X]^{\text{inv}}$  will not describe how the group  $G$  acts on  $X$  in any real way.

An example to keep in mind is when  $X = BH$  for another group  $H$ . Then the genuine group actions of  $G$  on  $BH$  (i.e. the objects of  $\text{Ho}(\text{Top}^{BG})$  with underlying space  $BH$ , up to  $G$ -weak equivalences) would correspond to extensions  $0 \rightarrow H \rightarrow \hat{G} \rightarrow G \rightarrow 0$  (up to isomorphisms of extensions that are the identity on  $G$ ), whereas the other notion would just be a morphism  $G \rightarrow \text{Out}(H)$  to the outer automorphisms of  $H$  (up to global conjugation by elements in  $\text{Out}(H)$ ). Exercise: Convince yourself that these two types of structures are *not* in bijection.

We can also give another model for  $\text{Ho}(\text{Top}^{BG})$ :

**PROPOSITION 4.1.20.** *Denote by  $BG$  the classifying space of a group  $G$ . Consider the category  $\text{Top} \downarrow BG$  with objects topological spaces together with a map to  $BG$  and with morphisms being maps between topological spaces that commute with the map to  $BG$ . Endow this category with a notion of weak equivalence by saying that a map  $f : (X \rightarrow BG) \rightarrow (Y \rightarrow BG)$  is a weak equivalence if the underlying map of spaces  $f : X \rightarrow Y$  is a weak equivalence.*

*We have a functor  $\text{Top}^{BG} \rightarrow \text{Top} \downarrow BG$  sending an object  $X \in \text{Top}^{BG}$  to the object  $EG \times_G X = (EG \times X)/G \rightarrow BG$  in  $\text{Top} \downarrow BG$ . Likewise we have a functor  $\text{Top} \downarrow BG \rightarrow \text{Top}^{BG}$  sending an object  $X \rightarrow BG$  to the pullback  $EG \times_{BG} X \subset EG \times X$ , equipped with the restriction of the  $G$ -action on  $X \times EG$  obtained as product of the trivial action on  $X$  and the canonical, free action on  $EG$ . These functors define equivalences of homotopy categories.*

We will not give the proof here. The necessary techniques are similar to the ones used in Section 2.11.

As it is good to have a set of homotopy categories to keep in mind, we also cannot resist mentioning the following example.

**THEOREM 4.1.21 (Elmendorf).** *Let  $G$  be a finite group. Say that a map of  $G$ -CW complexes  $f : X \rightarrow Y$  is a  $G$ -homotopy equivalence if there is a  $G$ -equivariant map  $g : Y \rightarrow X$  such that  $gf$  and  $fg$  are  $G$ -equivariantly homotopic to the identities of  $X$  and  $Y$  respectively (this is stronger than just asking  $f$  to be a weak equivalence between the spaces  $X$  and  $Y$ ). Then*

$$\text{Ho}(G\text{-CW complexes, } G\text{-homotopy equivalence}) \xrightarrow{\cong} \text{Ho}(\text{Top}^{\mathcal{O}(G)^{\text{op}}})$$

where  $\mathcal{O}(G)$  is the orbit category of  $G$  with objects transitive  $G$ -sets and morphisms  $G$ -maps. The functor is given by assigning to  $X$  the functor  $\mathcal{O}(G)^{\text{op}} \rightarrow \text{Top}$  which sends  $G/H$  to the fixed points  $X^H$ .

We will not prove this now, but may return to it later, when we have the language to easily describe the inverse (as a teaser: how can we construct out of a functor  $\mathcal{O}(G)^{\text{op}} \rightarrow \text{Top}$  a single  $G$ -space with prescribed  $H$ -fixed points for all  $H \subset G$ ?).

**nolimits**

**EXAMPLE 4.1.22.** Limits and colimits do not exist in the homotopy category  $\text{Ho}(\mathcal{C})^{\mathcal{I}}$  in general, except in very special cases like products and coproducts. Let's for instance see that  $* \rightarrow K(\mathbb{Z}, 3) \xleftarrow{p} K(\mathbb{Z}, 3)$  does not have a limit in the homotopy category, where  $p$  is a prime number and we denote by  $p$  also the self-map of  $K(\mathbb{Z}, 3)$  (unique up to homotopy) whose action on  $\pi_3 = \mathbb{Z}$  is multiplication by  $p$ . The

diagram has a limit if and only if the functor  $F(-) = \ker(H^3(-; \mathbb{Z}) \xrightarrow{p} H^3(-; \mathbb{Z}))$  is representable in the form  $[-, P]$  for some space  $P$ . However,  $F$  is not exact in the middle on cofibration sequences, so this is impossible.

[Exercise: Construct a concrete example. Hint: Try to realize the sequence  $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2$  on  $H^3$  of a cofibration sequence.]

EXERCISE 4.1.23. Can you say something about the homotopy type of  $B\mathcal{O}(G)$ ? Hint: the transitive  $G$ -set  $*$  is a terminal object in  $\mathcal{O}G$ .

EXERCISE 4.1.24. As a warm-up to Theorem 4.1.21 prove the equivariant Whitehead Theorem: a  $G$ -map between  $G$ -CW complexes is a  $G$ -homotopy equivalence iff it induces a homotopy equivalence on  $H$ -fixed-points for all  $H \leq G$ . (If you get stuck, you can e.g., look up [Ben91].)

derivedfunctors

## 4.2. Derived functors via deformations

There is a natural notion of morphism between homotopical categories:

DEFINITION 4.2.1. Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are homotopical categories. We say a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is *homotopical* if it takes weak equivalences in  $\mathcal{C}$  to weak equivalences in  $\mathcal{D}$ . A homotopical functor induces a unique functor  $\mathrm{Ho}\mathcal{C} \rightarrow \mathrm{Ho}\mathcal{D}$  (which we also call  $F$ ) that fits in a commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \gamma_{\mathcal{C}} & & \downarrow \gamma_{\mathcal{D}} \\ \mathrm{Ho}\mathcal{C} & \xrightarrow{F} & \mathrm{Ho}\mathcal{D} \end{array},$$

where  $\gamma_{\mathcal{C}}$  and  $\gamma_{\mathcal{D}}$  are the localizations.

However, many functors we are interested in are *not* homotopical — this is why we want derived functors.

derivedfunctor

DEFINITION 4.2.2. Let  $\mathcal{C}$  be a homotopical category, and let  $\mathcal{E}$  be some other category (playing the role of  $\mathrm{Ho}(\mathcal{D})$ ); consider  $\mathcal{E}$  as a homotopical category by declaring only isomorphisms in  $\mathcal{E}$  to be weak equivalences. Let  $F: \mathcal{C} \rightarrow \mathcal{E}$  be any functor. The *total left derived functor*  $\mathbb{L}F$  of  $F$  is a homotopical functor  $\mathbb{L}F: \mathcal{C} \rightarrow \mathcal{E}$  that best approximates  $F$  from the left.

More precisely, the total left derived functor is a pair  $(\mathbb{L}F, \lambda)$  consisting of a homotopical functor  $\mathbb{L}F: \mathcal{C} \rightarrow \mathcal{E}$  and a natural transformation  $\lambda: \mathbb{L}F \rightarrow F$  satisfying the following: whenever  $(G, \eta)$  is a pair of a homotopical functor  $G: \mathcal{C} \rightarrow \mathcal{E}$  and a natural transformation  $\eta: G \rightarrow F$ , there is a unique natural transformation  $\theta: G \rightarrow \mathbb{L}F$  factoring  $\eta$  through  $\lambda$ , i.e.  $\eta = \lambda \circ \theta$ .

The given universal property ensures that if a total left derived functor of  $F$  exists, then it is unique up to unique natural isomorphism.

REMARK 4.2.3. Note that specifying a homotopy invariant functor  $\mathcal{C} \rightarrow \mathcal{E}$  (i.e. a functor sending weak equivalences in  $\mathcal{C}$  to isomorphisms in  $\mathcal{E}$ ) is equivalent to specifying a functor on  $\mathrm{Ho}(\mathcal{C})$ . Hence, with this identification, the total *left* derived



functor is simply the *right* Kan extension  $\text{Ran}_{\gamma_{\mathcal{C}}} F$  of the functor  $\mathcal{C} \rightarrow \mathcal{E}$  along  $\gamma_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Ho}(\mathcal{C})$  (cf. Section 1.9.2).

derivedfunctor

DEFINITION 4.2.4. A *left derived functor* of a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between homotopical categories is a homotopical functor  $F' : \mathcal{C} \rightarrow \mathcal{D}$  together with a natural transformation  $F' \rightarrow F$  such that the induced functor  $\gamma_{\mathcal{D}} \circ F' : \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$  and natural transformation  $\gamma_{\mathcal{D}} F' \rightarrow \gamma_{\mathcal{D}} F$  is a total left derived functor of  $\gamma_{\mathcal{D}} \circ F$ .

Note that, a priori, there is no guarantee that a left derived functor for  $F$  be *unique* up to unique isomorphism. In fact, in the case  $\mathcal{C} = \mathcal{D}$  and  $F = \text{id}_{\mathcal{C}}$ , we see that  $\text{id}_{\mathcal{C}}$  as well as any left deformation of  $\mathcal{C}$  are left derived functors of  $F$ . The homotopy colimit functor we will construct will be a left derived functor of the colimit functor.

REMARK 4.2.5. There is a natural dual notion of *right* derived functors. Given a random functor, you might wonder whether we should care about its left or right derived functor (assuming both exist, which is rare). Vaguely speaking, we typically want left derived functors of left adjoints and right derived functors of right adjoints — for one thing, as we will see later, we then get derived adjunctions.

We can use left deformations to construct left derived functors, provided the deformation is *compatible* with the functor in the following sense:

leftdeformation

DEFINITION 4.2.6. A *left deformation for a functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  between homotopical categories is a left deformation  $(Q, q)$  for  $\mathcal{C}$  with the additional property that the restriction of  $F$  to some full subcategory of  $\mathcal{C}$  containing all objects in the image of  $Q$  is homotopical. If a left deformation for  $F$  exists, we call  $F$  *left deformable*.

EXAMPLE 4.2.7. The category  $\text{Ch}_{\geq 0}(R)$  of non-negatively graded chain complexes of modules over a ring  $R$  becomes a homotopical category with quasi-isomorphisms as weak equivalences. An additive functor  $F$  from  $R$ -modules to  $S$ -modules induces a functor  $F_{\bullet} : \text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(S)$ . It is possible to build a functorial projective resolution, i.e. a functor  $Q : \text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(R)$  sending each complex  $X \in \text{Ch}_{\geq 0}(R)$  to a complex  $QX$  which is degree-wise a projective  $R$ -module and comes with a quasi-isomorphism  $q_X : QX \rightarrow X$  natural in  $X$ . This is a deformation for  $\text{Ch}_{\geq 0}(R)$  and, in fact, it is adapted to  $F_{\bullet}$  and hence a *deformation for  $F_{\bullet}$* . The reason for this is that  $F_{\bullet}$  preserves chain homotopy equivalences (because  $F$  is additive) and that any quasi-isomorphism between non-negatively graded complexes of projective  $R$ -modules is a chain homotopy equivalence.

prop: defderftr

PROPOSITION 4.2.8 (Dwyer–Hirschhorn–Kan–Smith, cf. [Rie14, Thm. 2.2.8]). *Let  $(Q, q)$  be a left deformation for a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between homotopical categories. Then  $(FQ, Fq)$  is a left derived functor of  $F$ .*

PROOF. We must show that given any homotopical functor  $G : \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$  equipped with a natural transformation  $\alpha : G \rightarrow \gamma_{\mathcal{D}} F$ , the natural transformation  $\alpha$  factors uniquely through  $\gamma_{\mathcal{D}} Fq : \gamma_{\mathcal{D}} FQ \rightarrow \gamma_{\mathcal{D}} F$ . To see that such a factorization exists,

consider the commutative diagram

$$\begin{array}{ccc} GQ & \xrightarrow{\alpha_Q} & \gamma_{\mathcal{D}}FQ \\ \downarrow Gq & & \downarrow \gamma_{\mathcal{D}}Fq \\ G & \xrightarrow{\alpha} & \gamma_{\mathcal{D}}F. \end{array}$$

Here the left vertical arrow is an isomorphism, since  $q$  is a natural weak equivalence and  $G$  is homotopical. Thus  $\alpha$  factors as  $\gamma_{\mathcal{D}}Fq \circ \bar{\alpha}$  where  $\bar{\alpha} := \alpha_Q \circ (Gq)^{-1}$ .

Now suppose we are given any factorization of  $\alpha$  as  $\gamma_{\mathcal{D}}Fq \circ \alpha'$ . We first show that  $\alpha'_Q = \bar{\alpha}_Q$ . To see this, consider the diagram

$$\begin{array}{ccc} GQ & \xrightarrow{\alpha'_Q} & \gamma_{\mathcal{D}}FQQ \\ & \searrow \alpha_Q & \downarrow \gamma_{\mathcal{D}}Fq_Q \\ & & \gamma_{\mathcal{D}}FQ. \end{array}$$

Here  $\gamma_{\mathcal{D}}Fq_Q$  is an isomorphism (objectwise) because  $Fq_Q$  is a weak equivalence (objectwise), because  $(Q, q)$  is a deformation for  $F$ . So  $\alpha'_Q = (\gamma_{\mathcal{D}}Fq_Q)^{-1} \circ \alpha_Q$ . This holds for any  $\alpha'$ , so in particular  $\alpha'_Q = \bar{\alpha}_Q$ .

Next consider the commutative square

$$\begin{array}{ccc} GQ & \xrightarrow{\alpha'_Q} & \gamma_{\mathcal{D}}FQQ \\ \downarrow Gq & & \downarrow \gamma_{\mathcal{D}}Fq_Q \\ G & \xrightarrow{\alpha'} & \gamma_{\mathcal{D}}FQ. \end{array}$$

This implies  $\alpha' = (\gamma_{\mathcal{D}}Fq) \circ \alpha'_Q \circ (Gq)^{-1} = (\gamma_{\mathcal{D}}Fq) \circ \bar{\alpha}_Q \circ (Gq)^{-1} = \bar{\alpha}$ , as required.  $\square$

prop:ho-adj

PROPOSITION 4.2.9. *Suppose  $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$  is an adjunction between homotopical categories,  $Q$  is a left deformation of  $\mathcal{C}$  compatible with  $F$  (in the sense of Definition 4.2.6), and, dually,  $R$  is a right deformation of  $\mathcal{D}$  compatible with  $G$  (this is called a deformable adjunction). Then the total derived functors  $\mathbb{L}F := \mathbb{L}(\gamma_{\mathcal{D}} \circ F): \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$  and  $\mathbb{R}G := \mathbb{R}(\gamma_{\mathcal{C}} \circ G): \mathcal{D} \rightarrow \text{Ho } \mathcal{C}$  (which can be computed, for example, via  $F \circ Q$  and  $G \circ R$ ), form an adjunction*

$$\mathbb{L}F: \text{Ho } \mathcal{C} \rightleftarrows \text{Ho } \mathcal{D}: \mathbb{R}G.$$

For the proof, we will need the following Lemma which can be extracted from the proof of Lemma 2.2.13 in [Rie14]:

kanextension

LEMMA 4.2.10. *Let  $\mathcal{C}$  be a homotopical category and let  $\mathcal{D}$  be a category (considered as homotopical by taking isomorphisms as weak equivalences). The total left derived functor of a left deformable functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  (i.e. a functor for which a compatible left deformation  $(Q, q)$  of  $\mathcal{C}$  exists) is an absolute right Kan extension (of  $F \circ Q$ ) along  $\gamma_{\mathcal{C}}: \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$ , i.e. a right Kan extension which is preserved by the postcomposition with any functor: for any functor of categories  $G: \mathcal{D} \rightarrow \mathcal{E}$ , we have that  $\text{Ran}_{\gamma_{\mathcal{C}}}(G \circ F)$  is naturally isomorphic to  $G \circ (\text{Ran}_{\gamma_{\mathcal{C}}} F)$ . The dual statement holds for right deformable functors.*

PROOF OF PROPOSITION 4.2.9. We can obtain the counit of the adjunction  $\mathbb{L}F \dashv \mathbb{R}G$  by

$$\begin{aligned}
\mathbb{L}F \circ \mathbb{R}G &:= \mathbb{L}(\gamma_{\mathcal{D}} \circ F) \circ \mathbb{R}(\gamma_{\mathcal{C}} \circ G) \\
&\cong \mathbb{L}(\gamma_{\mathcal{D}} \circ F) \circ \text{Lan}_{\gamma_{\mathcal{D}}}(\gamma_{\mathcal{C}} \circ G) \quad (\text{Lemma 4.2.10}) \\
&\cong \text{Lan}_{\gamma_{\mathcal{D}}}(\mathbb{L}(\gamma_{\mathcal{D}} \circ F) \circ \gamma_{\mathcal{C}} \circ G) \quad (\text{Lemma 4.2.10}) \\
&\xrightarrow{\text{transformation that is part of the total left derivative}} \text{Lan}_{\gamma_{\mathcal{D}}}(\gamma_{\mathcal{D}} \circ F \circ G) \\
&\xrightarrow{\text{counit of } F \dashv G} \text{Lan}_{\gamma_{\mathcal{D}}} \gamma_{\mathcal{D}} \\
&\xrightarrow{\text{transformation given by universal property of } \text{Lan}_{\gamma_{\mathcal{D}}} \text{ and by equality isomorphism } \gamma_{\mathcal{D}} \cong \text{id}_{\text{Ho } \mathcal{D}} \circ \gamma_{\mathcal{D}}} \text{id}_{\text{Ho } \mathcal{D}} .
\end{aligned}$$

Dually, one obtains the unit. One then needs to verify that these are really unit and counit of an adjunction, see Proposition 1.3.5 for this.  $\square$

PROPOSITION 4.2.11. *Let  $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$  be composable functors between homotopical categories. Suppose that  $F$  and  $G$  are left deformable by  $(Q, q)$  and  $(R, r)$ , respectively, and let  $\mathcal{C}_Q \subset \mathcal{C}$  and  $\mathcal{D}_R \subset \mathcal{D}$  be full subcategories containing the images of  $Q$  and  $R$  and on which  $F$  and  $G$  are homotopical. Assume also  $F(\mathcal{C}_Q) \subset \mathcal{D}_R$ . Then there is a natural isomorphism of total left derived functors*

$$\mathbb{L}(G \circ F) \cong \mathbb{L}G \circ \mathbb{L}F.$$

PROOF. For  $x \in \mathcal{C}$  and the (non-total) left derived functors  $F' = FQ$  and  $G' = GR$ , we have a natural transformation

$$(4.2.1) \quad G'F'X = GRFQx \xrightarrow{G(r_{FQx})} GFQx .$$

By assumption  $r_{FQx} : RFQx \rightarrow FQx$  is a weak equivalence between objects in  $\mathcal{D}_R$  and hence it is sent to a weak equivalence by  $G$ . This implies that (4.2.1) is actually a natural weak equivalence.

We next show that  $Q$  is a deformation for  $GF$ : for this, it suffices to show that for any objects  $x, y \in \mathcal{C}$  and any weak equivalence  $f : Qx \rightarrow Qy$  in  $\mathcal{C}$ , also  $GFf$  is a weak equivalence in  $\mathcal{E}$ . First, since  $(Q, q)$  is a deformation for  $F$ , we have that  $Ff : FQx \rightarrow FQy$  is a weak equivalence. Second, by the previous argument together with 3-of-4 on the following diagram

$$\begin{array}{ccc}
GRFQx & \xrightarrow{G(r_{FQx})} & GFQx \\
\downarrow GRFf & & \downarrow GFf \\
GRFQy & \xrightarrow{G(r_{FQy})} & GFQy,
\end{array}$$

we conclude that also  $GFf$  is a weak equivalence.

Postcomposing (4.2.1) with  $\gamma_{\mathcal{E}}$  and then applying  $\text{Ran}_{\gamma_{\mathcal{C}}}(-)$  yields the desired natural isomorphism.  $\square$

REMARK 4.2.12. Any model category can be seen as homotopical category with the weak equivalences being the weak equivalences of the model structure (this means that we are forgetting fibrations and cofibrations). For a *Quillen adjunction*

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \rightleftarrows & \mathcal{D} \\ & G & \end{array}$$

between model categories (see Remark 2.10.6), we can obtain the total left derived functor through *cofibrant replacements*: A cofibrant replacement of an object  $X$  in a model category is a trivial fibration  $\tilde{X} \rightarrow X$ , where  $\tilde{X}$  is cofibrant. This gives us a left deformation for the Quillen left adjoint  $F$  (we will ignore here the question whether such a cofibrant replacement can be chosen in a functorial way). The total left derived functor of  $F$  is therefore given by  $\mathbb{L}F(X) = F(\tilde{X})$  with a cofibrant replacement  $\tilde{X} \rightarrow X$ . Dually,  $\mathbb{R}G(Y) = G(\tilde{Y})$ , where  $Y \rightarrow \tilde{Y}$  is a *fibrant replacement*, i.e. a trivial cofibration with  $\tilde{Y}$  fibrant. For more details, we refer to Section II.8 of [GJ99] and also [Hov99].

### 4.3. More exercises

## Homotopy limits and colimits

ch:hocolim

hocolim-sec

## 5.1. The homotopy colimit construction

In this section we will give a definition of the homotopy colimit as a functor  $\text{hocolim} : \text{Top}^{\mathcal{I}} \rightarrow \text{Top}$ , using a certain explicit construction, and we will look at some examples. Later we will justify this construction by proving that it is a left derived functor of the colimit functor. We first work in the full subcategory  $\text{CW}$  of  $\text{Top}$  consisting of spaces homotopy equivalent to CW complexes. We will show:

- (1) There is a natural transformation  $\iota : \text{hocolim} \rightarrow \text{colim}$  of functors  $\text{CW}^{\mathcal{I}} \rightarrow \text{CW}$ . (Proposition 5.1.18; see also Proposition 5.3.11.)
- (2) The functor  $\text{hocolim}$  is homotopical, i.e. it takes weak equivalences in  $\text{CW}^{\mathcal{I}}$  to weak equivalences in  $\text{CW}$ . (Theorem 5.2.1.)
- (3) For any homotopical functor  $G : \text{CW}^{\mathcal{I}} \rightarrow \text{CW}$  with a natural transformation  $\eta : G \rightarrow \text{colim}$ , then there is a unique factorization of  $\gamma\eta$  as

$$\gamma G \xrightarrow{\theta} \gamma \text{hocolim} \xrightarrow{\gamma\iota} \gamma \text{colim},$$

where  $\gamma = \gamma_{\text{CW}}$  is the localization functor  $\text{CW} \rightarrow \text{Ho}(\text{CW})$ , and  $\theta : \gamma G \rightarrow \gamma \text{hocolim}$  is a natural transformation. (Theorem 5.4.3.)

It then follows Definition 4.2.4 that  $\text{hocolim}$  is a left derived functor of  $\text{colim}$ . As part of doing this we will devise a functorial way  $Q : \text{CW}^{\mathcal{I}} \rightarrow \text{CW}^{\mathcal{I}}$  to replace any diagram by a “fattened-up” diagram, and we will then see that  $\text{hocolim} \cong \text{colim} \circ Q$  (Proposition 5.1.24), that  $Q$  is a left deformation of  $\text{CW}^{\mathcal{I}}$  (Proposition 5.1.23), and that  $\text{colim}$  is homotopical on the image of  $Q$ .

Note that it follows from Proposition 4.2.9 that  $\text{hocolim} : \text{Ho}(\text{CW}^{\mathcal{I}}) \rightarrow \text{Ho}(\text{CW})$  is then the left adjoint to the constant diagram functor  $\text{Ho}(\text{CW}) \rightarrow \text{Ho}(\text{CW}^{\mathcal{I}})$ . (Theorem 5.4.3.) By the uniqueness of adjoints this property specifies the functor uniquely up to unique natural isomorphism as a functor between the homotopy categories.

rem:DI

REMARK 5.1.1. The statement above can be immediately propagated to  $\text{Top}$  by redefining our  $\text{hocolim}$  functor to be the functor given by first applying  $|\text{Sing}_{\bullet}(-)|$  to land in  $\text{CW}$ , and then the old  $\text{hocolim}$ , cf. Proposition 4.1.10. Better yet, the above statement holds verbatim also for  $\text{Top}$  without applying  $|\text{Sing}_{\bullet}(-)|$ , even with the model for  $\text{hocolim}$  we have in mind. The proof of this last refinement however requires more point set topology than we want to get into here—we refer the reader to [DI04, App. A] instead for a proof of this.

REMARK 5.1.2. We work out our  $\text{hocolim}$  construction with topological spaces. However, we might as well have carried out all our constructions in  $\text{sSet}$ , and there are similar formulas in chain complexes—we may expand on this in the Exercises.

And indeed there are similar constructions in model categories (see [Hir03]) or indeed in any infinity category (where *hocolim* is just the colimit, see [Lur09, §1.2.13 and Ch.4]).

REMARK 5.1.3. We again stress what we have said earlier: While one can choose many point-set models for homotopy colimit, the derived functor of colim is unique if it exists, so all models will agree in the homotopy category. We choose to work with “our” point-set version of *hocolim*, because of its relative simplicity. And the fact that it is the geometric realization of a simplicial space in fact gives us a spectral sequence for computing the homology of a *hocolim*, which we will set up in a later section.

REMARK 5.1.4. Just because we abstractly know that *hocolim* is the “best” approximation to colim in the sense above doesn’t mean that it is very good in concrete cases — it may be quite far from the actual colim. The homotopy colimit of a point under the (trivial) action of a finite group  $G$  is for example the infinite dimensional space  $BG$ . We will provide loads of examples after giving the definition.

Before giving the definition of *hocolim*, we first need to say a few words about simplicial spaces.

### 5.1.1. Simplicial spaces.

DEFINITION 5.1.5. Recall that if  $X_\bullet: \Delta^{\text{op}} \rightarrow \text{Top}$  is a simplicial space, then the *geometric realization*  $|X_\bullet|$  is the coequalizer

$$\text{coeq} \left( \coprod_{(\phi: [n] \rightarrow [m]) \in \Delta} X_n \times \Delta_{\text{top}}^m \rightrightarrows \coprod_{[k] \in \Delta} X_k \times \Delta_{\text{top}}^k \right),$$

where the two maps are given on the factor indexed by  $\phi$  by  $X_n \times \Delta_{\text{top}}^m \xrightarrow{\phi^* \times \text{id}} X_m \times \Delta_{\text{top}}^m$  and  $X_n \times \Delta_{\text{top}}^m \xrightarrow{\text{id} \times \phi_*} X_n \times \Delta_{\text{top}}^n$ . (Here  $\Delta_{\text{top}}^n$  denotes the *topological*  $n$ -simplex.) More explicitly, this is the quotient space obtained from the disjoint union  $\coprod_n X_n \times \Delta_{\text{top}}^n$  by identifying  $(\phi^*(x), y) \in X_n \times \Delta_{\text{top}}^n$  with  $(x, \phi_* y) \in X_m \times \Delta_{\text{top}}^m$  for all  $\phi: [n] \rightarrow [m]$ ,  $x \in X_m$ , and  $y \in \Delta_{\text{top}}^n$ .

REMARK 5.1.6. Note that there is a natural map

$$|X_\bullet| \rightarrow \text{colim}_{\Delta^{\text{op}}} X_\bullet$$

induced by the taking coequalizers of the rows of the following diagram.

$$\begin{array}{ccc} \coprod_{\phi: [m] \rightarrow [n]} X_n \times \Delta_{\text{top}}^m & \rightrightarrows & \coprod_k X_k \times \Delta_{\text{top}}^k \\ \downarrow & & \downarrow \\ \coprod_{\phi: [m] \rightarrow [n]} X_n & \rightrightarrows & \coprod_k X_k, \end{array}$$

rem:maptocolim

where the vertical maps are given by projecting off the  $\Delta_{top}^n$ -factors. By definition the coequalizer of the top row is  $|X_\bullet|$ , whereas the coequalizer in the bottom row is  $\text{colim}_{\Delta^{op}} X_\bullet$ . In this way the geometric realization is a “fattened” version of the colimit of  $X_\bullet$  over  $\Delta^{op}$ , where we have stuck in some contractible spaces.

The following more economical description of  $\text{colim}_{\Delta^{op}} X_\bullet$  will also be used repeatedly later:

top:colimdelta

PROPOSITION 5.1.7.

$$\text{colim}_{\Delta^{op}} X_\bullet \cong \text{coeq}(d_0, d_1 : X_1 \rightrightarrows X_0)$$

SKETCH OF PROOF. You can probably convince yourself directly that this is true. A formal argument uses cofinality: First note that by cofinality (see Proposition 5.5.4),

$$\text{colim}_{\Delta^{op}} X_\bullet \cong \text{colim}_{(\Delta_{\leq 1})^{op}} X_\bullet$$

Now observe directly that the colimit over  $(\Delta_{\leq 1})^{op}$  can be further replaced by the colimit over  $(\Delta_{\leq 1}^{inj})^{op}$ , i.e. we can leave out the degeneracy  $s_0$  without changing the colimit. This is now the wanted coequalizer.  $\square$

### 5.1.2. The definition of the homotopy colimit.

hocolimdef

DEFINITION 5.1.8. For a functor  $F: \mathcal{I} \rightarrow \text{Top}$  define the associated simplicial space  $F_\bullet^\Delta$  by  $F_n^\Delta := \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0)$ ; for  $\phi: [m] \rightarrow [n]$  in  $\Delta$ , the map  $\phi^*: F_n^\Delta \rightarrow F_m^\Delta$  is given on the component indexed by  $i_0 \rightarrow \dots \rightarrow i_n$  by  $F(i_0 \rightarrow \dots \rightarrow i_{\phi(0)}): F(i_0) \rightarrow F(i_{\phi(0)})$  where the target is in the component indexed by  $i_{\phi(0)} \rightarrow i_{\phi(1)} \rightarrow \dots \rightarrow i_{\phi(m)}$ .

We then define the *homotopy colimit* of  $F$  as

$$\text{hocolim } F = \text{hocolim}_{\mathcal{I}} F = |F_\bullet^\Delta| \in \text{Top}.$$

We can unwind this definition to see that  $\text{hocolim } F$  is the quotient space

$$\text{hocolim } F = \left( \coprod_n \coprod_{i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{top}^n \right) / \sim$$

where  $\sim$  is generated by the usual identities  $(d_i x, t) \sim (x, d^i t)$  and  $(s_i x, t) \sim (x, s^i t)$ , with  $x \in \coprod_{i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n} F(i_0)$  and  $d_i x$  defined by:

$$d_0(i_0 \xrightarrow{k} i_1 \rightarrow \dots \rightarrow i_n, x \in F(i_0)) = (i_1 \rightarrow \dots \rightarrow i_n, k(x) \in F(i_1));$$

$$d_j(i_0 \rightarrow \dots \rightarrow i_n, x \in F(i_0)) = (i_0 \rightarrow \dots \rightarrow \hat{i}_j \rightarrow \dots \rightarrow i_n, x \in F(i_0)) \text{ for } j \neq 0.$$

REMARK 5.1.9. For arbitrary simplicial spaces, geometric realization is not so homotopically well-behaved, and does not agree with the homotopy colimit over  $\Delta^{op}$ . However, we will see that this is true under an extra point-set topological condition on a simplicial space called “Reedy cofibrancy”

REMARK 5.1.10. Note that the simplicial space  $F_\bullet^\Delta$  is really defined by the formula  $F_n^\Delta := \coprod_{i: [n] \rightarrow \mathcal{I}} F(i(0))$ , where  $[n]$  is the ordered set  $0 < \dots < n$ ; hence the set parametrising the disjoint union is the same as the set  $(N\mathcal{I})_n$  appearing in the nerve of  $\mathcal{I}$ .

The following proposition is immediate, and its proof is left as exercise.

naturalityhocolim

PROPOSITION 5.1.11. *Let  $\mathcal{I}$  be a small category and let  $F: \mathcal{I} \rightarrow \text{Top}$  be a functor.*

- (1) *If  $G: \mathcal{I}' \rightarrow \mathcal{I}$  is a functor between small categories, then there is a natural map of spaces  $\text{hocolim}_{\mathcal{I}'}(F \circ G) \rightarrow \text{hocolim}_{\mathcal{I}} F$ , induced by sending the copy of  $F \circ G(i'(0))$  corresponding to  $i': [n] \rightarrow \mathcal{I}'$  to the copy of  $F(G \circ i'(0))$  corresponding to  $G \circ i'$  via the identity map of spaces.*
- (2) *If  $F': \mathcal{I} \rightarrow \text{Top}$  is another functor and  $\eta: F \rightarrow F'$  is a natural transformation, then there is a natural map  $\text{hocolim}_{\mathcal{I}} F \rightarrow \text{hocolim}_{\mathcal{I}} F'$  induced by sending the copy of  $F(i(0))$  corresponding to  $i$  to the copy of  $F'(i(0))$  corresponding to  $i$  via the map  $\eta_{i(0)}$ .*

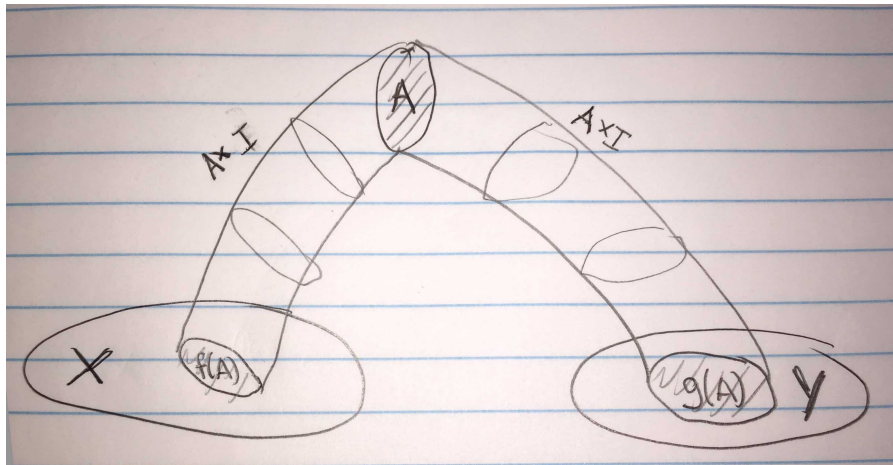
Now let us look at some examples:

EXAMPLE 5.1.12. Suppose  $F: \mathcal{I} \rightarrow \text{Top}$  is the constant functor with value  $*$ . Then  $F^\Delta$  is precisely the nerve  $N\mathcal{I}$  (viewed as a discrete simplicial space) and so  $\text{hocolim}_{\mathcal{I}} F$  is the classifying space  $B\mathcal{I}$ . More generally if  $F$  is constant with value  $X$  then  $\text{hocolim}_{\mathcal{I}} F \cong X \times B\mathcal{I}$ .<sup>1</sup>

EXAMPLE 5.1.13. Consider the pushout diagram:  $X \xleftarrow{f} A \xrightarrow{g} Y$ . The definition reveals that the homotopy colimit is a double mapping cylinder construction. In more details:

$$\text{hocolim}(X \xleftarrow{f} A \xrightarrow{g} Y) \cong ((X \amalg A \amalg Y) \amalg (A \times I \amalg A \times I)) / \sim$$

where  $\sim$  identifies certain points as in the following picture:



To spell this out, in the left copy of  $A \times I$ , the “face”  $A \times 0$  is glued to  $A$  and the other “face”  $A \times 1$  is glued to  $X$  via  $f$ , i.e,  $A \times I \ni (a, 1) \sim f(a) \in X$ . In the right copy of  $A \times I$ ,  $A \times 0$  is glued to  $A$  and  $A \times 1$  is glued to  $Y$  via  $A \times I \ni (a, 1) \sim g(a) \in Y$ .

Note that we have obtained  $\text{hocolim}(X \xleftarrow{f} A \xrightarrow{g} Y)$  as the quotient of the disjoint union of only 5 spaces of the form  $S \times \Delta_{top}^m$ ; and on the other hand, the nerve of

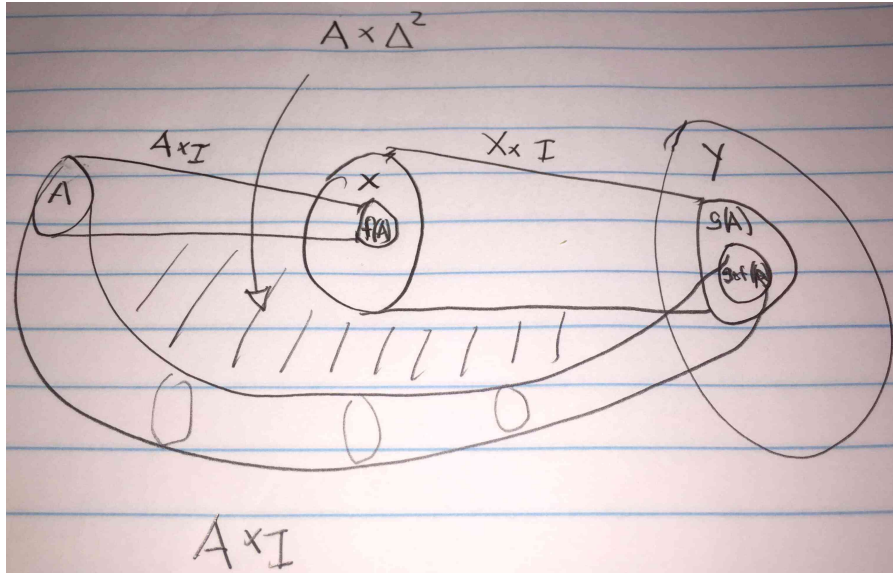
<sup>1</sup>Perhaps we need to assume  $X$  locally compact, or work within compactly generated Hausdorff spaces...



the pushout category  $N(\bullet \leftarrow \bullet \rightarrow \bullet)$ , as a simplicial set, contains precisely 5 non-degenerate simplices. Can you find an explanation (amenable to generalisations) for this?

AXY-exercise

EXERCISE 5.1.14. For the diagram  $A \rightarrow X \rightarrow Y$ , check that the homotopy colimit looks as in the following picture:



In particular, show that  $\text{hocolim}(A \rightarrow X \rightarrow Y)$  as a quotient of  $A \times \Delta_{\text{top}}^2 \amalg X \times \Delta_{\text{top}}^1 \amalg Y$ , and show that  $\text{hocolim}(A \rightarrow X \rightarrow Y)$  deformation retracts onto  $Y$ .

EXERCISE 5.1.15. Check that the homotopy colimit of a group  $G$  acting on a space  $X$  is homotopy equivalent to the so-called Borel construction, i.e.

$$\text{hocolim}_G X \simeq (X \times EG)/G.$$

- EXERCISE 5.1.16. (1) Give the precise definition of mapping cylinder  $\text{Cyl}(F)$  of  $F$ , as a quotient of  $\coprod_{n \geq 0} F_n \times I$  by gluing  $F_n \times \{1\}$  to  $F_{n+1} \times \{0\}$  along the map  $F(n < n + 1): F_n \rightarrow F_{n+1}$ .
- (2) Identify  $\text{Cyl}(F)$  as a subspace of  $\text{hocolim}_{\mathbb{N}} F$ .
- (3) Prove that  $\text{hocolim}_{\mathbb{N}} F$  deformation retracts onto  $\text{Cyl}(F)$ .

**5.1.3. hocolim as a deformation.** We now embark on establishing the basic properties of  $\text{hocolim}$ . Our first item on the agenda is to relate the homotopy colimit to the ordinary colimit.

First note that Proposition 5.1.7 allows us to reexpress the colimit of  $F$  as a colimit of  $F^\Delta$ :

PROPOSITION 5.1.17.  $\text{colim}_{\mathcal{I}} F \cong \text{colim}_{\Delta^{\text{op}}} F^\Delta$ . □

With this, it is easy to see that the first requirement for a derived functor from Definition 4.2.4 holds for our definition of  $\text{hocolim}$ :

pp:hocolim-trans

PROPOSITION 5.1.18. *There is a natural transformation  $\text{hocolim} \rightarrow \text{colim}$ .*

PROOF. By Remark 5.1.6 we have for any simplicial space  $X \in \text{Top}^{\Delta^{\text{op}}}$  the map  $|X| \rightarrow \text{colim}_{\Delta^{\text{op}}} X$ . Applying this to  $F_{\bullet}^{\Delta}$  we obtain the desired map

$$\text{hocolim}_{\mathcal{I}} F = |F_{\bullet}^{\Delta}| \rightarrow \text{colim}_{\Delta^{\text{op}}} F^{\Delta} \cong \text{colim}_{\mathcal{I}} F.$$

□

An important special case of homotopy invariance is when the indexing category has a terminal object, and we start with that case.

terminalobject

PROPOSITION 5.1.19. *Let  $\mathcal{I}$  be a small category with a terminal object  $t$ . Then for any functor  $F: \mathcal{I} \rightarrow \text{Top}$ , the natural map  $\text{hocolim}_{\mathcal{I}} F \rightarrow \text{colim}_{\mathcal{I}} F \cong F(t)$  is a homotopy equivalence, in particular a weak equivalence.*

REMARK 5.1.20. We have already seen an instance of this situation in Exercise 5.1.14, and looking at the associated picture should give a good idea of why this is true. The formal proof is similar to the proof that the realization of the nerve of a category with a terminal object is contractible, but now keeping an extra “space-coordinate” around: Recall that that proof goes by defining a functor  $\mathcal{I} \times [1] \rightarrow \mathcal{I}$  as the identity on  $\mathcal{I} \times 0$ , constant  $t$  on  $\mathcal{I} \times 1$ , and sending the morphism  $(i, 0) \rightarrow (i, 1)$  to the unique map  $i \rightarrow t$ . Upon realization of nerve this gives the wanted homotopy.

PROOF OF PROPOSITION 5.1.19. We want to make a deformation retraction onto  $F(t)$ , i.e. a homotopy

$$\text{hocolim}_{\mathcal{I}} F \times I \rightarrow \text{hocolim}_{\mathcal{I}} F$$

contracting onto  $F(t) \subset \text{hocolim}_{\mathcal{I}} F$ . The geometric realization functor  $|-|: \text{Top}^{\Delta^{\text{op}}} \rightarrow \text{Top}$  commutes with products; in particular, since  $I \cong |\Delta^1|$ , we can describe the homotopy colimit  $\text{hocolim}_{\mathcal{I}} F \times I$  as the realization of the product simplicial space  $F^{\Delta} \times \Delta^1$ ,<sup>2</sup> remembering the definition of  $F^{\Delta}$ , we can describe  $F^{\Delta} \times \Delta^1$  as

$$(F^{\Delta} \times \Delta^1)_{n+1} := \coprod_{i: [n] \rightarrow (\mathcal{I} \times [1])} F(\text{pr}_{\mathcal{I}}(i(0))),$$

where  $[n] = (0 < \dots < n)$  (hence functors  $F: [n] \rightarrow \mathcal{I} \times [1]$  correspond to  $n$ -simplices in  $N(\mathcal{I} \times [1])$ ), and  $\text{pr}_{\mathcal{I}}: \mathcal{I} \times [1] \rightarrow \mathcal{I}$  is the projection functor

Let  $T$  be the functor  $T: \mathcal{I} \times [1] \rightarrow \mathcal{I} \times [1]$  restricting to the identity on  $\mathcal{I} \cong \mathcal{I} \times 0 \rightarrow \mathcal{I}$  and restricting to the constant functor on  $\mathcal{I} \times 1 \rightarrow \{t\}$ . Note that there is a natural transformation  $\eta: \text{id}_{\mathcal{I} \times [1]} \rightarrow T$  which takes an identity morphism on an object of the form  $x \times 0$  and takes the terminal map to  $t$  on an object of the form  $x \times 1$ .

We now define a map of simplicial spaces  $F^{\Delta} \times \Delta^1 \rightarrow F^{\Delta}$  as follows: given  $n \geq 0$  and  $i: [n] \rightarrow \mathcal{I} \times [1]$ , we map the copy of  $F(\text{pr}_{\mathcal{I}}(i(0)))$  corresponding to  $i$  to the copy of  $F(\text{pr}_{\mathcal{I}}(T \circ i(0)))$  corresponding to  $\text{pr}_{\mathcal{I}} \circ T \circ i$ , along the map  $F(\text{pr}_{\mathcal{I}}(\eta_{i(0)}))$ .

Check that this gives a map of simplicial spaces, and induces the desired deformation retract on realizations. □

<sup>2</sup>Here we consider the simplicial set  $\Delta^1$  as a discrete simplicial space.

REMARK 5.1.21. The above proof can be generalized to the situation called “having an extra degeneracy”; this is explained in homework.

We can consider Proposition 5.1.19 as an instance of how the homotopy colimit construction first replaces a diagram by a homotopy equivalent and “good” diagram, and then takes the ordinary colimit of this second diagram; this generalizes the mapping cylinder construction and the examples/exercises in the beginning of the section.

defn:Q

DEFINITION 5.1.22. For a functor  $F: \mathcal{I} \rightarrow \text{Top}$ , we define a new functor  $QF: \mathcal{I} \rightarrow \text{Top}$  by

$$(QF)(i) := \text{hocolim}_{\mathcal{I}/i} F = \left| n \mapsto \coprod_{i_0 \rightarrow \cdots \rightarrow i_n \rightarrow i} F(i_0) \right|.$$

prop:Qdef

PROPOSITION 5.1.23. The map  $QF(i) = \text{hocolim}_{\mathcal{I}/i} F \xrightarrow{\cong} F(i)$  induced by

$$F(i_0)_{i_0 \rightarrow \cdots \rightarrow i_n \rightarrow i} \xrightarrow{F(i_0 \rightarrow i)} F(i)$$

gives a natural transformation  $q_F: QF \rightarrow F$ , and is an objectwise homotopy equivalence by Proposition 5.1.19. In particular  $(Q, q)$  is a left deformation of  $\text{Top}^{\mathcal{I}}$ , and it restricts to a left deformation of  $\text{CW}^{\mathcal{I}}$ , in the sense of Definition 4.1.7.  $\square$

propn:colimQ

PROPOSITION 5.1.24. For any functor  $F$ , we have a natural isomorphism

$$\text{colim}_{\mathcal{I}} QF = \text{hocolim}_{\mathcal{I}} F.$$

SKETCH PROOF. We unravel the definitions, using the notation  $i_0 \rightarrow \cdots \rightarrow i_n \downarrow i$  for an  $n$ -fold composition in  $\mathcal{I}/i$ :

$$\begin{aligned} \text{colim}_{\mathcal{I}} QF &= \text{colim}_{i \in \mathcal{I}} \left( \left( \coprod_{i_0 \rightarrow \cdots \rightarrow i_n \downarrow i} F(i_0) \times \Delta_{top}^n \right) / \sim \right) \\ &= \left( \coprod_{i \in \mathcal{I}} \left( \left( \coprod_{i_0 \rightarrow \cdots \rightarrow i_n \downarrow i} F(i_0) \times \Delta_{top}^n \right) / \sim \right) \right) / \approx \\ &= \left( \coprod_{i_0 \rightarrow \cdots \rightarrow i_n} F(i_0) \times \Delta_{top}^n \right) / \sim \\ &= \text{hocolim}_{\mathcal{I}} F \end{aligned}$$

where  $\sim$  is the equivalence relations coming from the simplicial identities and  $\approx$  is the equivalence relation identifying each point in  $\left( \coprod_{i_0 \rightarrow \cdots \rightarrow i_n \downarrow j} F(i_0) \times \Delta_{top}^n \right) / \sim$  with the corresponding point in  $\left( \coprod_{i_0 \rightarrow \cdots \rightarrow i_n \downarrow k} F(i_0) \times \Delta_{top}^n \right) / \sim$ , for all morphisms  $f: j \rightarrow k$  in  $\mathcal{I}$ . One can check that  $\approx$  is generated by just identifying each factor  $F(i_0) \times \Delta_{top}^n$  indexed by  $i_0 \rightarrow \cdots \rightarrow i_n \downarrow i$  with the corresponding factor  $F(i_0) \times \Delta_{top}^n$  indexed by  $i_0 \rightarrow \cdots \rightarrow i_n \downarrow i_n$ , i.e. a factor in whose index the map “ $\downarrow$ ” is the identity of  $i_n$  (so that  $i_n \downarrow i_n$  is terminal in  $\mathcal{I}/i_n$ ). Points coming from such special factors have

no further identifications due to  $\approx$ , so we can compute  $\operatorname{colim}_{\mathcal{I}} QF$  as the quotient of  $\coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{top}^n \cong \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{top}^n$  by  $\sim$ , which is the definition of  $\operatorname{hocolim}_{\mathcal{I}} F$ .  $\square$

We will see another slick proof of Proposition 5.1.24 later, in Proposition 5.3.11 after we have introduced the two-sided bar construction.

## 5.2. Homotopy invariance of the homotopy colimit

Our goal in this section is to prove the homotopy invariance of the homotopy colimit. More precisely, we will show:

hm:hocoliminv

**THEOREM 5.2.1.** *Suppose  $F$  and  $F'$  are functors  $\mathcal{I} \rightarrow \mathbf{CW}$  and  $\eta: F \rightarrow F'$  is a natural transformation such that for every  $i \in \mathcal{I}$  the morphism  $\eta_i: F(i) \rightarrow F'(i)$  is a weak homotopy equivalence. Then the induced map  $\operatorname{hocolim}_{\mathcal{I}} F \rightarrow \operatorname{hocolim}_{\mathcal{I}} F'$  is a weak equivalence.*

We will deduce Theorem 5.2.1 from a general result on homotopy invariance for the geometric realization of simplicial spaces. To state it we need some definitions:

**DEFINITION 5.2.2.** If  $X_{\bullet}$  is a simplicial space, its  $n^{\text{th}}$  *latching object*  $L_n X$  is the colimit of the composite functor

$$((\Delta_{<n}^{\text{surj}})_{[n]})^{\text{op}} \rightarrow \Delta^{\text{op}} \xrightarrow{X} \mathbf{Top},$$

where  $\Delta^{\text{surj}}$  is the subcategory of  $\Delta$  spanned by all objects and containing only surjective maps, and the awkward looking  $(\Delta_{<n}^{\text{surj}})_{[n]}$  is the (full) subcategory of the under category  $(\Delta^{\text{surj}})_{[n]}$  spanned by all surjections  $[n] \rightarrow [i]$  with  $i < n$  (that is  $i \neq n$ ). In other words  $L_n X$  is just the colimit of  $X_i$  over all the degeneracies  $[n] \rightarrow [i]$ ,  $i < n$ .

**REMARK 5.2.3.** We can think of  $L_n X$  as the collection of all *degenerate* simplices inside  $X_n$ . If  $X$  is a simplicial set, this is literally true. In general we only have a well-defined (and injective) map  $L_n X \rightarrow X_n$ , adjoint to the natural transformation  $X \rightarrow \delta(X_n)$  of functors  $(\Delta_{<n}^{\text{surj}})_{[n]} \rightarrow \mathbf{Top}$  given on the object  $(\theta: [n] \rightarrow [i])$  by  $\theta^*: X_i \rightarrow X_n$ .

**REMARK 5.2.4.** Notice that  $s_i: X_{n-1} \rightarrow X_n$  has a left inverse  $d_i$ , by the simplicial identities, so it is an injection and a homeomorphism onto its image  $s_i(X_{n-1})$  (since for any open  $U \subseteq X_{n-1}$ ,  $U = s_i^{-1} d_i^{-1}(U)$ ). Thus we can identify the colimit  $L_n X$  as the union  $\bigcup_{i=0}^n s_i(X_{n-1})$  inside  $X_n$ , at least as a set.

Moreover, if  $X_n$  is Hausdorff, then the equality of subsets  $s_i(X_{n-1}) = \{x \in X_n : s_i d_i x = x\}$  implies that  $s_i(X_{n-1})$  is a closed subset of  $X_n$ .<sup>3</sup> Similarly, one can show that for all  $\theta: [n] \rightarrow [i]$  with  $n > i$  one has that  $\theta^*(X_i) \subset X_n$  is closed. Now  $L_n X$  has the topology of a *finite* colimit of closed inclusions of topological spaces, all of which are closed subspaces of  $X_n$ : it then follows that  $L_n X \subset X$  is a closed embedding (homeomorphism with image + closed image).

<sup>3</sup>Recall that if  $f, g: X \rightarrow Y$  are continuous and  $Y$  is Hausdorff, then  $\{x : f(x) = g(x)\}$  is always a closed subset of  $X$ .

DEFINITION 5.2.5. A simplicial space  $X_\bullet$  is *Reedy cofibrant* if for every  $n$  the map  $L_n X \rightarrow X_n$  is a (closed) cofibration of topological spaces (i.e. a closed embedding with the homotopy extension property).

Let us check that the main example we have in mind is indeed Reedy cofibrant.

em:Fdeltareedy

LEMMA 5.2.6. *For any functor  $F: \mathcal{I} \rightarrow \text{Top}$ , the simplicial space  $F^\Delta$  is Reedy cofibrant.*

PROOF. The latching object  $L_n(F^\Delta)$  can be identified with the part of the disjoint union  $F_n^\Delta = \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{top}^n$  corresponding to sequences  $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n$  where at least one of the maps is an identity. Thus  $F_n^\Delta$  is a coproduct of  $L_n(F^\Delta)$  and the coproduct over the remaining  $n$ -simplices<sup>4</sup> in  $N\mathcal{I}$ , and the inclusion is obviously a closed cofibration.  $\square$

The result about simplicial spaces that we are aiming at for now is:

reedycofibrant

THEOREM 5.2.7. *Let  $X_\bullet$  and  $Y_\bullet$  be Reedy cofibrant simplicial spaces such that  $X_n$  and  $Y_n$  are CW complexes for all  $n$ , and let  $f_\bullet: X_\bullet \rightarrow Y_\bullet$  be a levelwise weak equivalence. Then  $|f|: |X| \rightarrow |Y|$  is a weak equivalence.*

Before embarking on the proof, let us see that it implies Theorem 5.2.1.

PROOF OF THEOREM 5.2.1 ASSUMING THEOREM 5.2.7. We have already seen that  $F_\bullet^\Delta$  is Reedy cofibrant in Lemma 5.2.6. Furthermore by assumption

$$F^\Delta([n]) = \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \rightarrow \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F'(i_0) = F'^\Delta([n])$$

is a weak homotopy equivalence, so it follows from Theorem 5.2.7 that

$$\text{hocolim}_{\mathcal{I}} F = |F_\bullet^\Delta| \rightarrow |F'_\bullet^\Delta| = \text{hocolim}_{\mathcal{I}} F'$$

is a weak equivalence as wanted.  $\square$

To prove Theorem 5.2.7 we need some input from more classical homotopy theory, which we summarize in a lemma:

em:reedyinput

LEMMA 5.2.8.

- (i) *Suppose  $X \hookrightarrow Y$  is a cofibration and  $X \rightarrow X'$  is a homotopy equivalence. Then the pushout  $Y \rightarrow Y \cup_X X'$  is also a homotopy equivalence.*
- (ii) *Suppose we are given a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & A & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longleftarrow & A' & \xleftarrow{i'} & Y' \end{array}$$

*where  $i$  and  $i'$  are cofibrations and the vertical maps are homotopy equivalences. Then the induced map on pushouts  $X \cup_A Y \rightarrow X' \cup_{A'} Y'$  is a homotopy equivalence.*

<sup>4</sup>...which are precisely the non-degenerate  $n$ -simplices in  $N\mathcal{I}$ .

(iii) Suppose we are given a map of sequences

$$\begin{array}{ccccccc} X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \cdots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ Y_0 & \hookrightarrow & Y_1 & \hookrightarrow & Y_2 & \hookrightarrow & \cdots, \end{array}$$

where the maps  $X_i \hookrightarrow X_{i+1}$  and  $Y_i \hookrightarrow Y_{i+1}$  are all cofibrations, and the maps  $f_i: X_i \rightarrow Y_i$  are all homotopy equivalences. Then the induced map on colimits  $f: \operatorname{colim}_{n \rightarrow \infty} X_n \rightarrow \operatorname{colim}_{n \rightarrow \infty} Y_n$  is a homotopy equivalence.

(iv) Suppose  $A \hookrightarrow A'$  and  $B \hookrightarrow B'$  are closed cofibrations. Then the induced map  $A \times B' \amalg_{A \times B} A' \times B \rightarrow A' \times B'$  is also a closed cofibration.

PROOF. For (i) see [Hat02, Exercise 0.27]. (ii) follows formally from (i) — see [MP12, Proposition 15.4.4]. For (iii) we can use Lemma 4.1.16 to inductively build a sequence of compatible homotopy inverses  $Y_n \rightarrow X_n$  of  $f_n$ , in the limit these then give a homotopy inverse to  $f$ . For (iv), see [May99, §6.4].  $\square$

We also need to introduce the notion of  $n$ -skeleton for simplicial spaces, and to prove some basic properties of these:

DEFINITION 5.2.9. For  $X_\bullet$  a simplicial space, let  $\operatorname{sk}_n |X|$  denote the analogue of the geometric realization where we only consider the objects  $[k] \in \mathbf{\Delta}$  where  $k \leq n$ , i.e.

$$\operatorname{coeq} \left( \coprod_{\substack{[m] \rightarrow [k] \in \mathbf{\Delta} \\ m, k \leq n}} X_k \times \Delta_{\operatorname{top}}^m \rightrightarrows \coprod_{\substack{[l] \in \mathbf{\Delta} \\ l \leq n}} X_l \times \Delta_{\operatorname{top}}^l \right).$$

**lem:skn**

LEMMA 5.2.10. Suppose  $X_\bullet$  is a simplicial space.

(i) The geometric realization  $|X|$  is the colimit in  $\mathbf{Top}$  of the natural maps

$$\operatorname{sk}_0 |X| \rightarrow \operatorname{sk}_1 |X| \rightarrow \cdots.$$

(ii) For every  $n$  there is a natural pushout square in  $\mathbf{Top}$

$$\begin{array}{ccc} L_n X \times \Delta_{\operatorname{top}}^n \amalg_{L_n X \times \partial \Delta_{\operatorname{top}}^n} X_n \times \partial \Delta_{\operatorname{top}}^n & \longrightarrow & \operatorname{sk}_{n-1} |X| \\ \downarrow & & \downarrow \\ X_n \times \Delta_{\operatorname{top}}^n & \longrightarrow & \operatorname{sk}_n |X|. \end{array}$$

(iii) If  $X$  is Reedy cofibrant, then the maps  $\operatorname{sk}_n |X| \rightarrow \operatorname{sk}_{n+1} |X|$  are all closed cofibrations.

PROOF. (i) is formal since colimits commute and  $\mathbf{\Delta}$  is the union of the subcategories with objects  $\leq n$ . It is also geometrically intuitive since every point in  $|X|$  lies in  $X_n \times \Delta_{\operatorname{top}}^n$  for some (unique) minimal  $n$ , and so  $|X|$  is the union of the subspaces  $\operatorname{sk}_n |X|$ .

(ii) makes sense geometrically since the  $n$ -skeleton is obtained from the  $(n-1)$ -skeleton by gluing on  $X_n \times \Delta_{\operatorname{top}}^n$ . This meets the  $(n-1)$ -skeleton precisely along

the images of  $X_{n-1} \times \Delta_{top}^n$  along a degeneracy in the first factor, and the images of  $X_n \times \Delta_{top}^{n-1}$  along the inclusion of a face of  $\Delta_{top}^n$  in the second factor. These are glued to the  $(n-1)$ -skeleton via the corresponding codegeneracy  $X_{n-1} \times \Delta^n \rightarrow X_{n-1} \times \Delta^{n-1}$  in the second factor and the corresponding face map  $X_n \times \Delta^{n-1} \rightarrow X_{n-1} \times \Delta^{n-1}$  in the first factor, respectively. Taking the union of these over all degeneracies and faces we get  $L_n X \times \Delta^n$  and  $X_n \times \partial \Delta^n$ , which meet in  $L_n X \times \partial \Delta^n$ , giving the pushout in the top left corner. See Remark 5.2.11 below for a formal argument in simplicial spaces.

(iii) follows from (ii) and the assumption of Reedy cofibrancy, since closed cofibrations are preserved under pushouts and the map  $L_n X \times \Delta^n \amalg_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n \rightarrow X_n \times \Delta^n$  is a closed cofibration by Lemma 5.2.8(iv).  $\square$

formal-argument

REMARK 5.2.11. Let us give a more formal argument for the pushout square (ii). Let  $\mathbf{\Delta}_{\leq n}$  denote the full subcategory of  $\mathbf{\Delta}$  with objects  $[k]$  where  $k \leq n$ , and write  $i_n: \mathbf{\Delta}_{\leq n} \rightarrow \mathbf{\Delta}$  for the inclusion functor. If  $X \in \mathbf{Top}^{\Delta^{op}}$  is a simplicial space, we can define its  $n$ -skeleton  $sk_n X \in \mathbf{Top}^{\Delta^{op}}$  as the left Kan extension  $i_{n,!} i_n^* X$  (where  $i_n^* X$  is the restriction of  $X$  to the subcategory  $\mathbf{\Delta}_{\leq n}^{op}$ ). Notice that everything in  $(sk_n X)_i$  is degenerate for  $i > n$ , so the realization  $|sk_n X|$  is homeomorphic to  $sk_n |X|$  as we defined this above. Note also that  $(sk_n X)_{n+1}$  is precisely  $L_{n+1} X$ . Since geometric realization is a left adjoint, it preserves colimits; and as it also preserves products, it is enough to show that we have a pushout of simplicial spaces

$$\begin{array}{ccc} L_n X \times \Delta^n \amalg_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n & \longrightarrow & sk_{n-1} X \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & sk_n X, \end{array}$$

where  $\Delta^n$  and  $\partial \Delta^n$  here denote simplicial sets (viewed as discrete simplicial spaces) rather than spaces. These simplicial spaces are all  $n$ -skeletal (i.e. are left Kan extensions of their restrictions to  $\mathbf{\Delta}_{\leq n}^{op}$ ), so as  $i_{n,!}$  preserves colimits (being a left adjoint) it suffices to prove we have a levelwise pushout when evaluated at  $[k]$  for  $k \leq n$ . Evaluating at  $k < n$  we have  $(\Delta^n)_k = (\partial \Delta^n)_k$  so the top left space is

$$L_n X \times (\Delta^n)_k \amalg_{L_n X \times (\Delta^n)_k} X_n \times (\Delta^n)_k \cong X_n \times (\Delta^n)_k;$$

hence both vertical arrows are isomorphisms, and the square is indeed a pushout. Evaluating at  $k = n$ , we have  $(\Delta^n)_n = \{*\} \amalg (\partial \Delta^n)_n$  so the top left object is  $L_n X \amalg Y$  where  $Y := X_n \times (\partial \Delta^n)_n$ . On the other hand  $X_n \times (\Delta^n)_n = X_n \amalg Y$  and so the diagram is

$$\begin{array}{ccc} L_n X \amalg Y & \longrightarrow & L_n X \\ \downarrow & & \downarrow \\ X_n \amalg Y & \longrightarrow & X_n, \end{array}$$

which is clearly a pushout.

PROOF OF THEOREM 5.2.7. Using Lemma 5.2.8(iii) and Lemma 5.2.10(iii) we see that it is enough to prove that  $\text{sk}_n|f|: \text{sk}_n|X| \rightarrow \text{sk}_n|Y|$  is a homotopy equivalence for all  $n$ . Now applying Lemma 5.2.8(ii) to the natural pushout squares from Lemma 5.2.10(ii) (where by Lemma 5.2.8(iv) the left vertical maps are closed cofibrations) we see that  $\text{sk}_n|f|$  is a homotopy equivalence provided the maps  $\text{sk}_{n-1}|f|$ ,  $f_n \times \Delta_{top}^n$ , and  $L_n f \times \Delta_{top}^n \amalg_{L_n f \times \partial \Delta_{top}^n} f_n \times \partial \Delta_{top}^n$  are all homotopy equivalences.<sup>5</sup> For  $f_n \times \Delta_{top}^n$  this holds by assumption (since  $X_n$  and  $Y_n$  are assumed to be CW complexes), so we can complete the proof by induction if we can show that the map  $L_n f \times \Delta_{top}^n \amalg_{L_n f \times \partial \Delta_{top}^n} f_n \times \partial \Delta_{top}^n$  is a homotopy equivalence. Using Lemma 5.2.8(ii) again, we see this will be true if  $L_n f: L_n X \rightarrow L_n Y$  is a homotopy equivalence. But  $L_n X$  can be naturally written as an iterated pushout of copies of  $X_{n-1}$  along inclusions of  $L_{n-1} X$ , so again this follows by induction and Lemma 5.2.8(ii).  $\square$

As we already showed that Theorem 5.2.7 implies Theorem 5.2.1, this concludes the proof of homotopy invariance.

### 5.3. Coends and homotopy coends

To prove the other properties of the homotopy colimits we are interested in, it turns out to be convenient to put our construction of the homotopy colimit into a slightly more general context. We start by the non-homotopical version of this, which is the notion of *coends*.

subsec:coends

**5.3.1. Coends and functor tensor products.** Recall our expression for the colimit of a functor  $F: \mathcal{I} \rightarrow \mathcal{C}$  as

$$\text{colim } F \cong \text{coeq} \left( \coprod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} F(i) \rightrightarrows \coprod_{k \in \text{Ob}(\mathcal{I})} F(k) \right).$$

Now we introduce a variant of this construction: given a functor  $\Phi: \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \mathcal{C}$ , its *coend* is the coequalizer

$$\text{coend } \Phi := \text{coeq} \left( \coprod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} \Phi(j, i) \rightrightarrows \coprod_{k \in \text{Ob}(\mathcal{I})} \Phi(k, k) \right).$$

where the two morphisms are given on the component  $\Phi(j, i)$  corresponding to  $f: i \rightarrow j$  by  $\Phi(f, \text{id}): \Phi(j, i) \rightarrow \Phi(i, i)$  and  $\Phi(\text{id}, f): \Phi(j, i) \rightarrow \Phi(j, j)$ . The coend of  $\Phi$  is sometimes denoted  $\int^{\mathcal{I}} \Phi$ ; we will generally avoid this notation though.

rem:twistedarr

REMARK 5.3.1. You might ask what this construction “means”, or more precisely whether it has a universal property. In the literature this is usually discussed in terms of a rather obscure notion of “extranatural transformations”. However, there is a very natural way to understand coends as ordinary colimits: If  $\mathcal{I}$  is a category, the *twisted*

<sup>5</sup>It is important to notice that Lemma 5.2.8(ii) requires some maps to be actual homotopy equivalences, and not just weak equivalences.



arrow category  $\text{Tw}(\mathcal{I})$  of  $\mathcal{I}$  is the category whose objects are the morphisms of  $\mathcal{I}$  and whose morphisms from  $i \rightarrow j$  to  $i' \rightarrow j'$  are the commutative diagrams of the form

$$\begin{array}{ccc} i & \longrightarrow & i' \\ \downarrow & & \downarrow \\ j & \longleftarrow & j'. \end{array}$$

There is an obvious forgetful functor  $\text{Tw}(\mathcal{I}) \rightarrow \mathcal{I}^{\text{op}} \times \mathcal{I}$ , given by taking target and source of morphisms, and the coend of a functor  $\Phi$  can be identified with the colimit of the composite functor

$$\text{Tw}(\mathcal{I}) \rightarrow \mathcal{I}^{\text{op}} \times \mathcal{I} \xrightarrow{\Phi} \mathcal{C}.$$

(This is not completely obvious, it requires a cofinality argument.)

A key special case of the coend construction is the so-called *functor tensor product*: given functors  $W: \mathcal{I}^{\text{op}} \rightarrow \mathcal{C}$  and  $F: \mathcal{I} \rightarrow \mathcal{C}$ , and assuming that  $\mathcal{C}$  admits products, we write

$$W \otimes_{\mathcal{I}} F = \text{coend}(W \times F)$$

where  $W \times F: \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \mathcal{C}$ .

exs:coend

EXAMPLES 5.3.2. The following hold.

eq:coend-colim

- (i) If  $F: \mathcal{I} \rightarrow \text{Top}$  is any functor, then  $* \otimes_{\mathcal{I}} F$  is the colimit of  $F$  — this is the description of the colimit we have repeatedly used before.
- (ii) If  $X_{\bullet}$  is a simplicial space, then the geometric realization  $|X|$  is nothing but the tensor  $\Delta^{\bullet} \otimes_{\Delta^{\text{op}}} X_{\bullet}$ , or else  $X_{\bullet} \otimes_{\Delta} \Delta^{\bullet}$ .
- (iii) If  $F$  takes values in  $\text{Top}$ , then for  $d \in \mathcal{I}$  we have a functor  $\mathcal{I}(-, d): \mathcal{I}^{\text{op}} \rightarrow \text{Set} \subset \text{Top}$ , and then we have  $\mathcal{I}(-, d) \otimes_{\mathcal{I}} F \cong \text{colim}_{\mathcal{I}/d} F \xrightarrow{\cong} F(d)$ , as the expression for  $\mathcal{I}(-, d) \otimes_{\mathcal{I}} F$  can be rewritten as

eq:coend-ev

$$\text{coeq} \left( \coprod_{i \rightarrow j \rightarrow d} F(i) \rightrightarrows \coprod_{k \rightarrow d} F(k) \right).$$

- (iv) More generally, given functors  $F: \mathcal{I} \rightarrow \text{Top}$  and  $\phi: \mathcal{I} \rightarrow \mathcal{J}$ , the left Kan extension  $\phi_! F: \mathcal{J} \rightarrow \text{Top}$  (cf. Section 1.9.2) can by a similar argument be described as

$$d \mapsto \mathcal{J}(\phi(-), d) \otimes_{\mathcal{I}} F.$$

We will need the fact that coends, just like colimits, commute with each other — Using the integral notation for coends this looks as follows, and is sometimes called the “Fubini theorem”:

propn:fubini

PROPOSITION 5.3.3. For a functor  $\Phi: \mathcal{I}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{J} \times \mathcal{I} \rightarrow \mathcal{C}$  there are natural isomorphisms

$$\int^{\mathcal{I}} \int^{\mathcal{J}} \Phi \cong \int^{\mathcal{I} \times \mathcal{J}} \Phi \cong \int^{\mathcal{J}} \int^{\mathcal{I}} \Phi.$$

In particular, if  $\mathcal{C}$  admits products, given  $W: \mathcal{I}^{\text{op}} \rightarrow \mathcal{C}$ ,  $\Phi: \mathcal{I} \times \mathcal{J}^{\text{op}} \rightarrow \mathcal{C}$ , and  $F: \mathcal{J} \rightarrow \mathcal{C}$ , we have a natural isomorphism

$$(W \otimes_{\mathcal{I}} \Phi) \otimes_{\mathcal{J}} F \cong W \otimes_{\mathcal{I}} (\Phi \otimes_{\mathcal{J}} F).$$

PROOF. We have a commutative diagram<sup>6</sup>

$$\begin{array}{ccc} \coprod_{\substack{i \rightarrow i' \\ j \rightarrow j'}} \Phi(i', j', j, i) & \rightrightarrows & \coprod_{i, j \rightarrow j'} \Phi(i, j', j, i) \\ \Downarrow & & \Downarrow \\ \coprod_{i \rightarrow i', j} \Phi(i', j, j, i) & \rightrightarrows & \coprod_{i, j} \Phi(i, j, j, i). \end{array}$$

Taking the colimit of this in three different ways gives the three expressions: First, we can do the coequalizers of the two columns, to get

$$\coprod_{i \rightarrow i'} \int^{\mathcal{J}} \Phi(i', -, -, i) \rightrightarrows \coprod_i \int^{\mathcal{J}} \Phi(i, -, -, i),$$

and then take the coequalizer of this to get  $\int^{\mathcal{I}} \int^{\mathcal{J}} \Phi$ . (The first step is a left Kan extension along the functor from our original diagram shape to  $\bullet \rightrightarrows \bullet$  that collapses the columns, and the colimit of a left Kan extension is the colimit of the original diagram.) Second, we can do the same thing but interchanging rows and columns to get  $\int^{\mathcal{J}} \int^{\mathcal{I}} \Phi$ . And third, we can take the coequalizer of the four diagonal maps (by a cofinality argument), which gives  $\int^{\mathcal{I} \times \mathcal{J}} \Phi$ .  $\square$

ALTERNATIVE PROOF. Recall the twisted arrow category construction from Remark 5.3.1. Then there is a natural isomorphism of categories  $\text{Tw}(\mathcal{I}) \times \text{Tw}(\mathcal{J}) \cong \text{Tw}(\mathcal{I} \times \mathcal{J})$ , compatible along the target-source functors with the isomorphism  $(\mathcal{I}^{\text{op}} \times \mathcal{I}) \times (\mathcal{J}^{\text{op}} \times \mathcal{J}) \cong (\mathcal{I} \times \mathcal{J})^{\text{op}} \times (\mathcal{I} \times \mathcal{J})$ . We can then pullback  $\Phi$  to a functor  $\Phi': \text{Tw}(\mathcal{I}) \times \text{Tw}(\mathcal{J})$ , then the statement reduces to an instance of commutativity of colimits:

$$\text{colim}_{\text{Tw}(\mathcal{I})} \text{colim}_{\text{Tw}(\mathcal{J})} \Phi' \cong \text{colim}_{\text{Tw}(\mathcal{J})} \text{colim}_{\text{Tw}(\mathcal{I})} \Phi' \cong \text{colim}_{\text{Tw}(\mathcal{I} \times \mathcal{J})} \Phi'.$$

$\square$

**5.3.2. Homotopy coends aka the two-sided bar construction.** Now we introduce a homotopical version of coends, analogous to our construction of the homotopy colimit: Given  $\Phi: \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \text{Top}$  we define a simplicial space, denoted abusively<sup>7</sup>  $\Phi^{\Delta} = \Phi_{\bullet}^{\Delta}$ , by setting

$$\Phi_n^{\Delta} = \coprod_{i_0 \rightarrow \cdots \rightarrow i_n} \Phi(i_n, i_0) = \coprod_{i: [n] \rightarrow \mathcal{I}} \Phi(i(n), i(0)).$$

<sup>6</sup>The indexing category of this diagram is  $(\bullet \rightrightarrows \bullet) \times (\bullet \rightrightarrows \bullet)$ , in particular it has 4 objects and 12 non-identity morphisms, 4 of which (the ‘‘diagonal’’ ones) are not represented in the picture.

<sup>7</sup>For a generic functor  $F$  we have already introduced another simplicial space  $F_{\bullet}^{\Delta}$ , without using that the source category of  $F$  has the form  $\mathcal{I}^{\text{op}} \times \mathcal{I}$ .

For  $\phi: [m] \rightarrow [n]$  in  $\Delta$ , the structure map  $\phi^*: \Phi_n^\Delta \rightarrow \Phi_m^\Delta$  is given on the component  $\Phi(i_n, i_0)$  indexed by  $i_0 \rightarrow \cdots \rightarrow i_n$  by the map  $\Phi(i_n, i_0) \rightarrow \Phi(i_{\phi(m)}, i_{\phi(0)})$  with the target in the component indexed by  $i_{\phi(0)} \rightarrow \cdots \rightarrow i_{\phi(m)}$ . In other words, we use the map

$$\Phi(i(\phi(m) \leq n), i(0 \leq \phi(0))): \Phi(i(n), i(0))_{i: [n] \rightarrow \mathcal{I}} \rightarrow \Phi(i(\phi(m)), i(\phi(0)))_{i \circ \phi: [m] \rightarrow \mathcal{I}}.$$

The *homotopy coend* of  $\Phi$  is then the realization

$$\text{hocoend}(\Phi) = |\Phi_\bullet^\Delta|.$$

Just as the homotopy colimit is a “fattened” version of the ordinary colimit, this is a “fattened” version of the colimit of the simplicial diagram  $\Phi_\bullet^\Delta$ , which is the coend of  $\Phi$  by Proposition 5.1.7.

We are interested in the homotopy version of the functor tensor product; for historical reasons this has a special name:

DEFINITION 5.3.4. For functors  $F: \mathcal{I} \rightarrow \text{Top}$  and  $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$ , the *two-sided simplicial bar construction*  $\mathbb{B}_\bullet(W, \mathcal{I}, F)$  is the simplicial space  $(W \times F)^\Delta$ . The *two-sided bar construction*  $B(W, \mathcal{I}, F)$  is the geometric realization  $|\mathbb{B}_\bullet(W, \mathcal{I}, F)|$ , i.e.,

$$B(W, \mathcal{I}, F) = |(W \times F)^\Delta|$$

EXAMPLE 5.3.5.  $B(*, \mathcal{I}, *) = B\mathcal{I}$ . More generally  $B(*, \mathcal{I}, F) = \text{hocolim } F$ , as  $\mathbb{B}_\bullet(*, \mathcal{I}, F)$  is the simplicial space we previously denoted  $F^\Delta$ .

DEFINITION 5.3.6. As suggested by Example 5.3.2(iv), we define the *homotopy Kan extension* of  $F: \mathcal{I} \rightarrow \text{Top}$  along  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  using the two-sided bar construction as

$$d \mapsto B(\mathcal{J}(\phi(-), d), \mathcal{I}, F).$$

We will abbreviate this by  $\phi_! F: \mathcal{J} \rightarrow \text{Top}$ .

We will now prove some formal properties of the two-sided bar construction. First of all, from our result on the homotopy invariance of geometric realizations we immediately get:

PROPOSITION 5.3.7 (Homotopy invariance of the bar construction). *For any functor  $W: \mathcal{I}^{\text{op}} \rightarrow \text{CW}$ , the functor  $B(W, \mathcal{I}, -): \text{CW}^\mathcal{I} \rightarrow \text{CW}$  preserves weak equivalences, and similarly in the other variable.*

PROOF. This follows from Theorem 5.2.7, just as in the proof of Theorem 5.2.1:  $\mathbb{B}_\bullet(W, \mathcal{I}, F)$  is always Reedy cofibrant, and for a (objectwise) weak equivalence  $\eta: F \rightarrow F'$  of functors  $F, F' \in \text{CW}^\mathcal{I}$ , the map of simplicial spaces  $\mathbb{B}_\bullet(W, \mathcal{I}, \eta)$  is a level-wise weak equivalence.  $\square$

LEMMA 5.3.8. *The two-sided bar construction commutes with functor tensor products in each variable, i.e. given functors  $W: \mathcal{J} \times \mathcal{I}^{\text{op}} \rightarrow \text{Top}$ ,  $F: \mathcal{I} \times \mathcal{K}^{\text{op}} \rightarrow \text{Top}$ ,  $\Phi: \mathcal{J}^{\text{op}} \rightarrow \text{Top}$ , and  $\Psi: \mathcal{K} \rightarrow \text{Top}$ , there is a canonical isomorphism*

$$\Phi \otimes_{\mathcal{J}} B(W, \mathcal{I}, F) \otimes_{\mathcal{K}} \Psi \cong B(\Phi \otimes_{\mathcal{I}} W, \mathcal{I}, F \otimes_{\mathcal{K}} \Psi).$$

Here we consider, for instance,  $B(W, \mathcal{I}, F)$  as a functor  $\mathcal{J} \times \mathcal{K}^{\text{op}} \rightarrow \text{Top}$ , and  $\Phi \otimes_{\mathcal{I}} W$  as a functor  $\mathcal{I}^{\text{op}} \rightarrow \text{Top}$ .

homotopy-kan

homotopy-inv

barandtens

PROOF. The geometric realization commutes with colimits and products, so this amounts to expanding everything out and commuting some colimits, and is left as exercise.  $\square$

def:confusing

DEFINITION 5.3.9. For convenience, we will use the standard (but perhaps slightly confusing!) notation  $B(\mathcal{I}, \mathcal{I}, F) : \mathcal{I} \rightarrow \text{Top}$  for the functor  $i \mapsto B(\mathcal{I}(-, i), \mathcal{I}, F)$ , and  $B(W, \mathcal{I}, \mathcal{I}) : \mathcal{I}^{\text{op}} \rightarrow \text{Top}$  for  $i \mapsto B(W, \mathcal{I}, \mathcal{I}(i, -))$ .<sup>8</sup>

We use similar notation for (non-derived) functor tensor products, and note that  $\mathcal{I} \otimes_{\mathcal{I}} F \cong F$ , and  $W \otimes_{\mathcal{I}} \mathcal{I} \cong W$ .

By the above, we can think of the functors  $B(\mathcal{I}, \mathcal{I}, F)$  and  $B(W, \mathcal{I}, \mathcal{I})$  as “fattened up” versions of  $F$  and  $W$ . These functors have an alternative description that will be important to us:

Bmanipulation

LEMMA 5.3.10. *Let  $F : \mathcal{I} \rightarrow \text{Top}$ . There are natural isomorphisms of spaces*

$$B(\mathcal{I}(-, i), \mathcal{I}, F) \cong B(*, \mathcal{I}_{/i}, F) \cong \text{hocolim}_{\mathcal{I}_{/i}} F$$

and hence an equivalence of functors

$$B(\mathcal{I}, \mathcal{I}, F) \cong (i \mapsto \text{hocolim}_{\mathcal{I}_{/i}} F) = QF$$

where  $Q$  was introduced in Definition 5.1.22. Similarly, for  $W : \mathcal{I}^{\text{op}} \rightarrow \text{Top}$ ,

$$B(W, \mathcal{I}, \mathcal{I}) \cong (i \mapsto B(W, \mathcal{I}_{/i}, *)) \cong (i \mapsto \text{hocolim}_{(\mathcal{I}_{/i})^{\text{op}}} W) = (i \mapsto \text{hocolim}_{(\mathcal{I}^{\text{op}})_{/i}} W).$$

In particular  $B(\mathcal{I}, \mathcal{I}, *) \cong B(\mathcal{I}_{/-}) = |N(\mathcal{I}_{/-})|$  as functors  $\mathcal{I} \rightarrow \text{Top}$ , and similarly  $B(*, \mathcal{I}, \mathcal{I}) \cong B(\mathcal{I}_{/-})$ .

PROOF. Expanding out the definitions, we observe that the simplicial spaces  $\mathbb{B}_{\bullet}(\mathcal{I}(-, i), \mathcal{I}, F)$  and  $\mathbb{B}_{\bullet}(*, \mathcal{I}_{/i}, F)$  are isomorphic, as the space of  $n$ -simplices in both cases is  $\coprod_{i_0 \rightarrow \dots \rightarrow i_n \rightarrow i} F(i_0)$  (and structure maps are compatible). Similarly in the other case. The final statements are obtained by taking  $F = W = *$ .  $\square$

Using our preparation, we can now give several alternative descriptions of the homotopy colimit, including the expression  $\text{hocolim}_{\mathcal{I}} F \cong \text{colim}_{\mathcal{I}} QF$ , first sketched in Proposition 5.1.24.

a:colimQ-again

PROPOSITION 5.3.11. *For a functor  $F : \mathcal{I} \rightarrow \text{Top}$ , there are canonical isomorphisms*

$$\text{hocolim}_{\mathcal{I}} F \cong B(\mathcal{I}_{/-}) \otimes_{\mathcal{I}} F \cong \text{colim}_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) \cong \text{colim}_{i \in \mathcal{I}} \left( \text{hocolim}_{\mathcal{I}_{/i}} F \right).$$

and the canonical map  $\text{hocolim}_{\mathcal{I}} F \rightarrow \text{colim}_{\mathcal{I}} F$  identifies with the projection map

$$\text{hocolim}_{\mathcal{I}} F \cong B(\mathcal{I}_{/-}) \otimes_{\mathcal{I}} F \rightarrow * \otimes_{\mathcal{I}} F = \text{colim}_{\mathcal{I}} F$$

<sup>8</sup>A better but heavier notation could be  $B(\text{Hom}_{\mathcal{I}}, \mathcal{I}, F)$  and  $B(W, \mathcal{I}, \text{Hom}_{\mathcal{I}})$ , considering  $\text{Hom}_{\mathcal{I}}$  as a functor  $\mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \text{Set} \subset \text{Top}$ .

PROOF. We have a chain of isomorphisms

$$\mathrm{hocolim} F \cong B(*, \mathcal{I}, F) \cong B(*, \mathcal{I}, \mathcal{I} \otimes_{\mathcal{I}} F) \cong B(*, \mathcal{I}, \mathcal{I}) \otimes_{\mathcal{I}} F \cong B(\mathcal{I}_{-}) \otimes_{\mathcal{I}} F.$$

Here the first isomorphism is by comparing the original definition of  $F^{\Delta}$  with the abused definition of  $(* \times F)^{\Delta}$ , where  $*$ :  $\mathcal{I}^{\mathrm{op}} \rightarrow \mathrm{Top}$  is the constant functor: the two simplicial spaces are isomorphic. The second is the observation in Definition 5.3.9. The third uses Lemma 5.3.8. And the fourth is Lemma 5.3.10. This establishes the first isomorphism in the proposition.

On the other hand, we also have a chain of isomorphisms

$$B(*, \mathcal{I}, F) \cong B(* \otimes_{\mathcal{I}} \mathcal{I}, \mathcal{I}, F) \cong * \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) \cong \mathrm{colim}_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F)$$

now using the observation in Definition 5.3.9, Lemma 5.3.8, and Example 5.3.2(i). This establishes the second isomorphism of the proposition, from which the third follows, using Lemma 5.3.10.

That the map  $\mathrm{hocolim}_{\mathcal{I}} F \rightarrow \mathrm{colim} F$  has the prescribed form also follows from the definitions (check it as exercise!).  $\square$

We now prove an associativity result for two-sided bar construction, analogous to the associativity of the functor tensor product of Proposition 5.3.3, that we will use several times later:

PROPOSITION 5.3.12. *Suppose we are given functors  $F: \mathcal{J} \rightarrow \mathrm{Top}$ ,  $\Phi: \mathcal{I} \times \mathcal{J}^{\mathrm{op}} \rightarrow \mathrm{Top}$  and  $W: \mathcal{I}^{\mathrm{op}} \rightarrow \mathrm{Top}$ . Then there is a natural isomorphism*

$$B(W, \mathcal{I}, B(\Phi, \mathcal{J}, F)) \cong B(B(W, \mathcal{I}, \Phi), \mathcal{J}, F).$$

PROOF. For a generic bisimplicial space  $Y: \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \rightarrow \mathrm{Top}$ , one can define two simplicial spaces  $Y'$  and  $Y''$  by taking geometric realization along one of the two simplicial coordinates, i.e.  $Y': [n] \mapsto |Y([n], -)|$  and  $Y'': [m] \mapsto |Y(-, [m])|$ . One can then check

$$|Y'| \cong |Y''| \cong \left( \coprod_{n, m \geq 0} Y([n], [m]) \times \Delta_{\mathrm{top}}^n \times \Delta_{\mathrm{top}}^m \right) / \sim,$$

where  $\sim$  is generated by identifying, for all  $\phi: [n] \rightarrow [n']$ ,  $\psi: [m] \rightarrow [m']$ ,  $y \in Y([n'], [m'])$ ,  $s \in \Delta_{\mathrm{top}}^n$  and  $t \in \Delta_{\mathrm{top}}^m$ , the points  $((\phi, \psi)^*(y), (s, t)) \sim (y, \phi_*(s), \psi_*(t))$ . This is an instance of the ‘‘Fubini theorem’’, Proposition 5.3.3, with  $\mathcal{I} = \mathcal{J} = \Delta$  and  $\Phi: \mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \times \mathcal{I} \rightarrow \mathrm{Top}$  given by  $\Delta_{\mathrm{top}}^{\bullet} \times Y \times \Delta_{\mathrm{top}}^{\bullet}$ .

We apply the previous to the bisimplicial space  $X: \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \rightarrow \mathrm{Top}$  given by

$$\begin{aligned} X_{n, m} = X([n], [m]) &= \coprod_{\substack{i: [n] \rightarrow \mathcal{I} \\ j: [m] \rightarrow \mathcal{J}}} W(i(n)) \times \Phi(i(0), j(m)) \times F(j(0)) \\ &= \coprod_{\substack{i_0 \rightarrow \cdots \rightarrow i_n \\ j_0 \rightarrow \cdots \rightarrow j_m}} W(i_n) \times \Phi(i_0, j_m) \times F(j_0). \end{aligned}$$

The pair of functors  $\phi: [n] \rightarrow [n']$  and  $\psi: [m] \rightarrow [m']$  is sent to the map of spaces restricting to the map between the components indexed by  $(i, j)$  and  $(i \circ \phi, j \circ \psi)$ :

$$\begin{aligned} X(\phi, \psi) &:= W(i(\phi(n) \leq n')) \times \Psi(i(0 \leq \phi(0)), j(\psi(m) \leq m')) \times F(j(0 \leq \psi(0))) \\ &\quad : W(i(n')) \times \Phi(i(0), j(m')) \times F(j(0)) \\ &\rightarrow W(i \circ \phi(n)) \times \Phi(i \circ \phi(0), j \circ \psi(m)) \times F(j \circ \psi(0)). \end{aligned}$$

The two sides in the isomorphism of the statement are  $|X'|$  and  $|X''|$ , respectively.  $\square$

As a final preliminary, we record the following restatement and extension of Proposition 5.1.23:

lem:BIweq

PROPOSITION 5.3.13.

- (i) For a functor  $F: \mathcal{I} \rightarrow \text{Top}$ , there is a natural transformation  $B(\mathcal{I}, \mathcal{I}, F) \rightarrow F$  that is a levelwise homotopy equivalence; moreover we have a natural isomorphism  $QF \cong B(\mathcal{I}, \mathcal{I}, F)$ .
- (ii) For a functor  $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$ , there is a natural transformation  $B(W, \mathcal{I}, \mathcal{I}) \rightarrow W$  that is a levelwise homotopy equivalence.

PROOF. For (i), by Lemma 5.3.10 we can identify  $B(\mathcal{I}(-, i), \mathcal{I}, F)$  with  $B(*, \mathcal{I}_{/i}, F)$  and with  $\text{hocolim}_{\mathcal{I}_{/i}} F$ , naturally in  $i \in \mathcal{I}$ , and the map we want is just the canonical map  $\text{hocolim}_{\mathcal{I}_{/i}} F \rightarrow F(i)$ , which is a homotopy equivalence by Proposition 5.1.19. In (ii), we can similarly use the identification of  $B(W, \mathcal{I}, \mathcal{I}(i, -))$  with  $B(W, \mathcal{I}_{i/}, *)$  and  $\text{hocolim}_{(\mathcal{I}_{i/})^{\text{op}}} W$ , note that  $(\mathcal{I}_{i/})^{\text{op}}$  is the same as  $(\mathcal{I}^{\text{op}})_{/i}$ , and use again Proposition 5.1.19 to conclude that the canonical map  $\text{hocolim}_{(\mathcal{I}^{\text{op}})_{/i}} W \rightarrow W(i)$  is a homotopy equivalence.  $\square$

REMARK 5.3.14. Suppose  $X$  is a simplicial space. Then we now have a description of the homotopy colimit of  $X$  as  $B(\Delta_{/-}) \otimes_{\Delta^{\text{op}}} X$ . On the other hand, we can write the geometric realization as  $\Delta_{\text{top}}^{\bullet} \otimes_{\Delta^{\text{op}}} X$ .

A variant of our proof of homotopy invariance for geometric realizations implies that if  $X$  is Reedy cofibrant then the functor  $- \otimes_{\Delta^{\text{op}}} X$  preserves weak equivalences between cosimplicial CW complexes that satisfy an analogue of the Reedy cofibrancy condition: the map from the  $n^{\text{th}}$  “matching object” (which is obtained as the colimit over all the face maps into  $[n]$ ; e.g. for  $\Delta_{\text{top}}^{\bullet}$  we get  $\partial\Delta_{\text{top}}^n$ ) into the  $n^{\text{th}}$  space is a cofibration. This holds for the cosimplicial spaces  $\Delta_{\text{top}}^{\bullet}$  and  $B(\Delta_{/-})$ ; using this we can show that if  $X$  is a Reedy cofibrant simplicial space then  $\text{hocolim}_{\Delta^{\text{op}}} X$  is weakly equivalent to  $|X|$ .

#### 5.4. The homotopy colimit as a derived functor

We are now ready to prove that the homotopy colimit is indeed a derived functor. Recall that we proved in Proposition 5.1.23 (and again in Proposition 5.3.13) that  $(Q, q)$  is a deformation of  $\text{CW}^{\mathcal{I}}$ . Recall also that we proved in Lemma 5.3.10, that the functor  $Q: \text{Top}^{\mathcal{I}} \rightarrow \text{Top}^{\mathcal{I}}$  (and hence its restriction to  $\text{CW}^{\mathcal{I}}$ ) identifies with  $B(\mathcal{I}, \mathcal{I}, -)$ . And we proved in Theorem 5.2.1 that  $\text{hocolim}_{\mathcal{I}} = \text{colim}_{\mathcal{I}} \circ Q: \text{CW}^{\mathcal{I}} \rightarrow \text{CW}$  respects weak equivalences, i.e., that  $\text{hocolim}_{\mathcal{I}}$  is a homotopical functor.

To prove that  $\text{hocolim}_{\mathcal{I}}$  is a left derived functor of  $\text{colim}_{\mathcal{I}}$ , by Proposition 4.2.8 we are just left with establishing the following proposition:

im-agree-on-Q

PROPOSITION 5.4.1. *The map  $\text{colim}_{\mathcal{I}}(q_{QF}): \text{colim}_{\mathcal{I}} Q(QF) \rightarrow \text{colim}_{\mathcal{I}}(QF)$  is a weak equivalence of spaces for any  $F \in \text{CW}^{\mathcal{I}}$ .*

Once this proposition is proved, it is easy to check that  $\text{colim}_{\mathcal{I}}$  is homotopical on the essential image of  $Q$ : given  $F, G \in \text{CW}^{\mathcal{I}}$  and a weak equivalence  $\eta: QF \rightarrow QG$ , we have that  $\text{colim}_{\mathcal{I}}(\eta)$  is a weak equivalence by 3-of-4 in the following commutative square, where we use that  $\text{colim}_{\mathcal{I}} \circ Q$  is known to be homotopical:

$$\begin{array}{ccc} \text{colim}_{\mathcal{I}} Q(QF) & \xrightarrow[\simeq]{\text{colim}_{\mathcal{I}} q_{QF}} & \text{colim}_{\mathcal{I}}(QF) \\ \simeq \downarrow \text{colim}_{\mathcal{I}} Q(\eta) & & \downarrow \text{colim}_{\mathcal{I}}(\eta) \\ \text{colim}_{\mathcal{I}} Q(QG) & \xrightarrow[\simeq]{\text{colim}_{\mathcal{I}} q_{QG}} & \text{colim}_{\mathcal{I}}(QG) \end{array}$$

PROOF OF PROPOSITION 5.4.1. This will also follow from what we have proven so far, by writing out what this map is. By Proposition 5.3.11, we are considering the map

$$\text{hocolim}_{\mathcal{I}} QF = B(\mathcal{I}_{-j}) \otimes_{\mathcal{I}} QF \rightarrow * \otimes QF = \text{colim}_{\mathcal{I}} QF$$

induced by the projection  $B(\mathcal{I}_{-j}) \rightarrow *$ . But this can now be rewritten as

$$\begin{array}{ccccc} B(\mathcal{I}_{-j}) \otimes_{\mathcal{I}} QF & \cong & B(\mathcal{I}_{-j}) \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) & \cong & B(B(\mathcal{I}_{-j}), \mathcal{I}, F) \\ \downarrow & & \downarrow & & \downarrow \\ * \otimes_{\mathcal{I}} QF & \cong & * \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) & \cong & B(*, \mathcal{I}, F) \end{array}$$

using the observations of Lemma 5.3.8 and Definition 5.3.9. However, the right-hand vertical map is a homotopy equivalence by Proposition 5.3.7, as  $B\mathcal{I}_{i/j} \rightarrow *$  is a weak equivalence for all  $i$ , showing the claim.  $\square$

REMARK 5.4.2. The same argument shows that for any  $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$ , the functor

$$B(W, \mathcal{I}, -) \cong W \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, -)$$

is a left derived functor of  $W \otimes_{\mathcal{I}} -$ . (Here we use in the last step that  $B(W, \mathcal{I}, \mathcal{I}) \rightarrow W$  is a natural weak equivalence for any  $W$ , not just for  $W = *$ .)

Let us summarize what we have proved:

colim-derived

THEOREM 5.4.3. *The functor  $\text{hocolim}_{\mathcal{I}}(-) = \text{colim}_{\mathcal{I}}(Q(-))$ , is a left derived functor of  $\text{colim}_{\mathcal{I}}$ , and the induced functor  $\text{Ho}(\text{CW}^{\mathcal{I}}) \rightarrow \text{Ho}(\text{CW})$  is left adjoint to the diagonal functor  $\delta: \text{Ho}(\text{CW}) \rightarrow \text{Ho}(\text{CW}^{\mathcal{I}})$ .*

*Replacing  $Q$  by  $Q \circ |\text{Sing}_{\bullet}(-)|$  the same statements hold with  $\text{Top}$  in place of  $\text{CW}$ .*

PROOF. To see that it is a derived functor we want to check that the assumptions of Proposition 4.2.8 are satisfied:

— By Proposition 5.1.23 (or Proposition 5.3.13),  $(Q, q)$  is a deformation of  $\text{CW}^{\mathcal{I}}$ .

— By Theorem 5.2.1,  $\text{hocolim}_I = \text{colim}_I Q : \text{CW}^{\mathcal{I}} \rightarrow \text{CW}$  respects weak equivalences, i.e., it is homotopical.

— By Proposition 5.4.1 the map “ $\text{colim}_{\mathcal{I}} q_{QF}$ ” is a weak equivalence of CW complexes for all  $F \in \text{CW}^{\mathcal{I}}$ , and this in turn implies that  $\text{colim}_{\mathcal{I}}$  is homotopical on the essential image of  $Q$ .

This proves by Proposition 4.2.8 that  $\text{hocolim}$  is a left derived functor of  $\text{colim}$ . That the induced functor on homotopy categories is left adjoint to the diagonal follows formally from Proposition 4.2.9, noting that  $\delta : \text{CW} \rightarrow \text{CW}^{\mathcal{I}}$  is already homotopy invariant, so we can take “ $R$ ” to be the identity.

We have also already remarked in Remark 5.1.1 that  $Q \circ |\text{Sing}_{\bullet}(-)|$  is a deformation on  $\text{Top}^{\mathcal{I}}$  with the wanted properties. (And in fact, with more effort in the proof, we can keep  $Q$  unchanged.)  $\square$

### 5.5. Cofinality for homotopy colimits and Quillen’s Theorem A

The final standard property about homotopy colimits that we want to mention is cofinality. Given a functor  $\phi : \mathcal{I} \rightarrow \mathcal{J}$  it relates homotopy colimits indexed by  $\mathcal{J}$  to those indexed by  $\mathcal{I}$ . For this we recall the definition of the undercategory, already given in Definition 1.9.1.

DEFINITION 5.5.1. For  $\phi : \mathcal{I} \rightarrow \mathcal{J}$  and  $j \in \mathcal{J}$  we define the undercategory of  $\phi$  under  $j$  to be  $\mathcal{I}_{j/\phi} := \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$ . In other words, the category  $\mathcal{I}_{j/\phi}$  has objects pairs  $(i \in \mathcal{I}, (j \rightarrow \phi(i)) \in \mathcal{J})$  and morphisms  $(i, j \rightarrow \phi(i)) \rightarrow (i', j \rightarrow \phi(i'))$  are given by morphisms  $f : i \rightarrow i'$  such that the following diagram commutes

$$\begin{array}{ccc} & j & \\ & \swarrow & \searrow \\ \phi(i) & \xrightarrow{\phi(f)} & \phi(i'). \end{array}$$

We may sometimes suppress  $\phi$  from the notation  $\mathcal{I}_{j/\phi}$ , in particular when  $\phi$  is an inclusion of categories, hoping that this will not cause confusion with the ordinary undercategory, and write  $\mathcal{I}_{j/}$ .

Overcategories  $\mathcal{I}_{\phi/j}$  are defined dually, for a functor  $\phi : \mathcal{I} \rightarrow \mathcal{J}$  and  $j \in \mathcal{J}$ .

Before getting to homotopy cofinality, we discuss cofinality for ordinary colimits.

#### 5.5.1. Cofinality for colimits.

DEFINITION 5.5.2. A functor  $\phi : \mathcal{I} \rightarrow \mathcal{J}$  between small categories is *cofinal* if for every  $j \in \mathcal{J}$  the undercategory  $\mathcal{I}_{j/\phi}$  is non-empty and connected, i.e. we can connect any two objects by a finite zig-zag of morphisms; or, equivalently, the classifying space  $B\mathcal{I}_{j/\phi}$  is non-empty and connected.

EXERCISE 5.5.3. Prove that for a small (non-empty!) category  $\mathcal{C}$  it is indeed equivalent to require either that (1)  $B\mathcal{C}$  is a connected topological space, or (2) the equivalence relation on  $\text{Ob}(\mathcal{C})$  generated by  $x \sim y$  whenever there is a morphism  $x \rightarrow y$  admits a unique equivalence class.



colim-cofinal

PROPOSITION 5.5.4. *If a functor  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  is cofinal, then for every diagram  $F: \mathcal{J} \rightarrow \mathcal{C}$  the natural map  $\operatorname{colim}_{\mathcal{I}} F\phi \rightarrow \operatorname{colim}_{\mathcal{J}} F$  is an isomorphism (assuming either colimit exists, in which case both colimits do exist).*

PROOF. The reader should easily be able to convince themselves that this is true, by directly manipulating the definitions, but we give a formal proof in preparation for homotopy cofinality; for the following argument we assume that  $\mathcal{C}$  is equal to  $\mathbf{Top}$  (or at least it has finite products, coproducts, and is endowed with a preferred functor  $\mathbf{Set} \rightarrow \mathcal{C}$ , all satisfying convenient properties that are implicit in our manipulations):

Recall from Example 5.3.2(iii) that  $\mathcal{J}(-, \phi(i)) \otimes_{\mathcal{J}} F \cong F(\phi(i))$  for any  $i \in \mathcal{I}$ . Thus we can write the functor  $F \circ \phi$  as  $\mathcal{J}(-, \phi(-)) \otimes_{\mathcal{J}} F$ , giving

$$\operatorname{colim}_{\mathcal{I}} F\phi \cong * \otimes_{\mathcal{I}} F\phi \cong * \otimes_{\mathcal{I}} (\mathcal{J}(-, \phi(-)) \otimes_{\mathcal{J}} F),$$

by Example 5.3.2(i).

Now using associativity for the tensor product of functors, Proposition 5.3.3, we see that this is isomorphic to  $(* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))) \otimes_{\mathcal{J}} F$ . For  $j \in \mathcal{J}$ , the colimit  $\operatorname{colim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) = * \otimes_{\mathcal{I}} \mathcal{J}(j, \phi(-))$  is given by the coequalizer of

$$\coprod_{i \rightarrow i'} \mathcal{J}(j, \phi(i)) \rightrightarrows \coprod_i \mathcal{J}(j, \phi(i)).$$

But we can identify this with

$$\operatorname{Mor}(\mathcal{I}_{j/\phi}) \rightrightarrows \operatorname{Ob}(\mathcal{I}_{j/\phi})$$

where the two maps send a morphism to its source and target object. The coequalizer is thus the quotient of  $\operatorname{Ob}(\mathcal{I}_{j/\phi})$  by the equivalence relation  $\sim$  generated by  $x \sim y$  if there is a morphism from  $x$  to  $y$ . This quotient is precisely the set of connected components of  $\mathcal{I}_{j/\phi}$  (or equivalently of the space  $B\mathcal{I}_{j/\phi}$ ). Thus if  $\phi$  is cofinal we get that  $* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))$  is the constant functor  $*$ , and so

$$\operatorname{colim}_{\mathcal{I}} F\phi \cong (* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))) \otimes_{\mathcal{J}} F \cong * \otimes_{\mathcal{J}} F \cong \operatorname{colim}_{\mathcal{J}} F. \quad \square$$

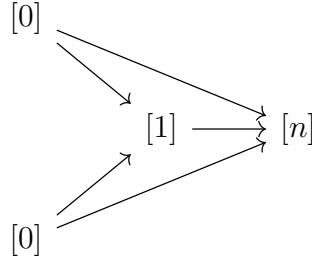
REMARK 5.5.5. In fact, this is an if and only if statement: the functors that induce isomorphisms on all colimits are precisely the cofinal functors. To see this, take  $F$  to be  $\mathcal{J}(j, -): \mathcal{J} \rightarrow \mathbf{Set}$ ; then as we saw above  $\operatorname{colim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) \cong \pi_0 B\mathcal{I}_{j/\phi}$  whereas  $\operatorname{colim}_{\mathcal{J}} \mathcal{J}(j, -) \cong \pi_0 B\mathcal{J}_{j/\operatorname{id}_{\mathcal{J}}} \cong *$ .

This criterion can be used to prove a lot of assertions in the literature that various colimits are “obviously” the same. Here is an example we have made use of already:

LEMMA 5.5.6. *The inclusion  $\Delta_{\leq 1}^{\operatorname{op}} \hookrightarrow \Delta^{\operatorname{op}}$  is cofinal.*

PROOF. We need to show that for every  $[n] \in \Delta^{\operatorname{op}}$ , the category  $(\Delta_{\leq 1}^{\operatorname{op}})_{[n]/} \cong ((\Delta_{\leq 1})_{[n]})^{\operatorname{op}}$  is connected, hence we can equivalently show that  $(\Delta_{\leq 1})_{[n]}$  is connected. For this note that every map  $[1] \rightarrow [n]$  is connected to a map  $[0] \rightarrow [n]$  by precomposing with either of the two maps  $[0] \rightarrow [1]$ . Furthermore, any two maps

$[0] \rightarrow [n]$  are connected by a zigzag



where if the top map  $[0] \rightarrow [n]$  takes 0 to  $i$ , and the bottom map  $[0] \rightarrow [n]$  takes 0 to  $j$ , then the map  $[1] \rightarrow [n]$  is given by sending  $(0, 1)$  to  $(i, j)$ , assuming without loss of generality that  $i \leq j$ .  $\square$

**5.5.2. Cofinality for homotopy colimits.** Now we want to prove an analogous result about homotopy colimits:

DEFINITION 5.5.7. A functor  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  between small categories is *homotopy cofinal* if for every  $j \in \mathcal{J}$  the category  $\mathcal{I}_{j/\phi} := \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_j$  is weakly contractible, i.e. its classifying space  $B\mathcal{I}_{j/\phi}$  is contractible.

Let us record the following lemma

LEMMA 5.5.8. *For a functor  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  of small categories we have an isomorphism of topological spaces*

$$B(*, \mathcal{I}, \mathcal{J}(j, \phi(-))) \cong B\mathcal{I}_{j/\phi},$$

which is natural in  $j$ .

PROOF. Similar to Lemma 5.3.10, this is seen by noting that the two spaces are the geometric realization of isomorphic simplicial sets, having in particular as  $n$ -simplices the set of pairs  $((i: [n] \rightarrow \mathcal{I}) \in \mathcal{I}^{[n]}, (j \rightarrow \phi(i(0)) \in \mathcal{J}^{[1]})$ .  $\square$

THEOREM 5.5.9 (Cofinality for hocolim). *Suppose  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  is a homotopy cofinal functor. Then for every functor  $F: \mathcal{J} \rightarrow \text{Top}$  taking values in CW complexes, the natural map*

$$\text{hocolim}_{\mathcal{I}} F\phi \rightarrow \text{hocolim}_{\mathcal{J}} F$$

is a weak equivalence.

Before the proof let us note a very important special case, obtained by applying Theorem 5.5.9 to the constant functor  $\mathcal{J} \rightarrow \text{Top}$  with value  $*$ :

COROLLARY 5.5.10 (Quillen's Theorem A). *Suppose  $F: \mathcal{I} \rightarrow \mathcal{J}$  is a homotopy cofinal functor. Then the induced map on classifying spaces  $BF: B\mathcal{I} \rightarrow B\mathcal{J}$  is a weak equivalence.*  $\square$

PROOF OF THEOREM 5.5.9. Our proof will be a homotopical version of the analogous proof for colim:

Recall first that we have a weak equivalence  $B(\mathcal{J}(-, j), \mathcal{J}, F) \rightarrow F(j)$ , natural in  $j \in \mathcal{J}$ , by Lemma 5.3.10. Hence we get a weak equivalence

$$\operatorname{hocolim}_{\mathcal{I}} F\phi \cong B(*, \mathcal{I}, F\phi) \xleftarrow{\sim} B(*, \mathcal{I}, B(\mathcal{J}(-, \phi(-)), \mathcal{J}, F)).$$

by homotopy invariance of the two-sided bar construction, Proposition 5.3.7. Now using associativity for the bar construction, Proposition 5.3.12, we get an isomorphism between this and the space  $B(B(*, \mathcal{I}, \mathcal{J}(-, \phi(-))), \mathcal{J}, F)$ . But by Lemma 5.5.8 we can again rewrite this as  $B(B\mathcal{I}_{-/ \phi}, \mathcal{J}, F)$ . So we can describe the map in the theorem as

$$\operatorname{hocolim}_{\mathcal{I}} F\phi \cong B(B\mathcal{I}_{-/ \phi}, \mathcal{J}, F) \rightarrow B(*, \mathcal{J}, F) \cong \operatorname{hocolim}_{\mathcal{J}} F,$$

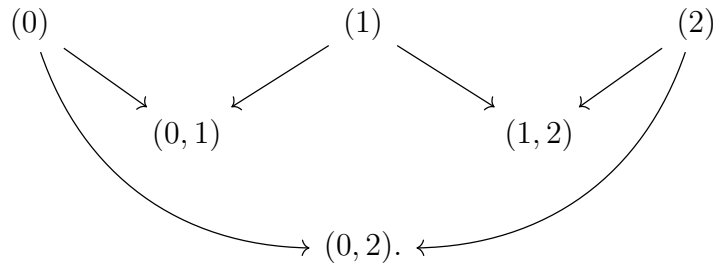
But now by homotopy invariance of the bar construction again (Proposition 5.3.7), the map is a weak equivalence if  $\phi$  is homotopy cofinal, since this means that we have a weak equivalence  $B\mathcal{I}_{-/ \phi} \simeq *$  of functors  $\mathcal{J}^{\text{op}} \rightarrow \text{CW}$ .  $\square$

REMARK 5.5.11. Again this is an if and only if statement: the functors that induce weak equivalences on all homotopy colimits are precisely the homotopy cofinal functors. As before, we see this by taking  $F$  to be  $\mathcal{J}(j, -)$ ; then we have  $\operatorname{hocolim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) \cong B\mathcal{I}_{j/\phi}$  whereas  $\operatorname{hocolim}_{\mathcal{J}} \mathcal{J}(j, -) \cong B\mathcal{J}_j \cong *$ , by Lemma 5.5.8.

EXAMPLE 5.5.12. Consider the simplex category  $\Delta\mathcal{I}$  with objects  $n$ -simplices  $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n$  and morphisms given by face and degeneracy maps. Formally,  $\Delta\mathcal{I}$  is the category of elements  $\text{el}(\text{Fun}(-, \mathcal{I}))$  associated with the functor  $\text{Fun}(-, \mathcal{I}) : \Delta^{\text{op}} \rightarrow \text{Set}$  which in turns sends  $[n]$  to the set of functors  $\text{Fun}([n], \mathcal{I})$ : see Definition 1.9.6.

The functor  $\phi : \Delta\mathcal{I} \rightarrow \mathcal{I}$  given by  $(i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n) \mapsto i_0$  is cofinal, and also homotopy cofinal, as for all  $i \in \mathcal{I}$  the undercategory  $(\Delta\mathcal{I})_{i/\phi}$  has as initial object “ $i \rightarrow i$ ”, which is formally the pair  $(([0] \rightarrow \mathcal{I}, 0 \mapsto i), i \xrightarrow{\text{id}_i} i)$ .

REMARK 5.5.13. The inclusion  $\Delta_{\leq 1}^{\text{op}} \hookrightarrow \Delta^{\text{op}}$  is *not* homotopy cofinal — the slice category  $(\Delta_{\leq 1})_{/[2]}$  is connected, but not simply connected: we get a non-trivial loop in  $B(\Delta_{\leq 1})_{/[2]}$  via the diagram<sup>9</sup>



<sup>9</sup>Do you notice a similarity between this diagram and the barycentric subdivision of  $\partial\Delta^2$ ?

More generally, it can be shown that for  $\Delta_{\leq n}^{\text{op}} \hookrightarrow \Delta^{\text{op}}$  the slice categories are  $(n-1)$ -connected, but *not*  $n$ -connected, so in order to compute homotopy colimits we cannot restrict to any finite part of  $\Delta^{\text{op}}$ .

EXERCISE 5.5.14. Identify the homotopy type of  $(\Delta_{\leq 1}^{\text{op}})_{/[2]}$  and more generally of  $(\Delta_{\leq n}^{\text{op}})_{/[m]}$ .

### 5.6. The Grothendieck construction as a homotopy colimit

As an application of the machinery we have set up, in this section we will give a more explicit description of homotopy colimits for diagrams of spaces  $\mathcal{I} \rightarrow \text{Top}$  that come from diagrams of categories  $\mathcal{I} \rightarrow \text{Cat}$  by taking classifying spaces. First we must introduce some notation:

DEFINITION 5.6.1. Suppose  $F: \mathcal{I} \rightarrow \text{Cat}$  is a functor. The *Grothendieck construction*  $\text{Gr}(F)$  is the category with objects given by pairs  $(i \in \mathcal{I}, x \in F(i))$ , and morphisms  $(i, x) \rightarrow (i', x')$  given by pairs of a map  $f: i \rightarrow i'$  in  $\mathcal{I}$  and a map  $\phi: F(f)(x) \rightarrow x'$  in  $F(i')$ . Note that there is an obvious projection functor  $\text{Gr}(F) \rightarrow \mathcal{I}$  that takes an object  $(i, x)$  to  $i$ .

REMARK 5.6.2. A special case of the Grothendieck construction is when the functor lands in  $\text{Set}$  (viewed as categories with no non-identity morphisms), in which case the morphism  $\phi$  above has to be the identity. In this case it is a classical construction known as the “translation category” or “category of elements” (see Definition 1.9.6). The objects consists of pairs  $(i \in \mathcal{I}, x \in F(i))$ , and a morphism is a map  $f: i \rightarrow i'$  such that  $F(f)(x) = x'$ .

We already saw an example of this special case, namely the simplex category  $\Delta\mathcal{I}$  of a category  $\mathcal{I}$ . More generally, for a simplicial set  $X: \Delta^{\text{op}} \rightarrow \text{Set}$ , one can define  $\Delta X = \text{el}(X)$  (and  $\Delta\mathcal{I}$  is nothing but  $\Delta(N\mathcal{I})$ ).

REMARK 5.6.3. It is possible to characterize the functors  $p: \mathcal{E} \rightarrow \mathcal{I}$  that arise as projections of Grothendieck constructions: namely, if  $p$  is a so-called “Grothendieck opfibration”, then there exists an essentially unique functor  $F: \mathcal{I} \rightarrow \text{Cat}$  such that  $\mathcal{E}$  is equivalent to  $\text{Gr}(F)$  over  $\mathcal{I}$ .

hm:Grishocolim

THEOREM 5.6.4 (Thomason [Tho79]). *Suppose we are given a functor  $F: \mathcal{I} \rightarrow \text{Cat}$ . Then the spaces  $\text{hocolim}_{i \in \mathcal{I}} BF(i)$  and  $B\text{Gr}(F)$  are weakly equivalent.*

EXAMPLE 5.6.5. Consider a group  $G$  acting on a collection of subgroups  $\mathcal{C}$ . We see that the classifying space of  $\mathcal{C}_G$  identifies with  $(BC)_{hG}$ . [ **Explain this example. Maybe  $\mathcal{C}$  is a category? What is  $\mathcal{C}_G$ ?** ]

We will prove Theorem 5.6.4 as a special case of a more general result that describes homotopy colimits indexed by Grothendieck constructions:

hm:hocolimofGr

THEOREM 5.6.6. *Suppose we are given a functor  $F: \mathcal{I} \rightarrow \text{Cat}$  and a functor  $\Phi: \text{Gr}(F) \rightarrow \text{Top}$ . Then we have a weak equivalence*

$$\text{hocolim}_{\text{Gr}(F)} \Phi \simeq \text{hocolim}_{i \in \mathcal{I}} \text{hocolim}_{F(i)} \Phi|_{F(i)}.$$

(Here we abusively denote “ $\Phi|_{F(i)}$ ” the composition of  $\Phi$  and the obvious functor  $F(i) \rightarrow \text{Gr}(F)$  sending  $x \mapsto (i, x)$  and  $(\phi: x \rightarrow x') \mapsto (\text{id}_i, \phi)$ .)

For the proof of Theorem 5.6.6, we first need an observation about homotopy left Kan extensions. Recall that for  $\phi: \mathcal{I} \rightarrow \mathcal{J}$  and  $F: \mathcal{I} \rightarrow \mathcal{C}$  the left Kan extension  $\phi_! F$  of  $F$  along  $\phi$  is given by

$$j \mapsto \mathcal{J}(\phi(-), j) \otimes_{\mathcal{I}} F \cong \text{colim}_{\mathcal{I}_{\phi/j}} F,$$

where  $\mathcal{I}_{\phi/j}$  denotes the fiber product category  $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}/j$  ( $\phi$  is implicit in the last notation). Similarly we define:

DEFINITION 5.6.7 (Homotopy left Kan extension). For  $F: \mathcal{I} \rightarrow \text{Top}$  we define the *homotopy left Kan extension*  $\phi_!^{\mathbb{L}} F$  of  $F$  along  $\phi$  by

$$j \mapsto B(\mathcal{J}(\phi(-), j), \mathcal{I}, F) \cong \text{hocolim}_{\mathcal{I}_{\phi/j}} F$$

Here the last rewriting is justified by expanding the definitions as in Lemma 5.5.8.

LEMMA 5.6.8. *Given functors  $\mathcal{I} \xrightarrow{\phi} \mathcal{J} \xrightarrow{\psi} \mathcal{K}$  and  $F: \mathcal{I} \rightarrow \text{Top}$ , there is a natural weak equivalence*

$$\psi_!^{\mathbb{L}} \phi_!^{\mathbb{L}} F \simeq (\psi\phi)_!^{\mathbb{L}} F.$$

PROOF. For  $k \in \mathcal{K}$ , the left-hand side is the iterated bar construction

$$B(\mathcal{K}(\psi(-), k), \mathcal{J}, B(\mathcal{J}(\phi(-), -), \mathcal{I}, F)).$$

By associativity this is naturally isomorphic to

$$B(B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(-), -)), \mathcal{I}, F).$$

But recall that for any  $j \in \mathcal{J}$  and for any  $W: \mathcal{J}^{\text{op}} \rightarrow \text{Top}$  we have a natural weak equivalence  $B(W, \mathcal{J}, \mathcal{J}(j, -)) \rightarrow W(j)$ , so in particular for  $W = \mathcal{K}(\psi(-), k)$  and putting  $j = \phi(i)$ , we have a weak equivalence of spaces (the second being discrete), natural in  $i \in \mathcal{I}^{\text{op}}$  and  $k \in \mathcal{K}$

$$B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(i), -)) \xrightarrow{\simeq} \mathcal{K}(\psi\phi(i), k).$$

By homotopy invariance of the bar construction, this gives a natural weak equivalence

$$B(B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(-), -)), \mathcal{I}, F) \rightarrow B(\mathcal{K}(\psi\phi(-), k), \mathcal{I}, F),$$

where the right-hand side is precisely  $(\psi\phi)_!^{\mathbb{L}} F$ .  $\square$

PROOF OF THEOREM 5.6.6. For  $\mathcal{C}$  a category, let us write  $\mathcal{C}_+$  for the category obtained from  $\mathcal{C}$  by freely adjoining a terminal object  $\infty$ . More precisely,  $\mathcal{C}_+$  has objects  $\text{Ob}(\mathcal{C}) \amalg \{\infty\}$  and its morphisms are defined by

$$\text{Hom}_{\mathcal{C}_+}(x, y) = \begin{cases} \text{Hom}_{\mathcal{C}}(x, y) & x, y \in \text{Ob}(\mathcal{C}), \\ *, & y = \infty, \\ \emptyset, & x = \infty, y \neq \infty. \end{cases}$$

For  $F: \mathcal{I} \rightarrow \text{Cat}$ , we similarly write  $F_+$  for the functor that takes  $i \in \mathcal{I}$  to  $F(i)_+$ . There is a fully faithful inclusion  $A: \text{Gr}(F) \hookrightarrow \text{Gr}(F_+)$  coming from the obvious

inclusions  $F(i) \hookrightarrow F(i)_+$ . And let  $*$   $\in$   $\text{Cat}$  denote the category with one object and one identity morphism.

Applying Lemma 5.6.8 to the homotopy Kan extensions obtained from diagram

$$\begin{array}{ccc} \text{Gr}(F) & \xrightarrow{\Phi} & \text{Top} \\ \downarrow A & & \\ \text{Gr}(F_+) & & \\ \downarrow & & \\ * & & \end{array}$$

we get a weak equivalence

$$\text{hocolim}_{\text{Gr}(F)} \Phi \simeq \text{hocolim}_{\text{Gr}(F_+)} A_!^\mathbb{L} \Phi.$$

Next, define  $\Omega: \mathcal{I} \rightarrow \text{Gr}(F_+)$  by  $i \mapsto (i, \infty \in F(i)_+)$ . Note that there exists exactly one functor  $\Omega$  with the given behaviour on objects. We claim that the functor  $\Omega$  is homotopy cofinal: Namely, for  $(i, x \in F(i)_+) \in \text{Gr}(F_+)$  the undercategory  $\mathcal{I}_{(i,x)/\Omega}$ , with objects  $(i', (i, x) \rightarrow (i', \infty))$ , has an initial object  $(i, (i, x) \rightarrow (i, \infty))$ , so it is in particular a contractible category.

Thus we have

$$\text{hocolim}_{\text{Gr}(F)} \Phi \simeq \text{hocolim}_{\mathcal{I}} A_!^\mathbb{L} \Phi \circ \Omega$$

by cofinality, Theorem 5.5.9. The functor  $A_!^\mathbb{L} \Phi \circ \Omega$  takes  $i \in \mathcal{I}$  to  $\text{hocolim}_{\text{Gr}(F)_{A/(i,\infty)}} \Phi$ .

There is a natural inclusion  $\iota_i: F(i) \rightarrow \text{Gr}(F)_{A/(i,\infty)}$  given by  $x \in F(i) \mapsto (i, x)$ . This functor is also homotopy cofinal: Objects of  $\text{Gr}(F)_{A/(i,\infty)}$  are essentially given by triples  $(i' \in \mathcal{I}, x \in F(i'), f: i' \rightarrow i)$  (we do not need to specify the terminal map  $F(f)(x) \rightarrow \infty \in F(i)_+$ ), and the undercategory  $F(i)_{(i',x,f)/\iota_i}$  has objects consisting of a pair  $(y \in F(i), F(f)(x) \rightarrow y)$ , such that the following triangle commutes in  $F(i) \subset \text{Gr}(F)$  (where  $t$  denotes any terminal morphism)

$$\begin{array}{ccc} (i, F(f)(x)) & \longrightarrow & (i, y) \\ \downarrow t & & \downarrow t \\ (i, F(f)(\infty)) & \xrightarrow{t} & (i, \infty); \end{array}$$

clearly, the diagram commutes for any chosen morphism  $F(f)(x) \rightarrow y$ , i.e. an object in  $F(i)_{(i',x,f)/\iota_i}$  is just given by a pair  $(y \in F(i), F(f)(x) \rightarrow y)$ . We see that  $(F(f)(x), F(f)(x) \xrightarrow{\text{id}} F(f)(x))$  is an initial object of this category, hence  $F(i)_{(i',x,f)/\iota_i}$  is weakly contractible.

It follows that

$$\text{hocolim}_{\text{Gr}(F)} \Phi \simeq \text{hocolim}_{i \in \mathcal{I}} \text{hocolim}_{F(i)} \Phi|_{F(i)},$$

as required.  $\square$

**PROOF OF THEOREM 5.6.4.** Taking  $\Phi$  in Theorem 5.6.6 to be the constant functor with value  $*$ , we get  $\text{BGr}(F) \xrightarrow{\sim} \text{hocolim}_{\mathcal{I}} \text{BF}(i)$ .  $\square$

### 5.7. Quillen's Theorem B

In this section we indicate how our results so far lead to a proof of Quillen's Theorem B. Recall that this says:

**THEOREM 5.7.1** (Quillen's Theorem B). *Suppose  $p: \mathcal{E} \rightarrow \mathcal{B}$  is a functor such that for all maps  $f: b \rightarrow b'$  in  $\mathcal{B}$ , the natural map  $B(\mathcal{E}_{/b}) \rightarrow B(\mathcal{E}_{/b'})$  is a (weak<sup>10</sup>) homotopy equivalence. Then the natural map from  $B(\mathcal{E}_{/b})$  to the homotopy fibre of  $B\mathcal{E} \rightarrow B\mathcal{B}$  at  $b \in \mathcal{B}$  is a weak equivalence.*

We will deduce this from our results in the last section together with the following result of Quillen [Qui73]:

**PROPOSITION 5.7.2.** *Suppose  $F: \mathcal{I} \rightarrow \text{Top}$  is a functor such that for every morphism  $\phi: i \rightarrow j$  in  $\mathcal{I}$ , the map  $F(\phi): F(i) \rightarrow F(j)$  is a homotopy equivalence. Then the homotopy fibre of the natural map*

$$\text{hocolim}_{\mathcal{I}} F \rightarrow \text{hocolim}_{\mathcal{I}} * \cong B\mathcal{I}$$

at  $i$  is weakly equivalent to  $F(i)$ .

**WORDS ABOUT THE (SKIPPED) PROOF.** Quillen's proof of this result is not hard, but we will not give it here as it uses some results about *quasifibrations* that we do not have time to discuss. A map of spaces is called a *quasifibration* if the natural maps from its strict fibres to its homotopy fibres are all weak equivalences. It is easy to see that each strict fibre of the above map is homeomorphic to the space  $F(i)$  for some  $i \in \mathcal{I}$  (for example, using that geometric realization commutes with pullbacks). Quillen shows that the map is a quasifibration using an induction over the skeleta of  $B\mathcal{I}$  and some basic results on quasifibrations from [DT58] (which incidentally is probably the most well-known paper in homotopy theory written in German).  $\square$

**REMARK 5.7.3.** Proposition 5.7.2 is a vast generalization of the fibration sequence  $X \rightarrow X_{hG} \rightarrow BG$  arising from the action of a group  $G$  on a space  $X$  via the Borel construction. The key point is that all the morphisms in  $BG$  induce homeomorphisms, hence weak equivalences.

Combining Proposition 5.7.2 with Theorem 5.6.4, we immediately get:

**COROLLARY 5.7.4.** *Suppose  $F: \mathcal{I} \rightarrow \text{Cat}$  is a functor such that for every map  $f: i \rightarrow i'$  the map  $BF(f): BF(i) \rightarrow BF(i')$  is a (weak) homotopy equivalence. Then the homotopy fibre of  $B\text{Gr}(F) \rightarrow B\mathcal{I}$  at  $i$  is  $BF(i)$ .  $\square$*

From this corollary, Theorem B follows easily:

**PROOF OF THEOREM B.** Let  $F: \mathcal{B} \rightarrow \text{Cat}$  be the functor  $b \mapsto \mathcal{E}_{/b} = \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{/b}$ , and set  $\bar{\mathcal{E}} := \text{Gr}(F)$ . Then the objects of  $\bar{\mathcal{E}}$  are triples  $(b \in \mathcal{B}, e \in \mathcal{E}, \phi: p(e) \rightarrow b)$  and a morphism  $(b, e, \phi) \rightarrow (b', e', \phi')$  is given by maps  $f: b \rightarrow b'$  and  $g: e \rightarrow e'$  such

<sup>10</sup>These are anyway CW complexes...

propn:quasifib

cor:Grhofib

that the square

$$\begin{array}{ccc} p(e) & \xrightarrow{p(g)} & p(e') \\ \downarrow \phi & & \downarrow \phi' \\ b & \xrightarrow{f} & b' \end{array}$$

commutes. Let  $\bar{p}$  denote the projection  $\bar{\mathcal{E}} \rightarrow \mathcal{B}$  that takes  $(b, e, \phi)$  to  $b$ ; there is also a projection  $q: \bar{\mathcal{E}} \rightarrow \mathcal{E}$  that takes this to  $e$ . Moreover, we have an inclusion  $i: \mathcal{E} \rightarrow \bar{\mathcal{E}}$  given by  $i(e) = (p(e), e, p(e)) \xrightarrow{\text{id}} p(e)$  and a commutative triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{i} & \bar{\mathcal{E}} \\ & \searrow p & \swarrow \bar{p} \\ & \mathcal{B} & \end{array}$$

Clearly  $qi = \text{id}_{\mathcal{E}}$ , and there is also a natural transformation  $iq \rightarrow \text{id}_{\bar{\mathcal{E}}}$  given at  $(b, e, \phi)$  by the map  $(pe, e, \text{id}_{p(e)}) \rightarrow (b, e, \phi)$  determined by  $\phi$  and  $\text{id}_e$ . But then  $Bq$  is a homotopy inverse to  $Bi$ , so  $Bi: B\mathcal{E} \rightarrow B\bar{\mathcal{E}}$  is a homotopy equivalence. Thus the homotopy fibres of  $\bar{p}$  and  $p$  are also (weakly) homotopy equivalent, and now applying Corollary 5.7.4 to  $F$  completes the proof.  $\square$

## 5.8. The homology spectral sequence of a simplicial space

In this section we will discuss a spectral sequence that computes the homology of the geometric realization of a simplicial space, and derive a description of its  $E^2$ -page. This is originally due to Segal [Seg68]; there is a more detailed presentation of Segal's result on the first differential in the book [KT06] (but unfortunately some details are still omitted). For precise statements, see Propositions 5.8.5 and 5.8.8.

Let us start by briefly recalling the spectral sequence of a filtered chain complex.

**5.8.1. Recollections on the spectral sequence of a filtered chain complex and filtered space.** A *filtered chain complex* is a sequence of inclusions of chain complexes

$$C(0) \hookrightarrow C(1) \hookrightarrow C(2) \hookrightarrow \dots$$

(One can also consider more general kinds of filtration, e.g. indexed by  $\mathbb{Z}$ , but this suffices for us.) Recall that from a filtered chain complex we get a spectral sequence of the form

$$E_{s,t}^1 = H_{s+t}(F(s)) \Rightarrow H_{s+t}(C)$$

where  $F(s)$  is the quotient  $C(s)/C(s-1)$  and  $C = \text{colim}_{i \rightarrow \infty} C(i)$  (which is an increasing union). The differentials are of the form

$$d_r: E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r.$$

The spectral sequence converges if, for example, we have  $H_{s+t}(F(s)) = 0$  for  $t < 0$  — this condition implies that the spectral sequence lives in the first quadrant.



REMARK 5.8.1. The first differential  $d_1$  can be described as follows: Given a class  $[x] \in H_{s+t}(F(s))$  let  $x \in F(s)_{s+t}$  be a cycle representing  $[x]$ . The map  $C(s) \rightarrow F(s)$  is surjective, so we can choose  $\bar{x} \in C(s)$  mapping to  $x$ . We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(s-1)_{s+t} & \longrightarrow & C(s)_{s+t} & \longrightarrow & F(s)_{s+t} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow d & & \downarrow d & & \downarrow \\ 0 & \longrightarrow & C(s-1)_{s+t-1} & \longrightarrow & C(s)_{s+t-1} & \longrightarrow & F(s)_{s+t-1} & \longrightarrow & 0. \end{array}$$

Thus, as  $dx = 0$  in  $F(s)_{s+t-1}$ , the element  $d\bar{x}$  must be the image of a (unique)  $x' \in C(s-1)_{s+t-1}$  (and  $dx' = 0$  since it maps to  $d^2\bar{x} = 0$  under an injective map). Also the image  $x''$  of  $x'$  in  $F(s-1)_{s+t-1}$  is a cycle. We define  $d_1[x] \in H_{s+t-1}F(s-1)$  to be class represented by the cycle  $x''$ .

EXERCISE 5.8.2. Check that the class  $d_1[x]$  in  $H_{s+t-1}(F(s-1))$  is independent of the choices of  $x$  and  $\bar{x}$  we made.

REMARK 5.8.3. If  $d_1[x] = 0 \in H_{s+t-1}(F(s-1))$  then this means we can choose  $y \in F(s-1)_{s+t}$  such that  $dy = x'' \in F(s-1)_{s+t-1}$ . Choose a  $\bar{y} \in C(s-1)_{s+t}$  mapping to  $y$  under quotient projection, then  $d\bar{y} \in C(s-1)_{s+t-1}$  maps to  $x''$  in  $F(s-1)_{s+t-1}$  under quotient projection, and so  $d\bar{y} - x'$  maps to 0 and by exactness lies in the image of  $C(s-2)_{s+t-1}$ . Let  $y' \in C(s-2)_{s+t-1}$  be the unique element that maps to  $d\bar{y} - x'$ ;  $y'$  is a cycle, and the image  $y''$  of  $y'$  in  $F(s-2)_{s+t-1}$  is also a cycle; moreover  $d_1[y''] = 0$ . We define  $d_2[x]$  to be the class in  $E_{s,t}^2$  represented by  $y''$ . Iterating this process gives  $d_r$  for all  $r$ .

Now suppose we have a sequence of cofibrations of topological spaces

$$X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \dots$$

Let  $X = \text{colim}_{i \rightarrow \infty} X(i)$ .<sup>11</sup> Then taking singular chains gives a filtered chain complex

$$C_*(X(0)) \hookrightarrow C_*(X(1)) \hookrightarrow C_*(X(2)) \hookrightarrow \dots$$

The associated spectral sequence is of the form

$$E_{s,t}^1 = H_{s+t}(X(s), X(s-1)) \Rightarrow H_{s+t}(X);$$

it converges if, for instance, we assume that  $H_{s+t}(X(s), X(s-1))$  vanishes for  $t < 0$ , i.e. the quotient  $X(s)/X(s-1)$  has no *reduced* homology in degrees  $\leq (s-1)$ .

LEMMA 5.8.4. *In this spectral sequence the first differential  $d_1$  is given by*

$$H_{s+t}(X(s), X(s-1)) \xrightarrow{\partial} H_{s+t-1}(X(s-1)) \rightarrow H_{s+t-1}(X(s-1), X(s-2)),$$

*i.e. the composite of the connecting homomorphism in the LES of  $(X(s), X(s-1))$  and the relativisation morphism of  $(X(s-1), X(s-2))$ .*

PROOF. Immediate from the definition of  $d_1$  above and the definition of the boundary map  $\partial$ . □

<sup>11</sup>This colimit is taken in Top: it is the space whose underlying set is the increasing union of the  $X(i)$ 's, and the topology is the weak topology on the union.

**5.8.2. The homology spectral sequence of a simplicial space.** We now want to apply this to the geometric realization  $|X|$  of a simplicial space  $X_\bullet$ . Suppose  $X$  is Reedy cofibrant, so the inclusion  $L_n X \hookrightarrow X_n$  is a closed cofibration for all  $n \geq 1$ . Then, as we saw before, in the pushout square

$$\begin{array}{ccc} L_n X \times \Delta_{top}^n \amalg_{L_n X \times \partial \Delta_{top}^n} X_n \times \partial \Delta_{top}^n & \longrightarrow & \text{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ X_n \times \Delta_{top}^n & \longrightarrow & \text{sk}_n|X|. \end{array}$$

from Lemma 5.2.10(ii) the left vertical map is also a cofibration, hence so is the right vertical map. Thus the skeletal filtration

$$\text{sk}_0|X| \hookrightarrow \text{sk}_1|X| \hookrightarrow \dots \hookrightarrow |X|$$

is given by cofibrations. We then have a spectral sequence of the form

$$E_{s,t}^1 = H_{s+t}(\text{sk}_s|X|, \text{sk}_{s-1}|X|) \Rightarrow H_{s+t}(|X|).$$

The above pushout square also implies that we have isomorphisms

$$\begin{aligned} H_{s+t}(\text{sk}_s|X|, \text{sk}_{s-1}|X|) &\cong H_{s+t}(X_s \times \Delta_{top}^s, L_s X \times \Delta_{top}^s \amalg_{L_s X \times \partial \Delta_{top}^s} X_s \times \partial \Delta_{top}^s) \\ &\cong H_{s+t}((X_s, L_s X) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s)). \end{aligned}$$

Here we use the notation  $(A, B) \wedge (A', B') = (A \times A', A \times B' \amalg_{B \times B'} B \times A')$  — note that  $(A \times A') / (A \times B' \amalg_{B \times B'} B \times A') \cong A/B \wedge A'/B'$  for the usual smash product of pointed spaces; in particular, using the isomorphism  $H_*(A, B) \cong \tilde{H}_*(A/B)$  for a cofibration  $B \hookrightarrow A$ , we obtain an isomorphism  $H_{s+t}((X_s, L_s X) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s))$ . In our case we thus have

$$\begin{aligned} H_{s+t}(\text{sk}_s|X|, \text{sk}_{s-1}|X|) &\cong H_{s+t}((X_s, L_s X) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s)) \\ &\cong \tilde{H}_{s+t}((X_s/L_s X) \wedge (\Delta_{top}^s/\partial \Delta_{top}^s)) \\ &\cong \tilde{H}_{s+t}(\Sigma^s(X_s/L_s X)) \\ &\cong \tilde{H}_t(X_s/L_s X) \\ &\cong H_t(X_s, L_s X). \end{aligned}$$

In particular this clearly vanishes for  $t < 0$ , so the spectral sequence converges. In summary, we have:

open:simplspE1 **PROPOSITION 5.8.5.** *Suppose  $X_\bullet$  is a Reedy cofibrant simplicial space. Then there is a convergent spectral sequence of the form*

$$E_{s,t}^1 = H_t(X_s, L_s X) \Rightarrow H_{s+t}(|X|).$$

□

Our next goal is to identify the first differential in this spectral sequence: we will show that it is given by the alternating sum of the maps induced in homology by the face maps of  $X_\bullet$ .

As a warm-up to proving this, let us consider the case where  $X$  is a simplicial set. Then writing  $X_s^{\text{nd}}$  for the set of non-degenerate  $s$ -simplices, we have a pushout square

$$\begin{array}{ccc} X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s & \longrightarrow & \text{sk}_{s-1}|X| \\ \downarrow & & \downarrow \\ X_s^{\text{nd}} \times \Delta_{\text{top}}^s & \longrightarrow & \text{sk}_s|X|. \end{array}$$

Consider  $X_s^{\text{nd}} \times \Delta_{\text{top}}^s$  equipped with the skeletal filtration coming from  $\Delta_{\text{top}}^s$ , i.e. let  $X_s^{\text{nd}} \times \text{sk}_i\Delta_{\text{top}}^s$  be the  $i^{\text{th}}$  filtration space, for  $i \geq 0$ . With this filtration the map  $X_s^{\text{nd}} \times \Delta_{\text{top}}^s \rightarrow |X|$  is a map of filtered spaces. This gives a commutative diagram

(5.8.1)

$$\begin{array}{ccccc} H_i(X_s^{\text{nd}} \times \Delta_{\text{top}}^s, X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s) & \xrightarrow{\partial} & H_{i-1}(X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s) & \xrightarrow{\theta} & H_{i-1}(X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s, X_s^{\text{nd}} \times \text{sk}_{s-2}\Delta_{\text{top}}^s) \\ \downarrow \cong & & \downarrow & & \downarrow \rho \\ H_i(\text{sk}_s|X|, \text{sk}_{s-1}|X|) & \xrightarrow{\partial} & H_{i-1}(\text{sk}_{s-1}|X|) & \longrightarrow & H_{i-1}(\text{sk}_{s-1}|X|, \text{sk}_{s-2}|X|). \end{array}$$

Here the bottom row gives the  $E^1$ -differential  $d_1$ . On the other hand, we have

$$(X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s) / (X_s^{\text{nd}} \times \text{sk}_{s-2}\Delta_{\text{top}}^s) \cong \bigvee_{j=0}^s (X_s^{\text{nd}} \times \Delta_{\text{top}}^{s-1}) / (X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^{s-1}),$$

inducing in homology an isomorphism

(5.8.2)

$$\begin{aligned} H_{i-1}(X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s, X_s^{\text{nd}} \times \text{sk}_{s-2}\Delta_{\text{top}}^s) &\cong \bigoplus_{j=0}^s H_{i-1}(X_s^{\text{nd}} \times \Delta_{\text{top}}^{s-1}, X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^{s-1}) \\ &\cong \bigoplus_{j=0}^s H_{i-s}(X_s^{\text{nd}}). \end{aligned}$$

We can identify  $H_i(X_s^{\text{nd}} \times \Delta_{\text{top}}^s, X_s^{\text{nd}} \times \partial\Delta_{\text{top}}^s) \cong H_{i-s}(X_s^{\text{nd}})$  using the homology cross product with the relative fundamental class of the pair  $(\Delta_{\text{top}}^s, \partial\Delta_{\text{top}}^s)$ ; under this identification, the map  $\theta \circ \partial$ , i.e. the top row of diagram (5.8.1), becomes the direct product<sup>12</sup>

$$\theta = \prod_{j=0}^s (-1)^j \text{id}: H_{i-s}(X_s^{\text{nd}}) \rightarrow \bigoplus_{j=0}^s H_{i-s}(X_s^{\text{nd}})$$

On the other hand, we can identify  $H_{i-1}(\text{sk}_{s-1}|X|, \text{sk}_{s-2}|X|) \cong H_{i-s}(X_{s-1})$ , and the map  $\rho$ , under this identification and the one from (5.8.2), is given by the direct sum

$$\rho = \bigoplus_{j=0}^s (d_j)_*: \bigoplus_{j=0}^s H_{i-s}(X_s^{\text{nd}}) \rightarrow H_{i-s}(X_{s-1}).$$

<sup>12</sup>Here we use that a finite direct sum is both a coproduct and a product in abelian groups.

Composing the top row of diagram (5.8.1) with  $\rho$  we obtain the desired formula  $\sum_{j=0}^s (-1)^j (d_j)_*$  for the  $E^1$ -differential  $d_1$ .

For a simplicial set  $X$  we have of course an isomorphism

$$H_i(X_s, L_s X) \cong H_{i-s}(X_s^{\text{nd}}) \cong \begin{cases} \mathbb{Z}X_s^{\text{nd}} & t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the  $E^1$ -page of the spectral sequence is 0 except for the row  $t = i - s = 0$  where we have a chain complex

$$\cdots \rightarrow \mathbb{Z}X_s^{\text{nd}} \rightarrow \mathbb{Z}X_{s-1}^{\text{nd}} \rightarrow \cdots \rightarrow \mathbb{Z}X_0^{\text{nd}} = \mathbb{Z}X_0.$$

Since the spectral sequence converges, the homology of this chain complex is isomorphic to the homology  $H_*(|X|)$  of the realization. We can interpret this using the chain complex associated to the simplicial set  $X$ ; this uses a standard result on simplicial abelian groups, which is a reformulation of Lemma 3.1.3:

LEMMA 5.8.6. *Suppose  $A_\bullet$  is a simplicial abelian group, and write  $C(A)$  for the associated chain complex with  $C(A)_n = A_n$  and  $d = \sum_i (-1)^i d_i$ . Let  $D_k(A)$  be the subgroup generated by degenerate elements. Then the differential on  $C(A)$  restricts to a differential on  $D(A)$ . Moreover, the chain complex  $D(A)$  is contractible, so the projection  $C(A) \rightarrow \bar{C}(A) := C(A)/D(A)$  is a quasi-isomorphism.*

The chain complex  $E_{s,0}^1$  found above is isomorphic to  $\bar{C}(\mathbb{Z}X)$ , which is by the previous lemma quasi-isomorphic to the usual chain complex  $C(\mathbb{Z}X)$ . Thus, if we want we can add back in the degenerate simplices and as the spectral sequence converges we have proved an isomorphism

$$H_*(|X|) \cong H_*(C(\mathbb{Z}X)).$$

Before we turn to the case of general simplicial spaces, we state a lemma:

lem:HFrxs

LEMMA 5.8.7. *Let  $X_\bullet$  be a Reedy cofibrant simplicial space, and for  $0 \leq r \leq s$  let  $F_{-r}X_s$  denote the union of all subspaces of  $X_s$  of the form  $s_{i_1} \cdots s_{i_r} X_{s-r}$ ; let moreover  $F_{-r}H_*X_s \subseteq H_*(X_s)$  be generated by the images of  $H_*X_{s-r}$  under the degeneracies. Then*

- (1) *The natural map  $H_*(F_{-r}X_s) \rightarrow H_*(F_{1-r}X_s)$  is injective.*
- (2) *The image of  $H_*(F_{-r}X_s)$  in  $H_*X_s$  is precisely  $F_{-r}H_*X_s$ . In particular,  $H_*(L_s X) \cong D(H_*(X_\bullet))_s$ , considering  $H_*(X_\bullet)$  as a graded simplicial abelian group.*

This is supposed to follow from the simplicial identities, but we will not give a complete proof. In analogy with the proof of Lemma 2.3.2, one can show that  $H_*(X_s)$  splits as a direct sum

$$H_*(X_s) \cong \bigoplus_{\substack{m \leq s \\ \theta: [s] \rightarrow [m]}} H_*(\theta^*(X_m), \theta^*(L_m(X))),$$

using essentially the map  $(d_{i_1} \cdots d_{i_r})_*: H_*(X_s) \rightarrow H_*(X_{s-r}, L_{s-r}X)$  to witness the summand corresponding to  $\theta = s^{i_1} \cdots s^{i_r}: [s] \rightarrow [s-r]$ . Once this is proved, Lemma 5.8.7 follows from identifying  $H_*(F_{-r}X_s) \subseteq H_*(X_s)$  with the part of the direct sum corresponding to pairs  $(m, \theta)$  with  $m \leq s - r$ .

propn:d1

PROPOSITION 5.8.8. *Let  $X_\bullet$  be a Reedy cofibrant simplicial space. Then the  $t^{\text{th}}$  row in the  $E^1$ -page of the spectral sequence is  $\bar{C}(H_t(X_\bullet))$ . Thus we have*

$$E_{s,t}^2 \cong H_s(H_t(X_\bullet)),$$

where we consider  $H_t(X_\bullet)$  as a simplicial abelian group, and define its homology as in Definition 3.1.1.

PROOF. Filter  $X_s$  as before, and filter  $X_s \times \Delta_{\text{top}}^s$  by taking

$$F_k := F_k(X_s \times \Delta^s) = \bigcup_{i+j=k} F_i X_s \times \text{sk}_j \Delta_{\text{top}}^s \quad 0 \leq k \leq s.$$

Thus  $F_s = X_s \times \Delta_{\text{top}}^s$  and  $F_{s-1} = L_s X \times \Delta_{\text{top}}^s \amalg_{L_s X \times \partial \Delta_{\text{top}}^s} X_s \times \partial \Delta_{\text{top}}^s$ . With this filtration the map  $X_s \times \Delta_{\text{top}}^s \rightarrow |X|$  is a map of filtered spaces. This means we get a commutative diagram

(5.8.3)

$$\begin{array}{ccccc} H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}) & \xrightarrow{\bar{\theta}} & H_{i-1}(F_{s-1}, F_{s-2}) \\ \downarrow \cong & & \downarrow & & \downarrow \bar{\rho} \\ H_i(\text{sk}_s |X|, \text{sk}_{s-1} |X|) & \xrightarrow{\partial} & H_{i-1}(\text{sk}_{s-1} |X|) & \longrightarrow & H_{i-1}(\text{sk}_{s-1} |X|, \text{sk}_{s-2} |X|), \end{array}$$

and the left vertical map already identifies, for  $i = s+t$ , the entry  $E_{s,t}^1$  in the spectral sequence with  $H_{s+t}(F_s, F_{s-1})$ , which by Lemma 5.8.7 is isomorphic to  $\bar{C}_s(H_t(X_\bullet)) = H_t(X_s)/D(H_t(X_\bullet))_s$ .

We then note that the space  $F_{s-1}/F_{s-2}$  splits as a wedge sum

$$\left( (L_s X / F_{-2} X_s) \wedge (\Delta_{\text{top}}^s / \partial \Delta_{\text{top}}^s) \right) \vee \left( (X_s / L_s X) \wedge (\partial \Delta_{\text{top}}^s / \text{sk}_{s-2} \Delta_{\text{top}}^s) \right).$$

Correspondingly, we have a decomposition in homology

$$\begin{aligned} H_{i-1}(F_{s-1}, F_{s-2}) &\cong H_{i-1} \left( (L_s X, F_{-2} X_s) \wedge (\Delta_{\text{top}}^s, \partial \Delta_{\text{top}}^s) \right) \oplus H_{i-1} \left( (X_s, L_s X) \wedge (\partial \Delta_{\text{top}}^s, \text{sk}_{s-2} \Delta_{\text{top}}^s) \right) \\ &\cong H_{i-1-s}(L_s X, F_{-2} X_s) \oplus \bigoplus_{j=0}^s H_{i-s}(X_s, L_s X). \end{aligned}$$

The composition of the first row of diagram (5.8.3) can be identified with the help of the following commutative diagram, in which the comparison between middle and bottom row comes from a map of filtered spaces, the comparison between top and middle row comes from the homology cross product with the fundamental class of

$(\Delta_{top}^s, \partial\Delta_{top}^s)$ , and we abbreviate the pair  $(\Delta_{top}^s, \partial\Delta_{top}^s)$  by  $(\Delta_{top}^s, \partial)$ :

$$\begin{array}{ccccc}
H_{i-s}(X_s, L_s X) & \xrightarrow{\partial} & H_{i-s-1}(L_s X) & \longrightarrow & H_{i-s-1}(L_s X, F_{-2} X_s) \\
-\times[\Delta_{top}^s, \partial] \downarrow \cong & & -\times[\Delta_{top}^s, \partial] \downarrow \cong & & -\times[\Delta_{top}^s, \partial] \downarrow \cong \\
H_i((X_s, L_s X) \wedge (\Delta_{top}^s, \partial)) & \xrightarrow{\partial} & H_{i-1}(L_s X \times (\Delta_{top}^s, \partial)) & \longrightarrow & H_{i-1}((L_s X, F_{-2} X_s) \wedge (\Delta_{top}^s, \partial)) \\
\downarrow \cong & & \downarrow & & \downarrow \\
H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}, L_s X \times \partial\Delta_{top}^s) & \longrightarrow & H_{i-1}(F_{s-1}, F_{s-2}) \\
& \searrow \partial & \uparrow & \nearrow \bar{\theta} & \\
& & H_{i-1}(F_{s-2}) & & 
\end{array}$$

But here the top left boundary map  $H_t(X_s, L_s X) \rightarrow H_{t-1}(L_s X)$  is zero, since by Lemma 5.8.7 the map  $H_*(L_s X) \rightarrow H_*(X_s)$  is injective. Thus the composition  $\theta \circ \partial$ , i.e. the top row of diagram (5.8.3), has image inside the summand  $\bigoplus_{j=0}^s H_{i-s}(X_s, L_s X)$ .

We can then use the following diagram, where we abbreviate  $\Delta = \Delta_{top}^s$ ,  $\partial = \partial\Delta_{top}^s$  and  $\text{sk} = \text{sk}_{n-2}\partial\Delta_{top}^s$

$$\begin{array}{ccccc}
H_i((X_s, L_s X) \wedge (\Delta, \partial)) & \xrightarrow{\partial} & H_{i-1}((X_s, L_s X) \times \partial) & \rightarrow & H_{i-1}((X_s, L_s X) \wedge (\partial, \text{sk})) \\
\downarrow \cong & & \downarrow & & \downarrow \\
H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}, L_s X \times \partial) & \longrightarrow & H_{i-1}(F_{s-1}, F_{s-2}) \\
& \searrow \partial & \uparrow & \nearrow \bar{\theta} & \\
& & H_{i-1}(F_{s-2}) & & 
\end{array}$$

The composite of the top row of the last diagram is the identity of  $H_{i-s}(X_s, L_s X)$  tensored with the map  $H_s(\Delta_{top}^s, \partial\Delta_{top}^s) \rightarrow H_{s-1}(\partial\Delta_{top}^s, \text{sk}_{n-2}\partial\Delta_{top}^s)$  sending  $[\Delta_{top}^s, \partial\Delta_{top}^s]$  to  $\sum_{j=0}^s (-1)^j d_*^j [\Delta^{s-1}, \partial\Delta^{s-1}]$ . The composition of  $\tau$  with the map  $\bar{\rho}$  from diagram (5.8.3) is induced on the quotient  $H_{i-1}((X_s, L_s X) \wedge (\partial, \text{sk})) \cong H_{i-1}(X_s \times (\partial, \text{sk})) / H_{i-1}(L_s X \times (\partial, \text{sk}))$  by the map

$$H_{i-1}(X_s \times (\partial\Delta_{top}^s, \text{sk}_{s-2}\partial\Delta_{top}^s)) \cong \bigoplus_{j=0}^s H_{i-1}(X_s \times (\Delta_{top}^{s-1}, \partial\Delta_{top}^{s-1})) \rightarrow H_{i-1}(\text{sk}_{s-1}|X|, \text{sk}_{s-2}|X|)$$

which on the  $j^{\text{th}}$  summand is induced by the composite

of  $\bigoplus_{j=0}^s H_{i-s}(X_s, L_s X)$  to the map induced in homology by the composite

$$X_s \times (\Delta^{s-1} / \partial\Delta^{s-1}) \xrightarrow{d_j \times \text{id}} X_{s-1} \times (\Delta^{s-1} / \partial\Delta^{s-1}) \rightarrow \text{sk}_{s-1}|X| / \text{sk}_{s-2}|X|.$$

This concludes the identification of the  $E^1$ -differential.  $\square$

### 5.9. The homology spectral sequence of a homotopy colimit

We now want to give a more conceptual (and potentially more computable) description of the  $E^2$ -page of the homology spectral sequence for the simplicial space  $\mathbb{B}_\bullet(*, \mathcal{I}, F)$  whose geometric realization is the homotopy colimit of  $F$ . This will be in

terms of the left derived functors of the colimit functor for abelian groups. We will now introduce these, and then derive the required expression for relating them to the  $E^2$ -page.

First of all, observe that for any small category  $\mathcal{I}$ , the functor category  $\text{Ab}^{\mathcal{I}}$  is an abelian category. To define left derived functors (of covariant functors), we first need to know this category has enough projectives:

DEFINITION 5.9.1. Let  $\mathcal{I}$  be a small category. For  $i \in \mathcal{I}$ , write  $L_i$  for the functor  $\mathcal{I} \rightarrow \text{Ab}$  given by  $i \mapsto \mathbb{Z}\mathcal{I}(i, -)$ .

EXAMPLE 5.9.2. If  $\mathcal{I} = 1 \leftarrow 0 \rightarrow 2$ , then  $L_0 = (\mathbb{Z} \leftarrow \mathbb{Z} \rightarrow \mathbb{Z})$ ,  $L_1 = (\mathbb{Z} \leftarrow 0 \rightarrow 0)$ , and  $L_2 = (0 \leftarrow 0 \rightarrow \mathbb{Z})$ .

enoughproj

LEMMA 5.9.3. *The object  $L_i \in \text{Ab}^{\mathcal{I}}$  is projective.*

PROOF. This follows since  $L_i$  is “free”: For  $F \in \text{Ab}^{\mathcal{I}}$  we have

$$\text{Hom}_{\text{Ab}^{\mathcal{I}}}(L_i, F) \cong \text{Hom}_{\text{Set}^{\mathcal{I}}}(\mathcal{I}(i, -), F) \cong F(i)$$

by the adjunction  $\mathbb{Z}(-): \text{Set} \rightleftarrows \text{Ab}: U$  (which induces an adjunction between diagram categories) and by the Yoneda Lemma. The previous is in fact a chain of isomorphisms of abelian groups, natural in  $F \in \text{Ab}^{\mathcal{I}}$ . Hence  $\text{Hom}_{\text{Ab}^{\mathcal{I}}}(L_i, -): \text{Ab}^{\mathcal{I}} \rightarrow \text{Ab}$  preserves exact sequences, so  $L_i$  is projective.  $\square$

EXERCISE 5.9.4. This exercise is somewhat paranthetical, but some may enjoy it: Show that the argument in Lemma 5.9.3 can be seen directly as an instance of the Yoneda lemma in enriched category theory, where the enrichment is over abelian groups (and find out what these words mean). [Hint: Introduce the linear category  $\mathbb{Z}\mathcal{I}$  with objects  $\mathcal{I}$  and morphisms the  $\mathbb{Z}$ -linear span of the morphisms in  $\mathcal{I}$  ie,  $\text{Hom}_{\mathbb{Z}\mathcal{I}}(i, j) = \mathbb{Z}\text{Hom}_{\mathcal{I}}(i, j)$ , and observe that  $L_i(j) = \text{Hom}_{\mathbb{Z}\mathcal{I}}(i, j)$ .]

enoughproj

LEMMA 5.9.5. *The category  $\text{Ab}^{\mathcal{I}}$  has enough projectives.*

PROOF. Given  $F \in \text{Ab}^{\mathcal{I}}$  we can take  $P = \bigoplus_{i \in \mathcal{I}} \bigoplus_{x \in F(i)} L_i$ ; this is projective and has a surjective map  $P \rightarrow F$  given on the component corresponding to  $x \in F(i)$  by the map  $L_i \rightarrow F$  corresponding to  $x$ .  $\square$

The functor  $\text{colim}: \text{Ab}^{\mathcal{I}} \rightarrow \text{Ab}$  is clearly right exact (since it commutes with all colimits) and by Lemma 5.9.5 the category  $\text{Ab}^{\mathcal{I}}$  has enough projectives. We can therefore make the following definition:

DEFINITION 5.9.6. For  $s \geq 0$  define  $\text{colim}_s$  as the  $s^{\text{th}}$  left derived functor of  $\text{colim}$ . In other words, for a functor  $F \in \text{Ab}^{\mathcal{I}}$ , the abelian group  $\text{colim}_s F \in \text{Ab}$  is  $H_s(\text{colim } P_{\bullet})$  where  $P_{\bullet}$  is a projective resolution of  $F$  in  $\text{Ab}^{\mathcal{I}}$ .

EXAMPLE 5.9.7. If  $\mathcal{I} = \mathcal{B}G$ , meaning a group  $G$  viewed as a category with one object  $*$ , then the functor  $L_*$  is just the group algebra  $\mathbb{Z}G$  viewed as a free module of rank one, and  $\text{colim}_s M$  identifies with the group homology  $H_s(G; M)$  (defined, for instance, as  $\text{Tor}_s^{\mathbb{Z}G}(\mathbb{Z}, M)$ ).

ex:pushout

EXAMPLE 5.9.8. [This will probably be done in the exercise session.] Suppose  $\mathcal{I}$  is the category  $\bullet \leftarrow \bullet \rightarrow \bullet$ , and consider  $M = A \xleftarrow{f} C \xrightarrow{g} B$ . Then

$$\operatorname{colim}_s M \cong \begin{cases} \operatorname{colim} M \cong \operatorname{coker} \left( C \xrightarrow{(f, -g)} A \oplus B \right) & s = 0, \\ \ker \left( C \xrightarrow{(f, -g)} A \oplus B \right) & s = 1, \\ 0 & s > 1. \end{cases}$$

To see this, consider the exact sequence

$$0 \rightarrow (A \leftarrow 0 \rightarrow B) \rightarrow (A \leftarrow C \rightarrow B) \rightarrow (0 \leftarrow C \rightarrow 0) \rightarrow 0$$

By the long exact sequence of derived functors, the claim will follow from the following claims:

$$\operatorname{colim}_s((A \leftarrow 0 \rightarrow B)) = \begin{cases} A \oplus B & s = 0 \\ 0 & s > 0 \end{cases}$$

$$\operatorname{colim}_s((0 \leftarrow C \rightarrow 0)) = \begin{cases} 0 & s \neq 1 \\ C & s = 1 \end{cases}$$

For the first claim, we pick as projective resolution of  $(A \leftarrow 0 \rightarrow B)$  one of the form  $(P_\bullet^A \leftarrow 0 \rightarrow P_\bullet^B)$ , obtained by using projective resolutions of  $A$  and  $B$  in  $\operatorname{Ab}$ .

For the second claim, let  $P \xrightarrow{p} C$  be a projective cover in  $\operatorname{Ab}$ , and let  $K = \ker(p) \in \operatorname{Ab}$ ; note that  $K$  is also a free/projective abelian group. Let  $C_0 = (P \leftarrow P \rightarrow P)$  which maps onto  $(0 \leftarrow C \rightarrow 0)$  with kernel  $(P \leftarrow K \rightarrow P)$ .<sup>13</sup> Set

$$C_1 = ((K \leftarrow K \rightarrow K) \oplus (0 \leftarrow 0 \rightarrow P) \oplus (P \leftarrow 0 \rightarrow 0))$$

which maps onto  $(P \leftarrow K \rightarrow P)$  with kernel  $C_2 := (K \leftarrow 0 \rightarrow 0) \oplus (0 \leftarrow 0 \rightarrow K)$

We have hence produced a projective resolution of length 2. Taking  $\operatorname{colim}$  gives the chain complex of abelian groups

$$\dots 0 \rightarrow K \oplus K \rightarrow (K \oplus K)/\Delta(K) \oplus P \oplus P \rightarrow (P \oplus P)/\Delta(P) \rightarrow 0 \dots$$

which has homology groups

$$\dots 0; \quad 0; \quad C; \quad 0; \quad 0 \dots$$

[Alternative proof of the second part: First observe that the claim holds for diagrams  $(0 \leftarrow P \rightarrow 0)$  with  $P$  projective, and use the long the long exact sequence for

$$(0 \leftarrow K \rightarrow 0) \rightarrow (0 \leftarrow P \rightarrow 0) \rightarrow (0 \leftarrow C \rightarrow 0)$$

to prove it in general.]

---

<sup>13</sup>Don't be tempted to think that  $(P \leftarrow K \rightarrow P)$  is projective in  $\operatorname{Ab}^{\mathcal{I}}$ , just because  $P$  and  $K$  are projective in  $\operatorname{Ab}$ !



REMARK 5.9.9. Notice that  $\text{colim}_s$ , as a functor from pushouts of abelian groups to abelian groups, vanishes for  $s \geq 2$ , despite the fact that the projective dimension of the category of pushouts of abelian groups is 2. An explanation of this fact is given in Proposition 5.9.11, and has to do with considering  $\text{colim}_{\mathcal{I}} F = \mathbb{Z} \otimes_{\mathcal{I}} F$  as a bifunctor (together with the fact that  $\mathbb{Z} = \delta(\mathbb{Z}) \in \text{Ab}^{\mathcal{I}^{\text{op}}}$  admits a projective resolution of length 1 in  $\text{Ab}^{\mathcal{I}^{\text{op}}}$ , where  $\mathcal{I}$  is the pushout category).

DEFINITION 5.9.10. For  $F: \mathcal{I} \rightarrow \text{Ab}$ , let  $F^\Delta$  denote the simplicial abelian group where

$$F_n^\Delta := \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) = \bigoplus_{i: [n] \rightarrow \mathcal{I}} F(i(0)),$$

with the by now familiar structure maps. We write  $E(F) := C(F^\Delta)$  for the associated chain complex.

colimodel

PROPOSITION 5.9.11. For  $F: \mathcal{I} \rightarrow \text{Ab}$ , we have a natural isomorphism

$$\text{colim}_s F \cong H_s E(F).$$

PROOF. For  $W: \mathcal{I}^{\text{op}} \rightarrow \text{Ab}$  and  $F: \mathcal{I} \rightarrow \text{Ab}$ , in this proof we will write  $W \otimes_{\mathcal{I}} F$  for the coend of  $W \otimes F: \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \text{Ab}$ . Then, writing  $\mathbb{Z} = \delta(\mathbb{Z}) \in \text{Ab}^{\mathcal{I}^{\text{op}}}$ , we have

$$\text{colim}_{\mathcal{I}} F \cong \mathbb{Z} \otimes_{\mathcal{I}} F,$$

since  $\mathbb{Z} \in \text{Ab}$  is the unit for the tensor product of  $\text{Ab}$ .

Moreover,  $-\otimes_{\mathcal{I}}-$  is right exact in both variables (since it preserves colimits in both), and by the same double complex argument as for the derived functors of  $-\otimes_R -: \text{Mod}_R \times {}_R\text{Mod} \rightarrow \text{Ab}$  we can use a projective resolution in either variable to compute derived functors.

We claim that the functor  $i \mapsto E(L_i)$  is a projective resolution of  $\mathbb{Z}$  in  $\text{Ab}^{\mathcal{I}^{\text{op}}}$ . Note that  $i \mapsto L_i(x) = \mathbb{Z}\mathcal{I}(i, x) = \mathbb{Z}\mathcal{I}^{\text{op}}(x, i)$  is projective in  $\text{Ab}^{\mathcal{I}^{\text{op}}}$ , so  $E(L_{(-)})$  is a chain complex of projectives since each term is a direct sum of such functors: we have indeed  $E(L_{(-)})_n = \bigoplus_{i: [n] \rightarrow \mathcal{I}} L_{(-)}(i(0))$ .

Now by the usual rewriting we have  $E(L_i)_n = \bigoplus_{i \rightarrow i_0 \rightarrow \dots \rightarrow i_n} \mathbb{Z} \cong \mathbb{Z}(\text{N}\mathcal{I}_{i'})_n$ . Thus  $E(L_i) \cong C(\mathbb{Z}\text{N}\mathcal{I}_{i'})$  and so

$$H_* E(L_i) \cong H_* B\mathcal{I}_{i'} \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0, \end{cases}$$

since  $B\mathcal{I}_{i'}$  is contractible. A morphism  $i \rightarrow i'$  in  $\mathcal{I}$  induces a map of spaces  $B\mathcal{I}_{i'} \rightarrow B\mathcal{I}_i$  which on  $H_0$  induces the identity of  $\mathbb{Z}$ : thus by naturality we have that  $H_0(E(L_{(-)})) \cong \delta\mathbb{Z} \in \text{Ab}^{\mathcal{I}^{\text{op}}}$ .

It follows that  $\text{colim}_s F$  is computed by the chain complex  $E(L_{(-)}) \otimes_{\mathcal{I}} F$ . Now we need to identify this with  $E(F)$ . We have a chain of isomorphisms in  $\text{Ab}^{\mathcal{I}^{\text{op}}}$ , natural

in  $[n] \in \mathbf{\Delta}^{\text{op}}$

$$\begin{aligned}
(L_{(-)})_n^\Delta \otimes_{\mathcal{I}} F &\cong \text{coeq} \left( \bigoplus_{i \rightarrow i' \rightarrow \dots \rightarrow i_n} \bigoplus \mathbb{Z}\mathcal{I}(i', i_0) \otimes F(i) \rightrightarrows \bigoplus_i \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} \mathbb{Z}\mathcal{I}(i, i_0) \otimes F(i) \right) \\
&\cong \text{coeq} \left( \bigoplus_{i \rightarrow i' \rightarrow i_0 \rightarrow \dots \rightarrow i_n} F(i) \rightrightarrows \bigoplus_{i \rightarrow i_0 \rightarrow \dots \rightarrow i_n} F(i) \right) \\
&\cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} \text{coeq} \left( \bigoplus_{i \rightarrow i' \rightarrow i_0} F(i) \rightrightarrows \bigoplus_{i \rightarrow i_0} F(i) \right) \\
&\cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} \text{colim}_{\mathcal{I}/i_0} F \\
&\cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \\
&= F_n^\Delta.
\end{aligned}$$

Passing to associated chain complexes and to homology, we get  $\text{colim}_s F \cong H_s(E(L_{(-)}) \otimes_{\mathcal{I}} F) \cong H_s(E(F))$ , as required.  $\square$

**THEOREM 5.9.12.** *Given  $F: \mathcal{I} \rightarrow \text{Top}$  there is a convergent spectral sequence of the form*

$$E_{s,t}^2 = \text{colim}_s H_t F \Rightarrow H_{s+t}(\text{hocolim } F).$$

**PROOF.** We consider the homology spectral sequence of the simplicial space  $\mathbb{B}_\bullet(*, \mathcal{I}, F)$ . By Proposition 5.8.5 this converges, and by Proposition 5.8.8 it has  $E^2$ -term given by

$$E_{s,t}^2 \cong H_s(H_t(\mathbb{B}_\bullet(*, \mathcal{I}, F))).$$

But here  $H_t(\mathbb{B}_n(*, \mathcal{I}, F)) \cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} H_t F(i_0) \cong E(H_t F)_n$ , and the differentials clearly agree too. So

$$E_{s,t}^2 \cong H_s E(H_t F) \cong \text{colim}_s H_t F. \quad \square$$

**EXERCISE 5.9.13.** By Example 5.9.8, the  $E_2$ -term of the spectral sequence for a homotopy pushout  $(X \leftarrow A \rightarrow Y)$  degenerates to two lines:

$$\begin{aligned}
\text{colim}_0(H_i(-)) &= \text{coker}(H_i(C) \rightarrow H_i(X) \oplus H_i(Y)) \\
\text{colim}_1(H_i(-)) &= \text{ker}(H_i(C) \rightarrow H_i(X) \oplus H_i(Y))
\end{aligned}$$

Check that the corresponding long exact sequence identifies with the Mayer-Vietoris sequence.

## 5.10. The Čech decomposition of a space as a homotopy colimit

Notes from CatTop 2009.

## 5.11. More exercises

## CHAPTER 6

### Localization and completion

ch:loccom

In this chapter, we will construct localizations and completions for simply connected spaces via a quick-and-dirty explicit construction, working only in the homotopy category. After this we will discuss more sophisticated approaches.

locrefl

#### 6.1. Localizations and reflective subcategories

Before we start we will need to do a bit of abstract yoga. Recall from Proposition 4.1.2, that given a small category  $\mathcal{C}$  we can form a category  $\mathcal{C}[\mathcal{W}^{-1}]$ , which is the category where we formally invert the morphisms in  $\mathcal{W}$ , i.e., turn them into isomorphisms. We can also do this when  $\mathcal{C}$  is not small, although in this case it is not a priori clear that  $\mathcal{C}[\mathcal{W}^{-1}]$  is locally small (i.e., the “hom sets” may not be sets).

DEFINITION 6.1.1. Let  $\mathcal{W}$  be a collection of morphisms (not necessarily a set) in a category  $\mathcal{C}$ . We say an object  $c \in \mathcal{C}$  is  $\mathcal{W}$ -local if for every morphism  $\phi: x \rightarrow y$  in  $\mathcal{W}$ , the map

$$\mathrm{Hom}_{\mathcal{C}}(y, c) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}}(x, c)$$

is an isomorphism.

2of3saturated

REMARK 6.1.2. It does not harm to assume that  $\mathcal{W}$  satisfies 2-of-3. To see this, if  $\mathcal{W}$  is any collection of morphisms in  $\mathcal{C}$ , we can define  $\mathcal{W}'$  to be the smallest collection of morphisms in  $\mathcal{C}$  that contains  $\mathcal{W}$  and satisfies the 2-of-3 property. Then it is immediate to see that an object of  $\mathcal{C}$  is  $\mathcal{W}$ -local if and only if it is also  $\mathcal{W}'$ -local.

We will henceforth assume that  $\mathcal{W}$  satisfies the 2-of-3 property.

Moreover, for a collection  $\mathcal{W}$  satisfying 2-of-3, we can define a new, possibly larger collection  $\mathcal{W}''$  containing all morphisms  $f: x \rightarrow y$  inducing for all  $\mathcal{W}$ -local  $c$  a bijection  $\mathrm{Hom}_{\mathcal{C}}(y, c) \cong \mathrm{Hom}_{\mathcal{C}}(x, c)$ . Then it is immediate to see that an object of  $\mathcal{C}$  is  $\mathcal{W}$ -local if and only if it is also  $\mathcal{W}''$ -local.

We will henceforth assume that  $\mathcal{W}$  is “saturated” in the sense that it coincides with  $\mathcal{W}''$ .

DEFINITION 6.1.3. For  $x \in \mathcal{C}$ , a  $\mathcal{W}$ -localization of  $x$  is a morphism  $\lambda_x: x \rightarrow Lx$  where  $Lx$  is a  $\mathcal{W}$ -local object of  $\mathcal{C}$  and the map  $\lambda_x$  lies in  $\mathcal{W}$ .

We first note that maps in  $\mathcal{W}$  between  $\mathcal{W}$ -local objects are isomorphisms:

prop:Wmapiso

PROPOSITION 6.1.4. *If  $x, y$  are  $\mathcal{W}$ -local and  $\phi: x \rightarrow y$  is in  $\mathcal{W}$ , then  $\phi$  is an isomorphism. In particular, if  $x$  is  $\mathcal{W}$ -local, then a  $\mathcal{W}$ -localization  $\lambda_x: x \rightarrow Lx$  of  $x$  is an isomorphism.*

PROOF. By the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(y, x) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}}(x, x)$$

there exists  $f : y \rightarrow x$  such that  $f \circ \phi = 1_x$ . Hence under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(y, y) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}}(x, y),$$

both  $\phi \circ f$  and  $1_y$  go to  $\phi$ , and are thus equal; so  $f$  is a right inverse of  $\phi$  as well.

(Or slightly less directly one may appeal to our old friend the Yoneda lemma, observing that  $\mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(y, -) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(x, -)$  is an isomorphism of representable functors, with  $\mathcal{C}_{\mathcal{W}}$  the full subcategory of  $\mathcal{C}$  generated by  $\mathcal{W}$ -local objects.)  $\square$

initialfinal

LEMMA 6.1.5. *Suppose  $x \rightarrow Lx$  is a  $\mathcal{W}$ -localization.*

(i) *For any  $f : x \rightarrow y$  with  $y$   $\mathcal{W}$ -local there exists a unique  $f' : Lx \rightarrow y$  such that*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \lambda_x \downarrow & \nearrow f' & \\ Lx & & \end{array}$$

*commutes.*

(ii) *For any map  $f : x \rightarrow y$  in  $\mathcal{W}$  there exists a unique map  $\ell : y \rightarrow Lx$  such that*

$$\begin{array}{ccc} x & \xrightarrow{\lambda_x} & Lx \\ f \downarrow & \nearrow \ell & \\ y & & \end{array}$$

*commutes.*

PROOF. Since  $y$  is  $\mathcal{W}$ -local and  $\lambda_x$  is in  $\mathcal{W}$ , we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Lx, y) \xrightarrow{\lambda_x^*} \mathrm{Hom}_{\mathcal{C}}(x, y).$$

Then  $f$  has a unique preimage  $f'$ , which gives (i).

For (ii), observe that since  $Lx$  is  $\mathcal{W}$ -local and  $f$  is in  $\mathcal{W}$ , we have an isomorphism  $\mathrm{Hom}_{\mathcal{C}}(y, Lx) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(x, Lx)$ . Then  $\lambda_x$  has a unique preimage  $\ell$ .  $\square$

REMARK 6.1.6. We can reformulate Lemma 6.1.5 as: a  $\mathcal{W}$ -localization  $\lambda_x : x \rightarrow Lx$  is the initial map from  $x$  to a  $\mathcal{W}$ -local object, and the terminal map in  $\mathcal{W}$  out of  $x$ . In particular, we see that the  $\mathcal{W}$ -localization is unique (up to unique isomorphism) if it exists.

PROPOSITION 6.1.7. *Suppose every object of  $\mathcal{C}$  has a  $\mathcal{W}$ -localization, and let  $\mathcal{C}_{\mathcal{W}}$  denote the full subcategory of  $\mathcal{C}$  spanned by the  $\mathcal{W}$ -local objects. Then the inclusion  $i : \mathcal{C}_{\mathcal{W}} \hookrightarrow \mathcal{C}$  has a left adjoint  $L : \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{W}}$ , with unit of the adjunction given by  $\lambda_x : x \rightarrow Lx$  for a  $\mathcal{W}$ -localization of  $x$ .*

*The composite*

$$\mathcal{C}_{\mathcal{W}} \rightarrow \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

is an equivalence of categories with inverse equivalence given by the functor  $\mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{C}_{\mathcal{W}}$  induced by  $L : \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{W}}$ . In particular  $\mathcal{C}[\mathcal{W}^{-1}]$  is a locally small category in this case.<sup>1</sup>

PROOF. If  $x \rightarrow Lx$  is a  $\mathcal{W}$ -localization of  $x$ , then the induced isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(x, iy) \cong \mathrm{Hom}_{\mathcal{C}}(Lx, y) \cong \mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(Lx, y)$$

is natural in  $y \in \mathcal{C}_{\mathcal{W}}$ , so the functor  $\mathrm{Hom}_{\mathcal{C}}(x, i(-)) : \mathcal{C}_{\mathcal{W}} \rightarrow \mathrm{Set}$  is representable (by  $Lx$ ) for every  $x$ . By the Yoneda Lemma it follows that  $i$  has a left adjoint with  $x \rightarrow Lx$  as the unit map at  $x$ .

More explicitly, we can define the functor  $L$  on morphisms by taking  $L(f)$  for  $f : x \rightarrow y$  to be the unique map  $Lx \rightarrow Ly$  that factors  $\lambda_y \circ f : x \rightarrow Ly$  through  $Lx$ . Note also that  $L$  sends morphisms in  $\mathcal{W}$  to isomorphisms: if  $f \in \mathcal{W}$ , then by 2-of-3 also the composite  $\lambda_y \circ f = Lf \circ \lambda_x$  lies in  $\mathcal{W}$ , and again by 2-of-3 also  $Lf$  lies in  $\mathcal{W}$ ; by Proposition 6.1.4 we then obtain that  $Lf$  is an isomorphism.

Now suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor that takes the morphisms in  $\mathcal{W}$  to isomorphisms. Let  $F' := F \circ i : \mathcal{C}_{\mathcal{W}} \rightarrow \mathcal{D}$ . Then  $F \cong F'L = FiL$ : the unit transformation  $\lambda : \mathrm{id} \rightarrow iL$  is given by maps in  $\mathcal{W}$ , so since  $F$  takes these maps to isomorphisms,  $F\lambda : F \rightarrow FiL = F'L$  is a natural isomorphism. Thus  $F$  factors through  $L$  up to natural isomorphism.

To see that this factorization is unique up to *unique* natural isomorphism, suppose we have a natural isomorphism  $F \cong GL$  for some  $G : \mathcal{C}_{\mathcal{W}} \rightarrow \mathcal{D}$ . Then we get  $F' = Fi \cong GLi \cong G$  since  $Li \cong \mathrm{id}$ .

Moreover, if  $\eta : F' \xrightarrow{\cong} F'$  is a natural automorphism of  $F'$  such that  $\eta L : F'L \xrightarrow{\cong} F'L$  is the *identity* natural automorphism of  $F'L$ , then for  $x \in \mathcal{C}_{\mathcal{W}}$  we have  $\eta_{Lix} = \mathrm{id}_{F'Lix}$ , and the following commutative square, together with the fact that  $\epsilon_x : Lix \rightarrow x$  is an isomorphism, implies that  $\eta_x = \mathrm{id}_{F'x}$ :

$$\begin{array}{ccc} F'Lix & \xrightarrow{\eta_{Lix} = \mathrm{id}_{F'Lix}} & F'Lix \\ \downarrow F'\epsilon_x & & \downarrow F'\epsilon_x \\ F'x & \xrightarrow{\eta_x} & F'x. \end{array}$$

This shows that  $\mathcal{C}_{\mathcal{W}}$  satisfies the universal property of  $\mathcal{C}[\mathcal{W}^{-1}]$ .  $\square$

REMARK 6.1.8. Note that the above identifies the localization functor  $\mathcal{C} \rightarrow \mathcal{C}$  as the *composite* of a left adjoint  $L$  followed by a right adjoint  $i$ . This means that good properties are in general not just “formal”.

REMARK 6.1.9. When  $\mathcal{C} = \mathrm{Ho}(\mathcal{D})$  is a homotopy category, we can sometimes construct the localization as a deformation of the “underlying” category  $\mathcal{D}$ , in the sense of Chapter 4. This allows us to get a strict functor, rather than just a functor on the homotopy category.

DEFINITION 6.1.10. A full subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  such that the inclusion has a left adjoint is called a *reflective* subcategory.

<sup>1</sup>If you do not want to consider “large” categories at all, you can rephrase the statement as: if every object in  $\mathcal{C}$  has a  $\mathcal{W}$ -localization, then  $\mathcal{C}_{\mathcal{W}}$  has the universal property for being  $\mathcal{C}[\mathcal{W}^{-1}]$ .

EXAMPLE 6.1.11. Let  $S \subseteq \mathbb{Z}$  be a set of primes, and take  $\mathcal{W}$  to be the collection of maps  $f: A \rightarrow B$  in  $\text{Ab}$  such that the kernel and cokernel of  $f$  is  $S$ -torsion (equivalently,  $S^{-1}f: S^{-1}A \rightarrow S^{-1}B$  is an isomorphism). Then  $\mathcal{W}$ -localizations of objects in  $\text{Ab}$  exist and are given by the natural map  $A \rightarrow S^{-1}A$ . (This is just a reformulation for the usual universal property of  $S^{-1}A$ .)

REMARK 6.1.12. If  $\mathcal{C}$  is a very well-behaved category, namely a so-called *pre-sentable* (or sometimes “locally presentable”) category, then  $\mathcal{W}$ -localizations always exist for classes of morphisms  $\mathcal{W}$  that are in a suitable sense generated by a *set* of morphisms.

## 6.2. $R$ -localization of spaces

We are interested in  $\mathcal{W}$ -localizations in  $\text{Ho}(\text{Top})$  of the following type:

defn:WR

DEFINITION 6.2.1. Let  $R$  be a commutative ring. We denote by  $W_R$  the class of maps  $f: X \rightarrow Y$  in  $\text{Ho}(\text{Top})$  such that  $f_*: H_*(X; R) \rightarrow H_*(Y; R)$  is an isomorphism. We will call the maps in  $W_R$   *$R$ -equivalences*, and we will call  $W_R$ -local spaces  *$R$ -local spaces* and talk about  *$R$ -localization* of spaces.

lem:WR

LEMMA 6.2.2. *The class  $W_R$  satisfies the assumptions from Remark 6.1.2.*

PROOF. It is clear that  $H_*(-; R)$ -equivalences satisfy 2-of-3. For “saturation”, note that Lemma 6.2.6 in the following implies that  $K(M, n)$  is  $R$ -local for all  $R$ -module  $M$  and  $n \geq 1$ . Let  $f: X \rightarrow Y$  induce an isomorphism  $[Y, Z] \cong [X, Z]$  for all  $R$ -local  $Z$ ; then in particular  $f$  induces an isomorphism  $H^*(Y; M) \cong H^*(X; M)$ ; this is equivalent to the vanishing of  $H^*(Y, X; M)$  for all  $M \in \text{Mod}_R$  (we replace  $f$  by an inclusion via mapping cylinder), i.e. the chain complex of free  $R$ -modules  $C_*(Y, X; R)$  becomes acyclic under each functor  $\text{Hom}_R(-; M)$ ; this in turns implies that  $C_*(Y, X; R)$  is already acyclic (and being levelwise free and bounded below, it is actually chain null-homotopic), so that  $H_*(X; R) \cong H_*(Y; R)$  under  $f$ , i.e.  $f \in W_R$ .  $\square$

The category  $\text{Ho}(\text{Top})$  is far from being presentable (one requirement is the existence of colimits), so the existence of  $R$ -localizations is not formal. Nevertheless, we have the following result of Bousfield:

THEOREM 6.2.3 (Bousfield [Bou75]). *Any space  $X \in \text{Ho}(\text{Top})$  admits an  $R$ -localization.*

We will not prove this here. Instead, we will give explicit constructions of  $R$ -localizations for nice (more precisely, simply connected<sup>2</sup>) spaces in two important cases:

- $R = \mathbb{Z}_{(p)} := \mathbb{Z}[q^{-1} : q \neq p \text{ prime}]$ , for a fixed prime number  $p$ ; the  $\mathbb{Z}_{(p)}$ -localization is called  *$p$ -localization*, and we write  $X_{(p)}$  for the  $p$ -localization of a space  $X$ . More generally we will consider subrings  $R \subseteq \mathbb{Q}$ ; this gives for example the *rationalization*  $X_{\mathbb{Q}}$  of a space, i.e. its  $\mathbb{Q}$ -localization.

<sup>2</sup>Most of our arguments work in fact without changes for generic *simple* spaces, of which simply connected ones are a subclass. For the even larger class of *nilpotent* spaces more work is needed.

- $R = \mathbb{F}_p$ ;  $\mathbb{F}_p$ -localization is often called  $p$ -completion, and we write  $X_p^\wedge$  for the  $p$ -completion of  $X$ .

The general existence result gives an inexplicit description of the localization that is not useful for computations, so the explicit construction is in any case important. Note that with more work everything we do can be extended from simply connected spaces to *nilpotent* spaces.

REMARK 6.2.4.

- With more care one can construct  $R$ -localization as a functor on  $\text{Top}$  rather than  $\text{Ho}(\text{Top})$ , which is (very) often convenient since it is much better to have diagrams that commute strictly than only up to homotopy, as we have seen in the previous sections.
- The  $R$ -local objects can be shown to be the collection of fibrant objects in a model category, and this is often a convenient way to construct the localization.

REMARK 6.2.5. The name “ $p$ -completion” will make more sense when we understand  $\mathbb{F}_p$ -local objects. Note that there is a map from the  $p$ -localization to the  $p$ -completion (as well as to the rationalization) induced by the maps  $\mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p$  and  $\mathbb{Z}_{(p)} \rightarrow \mathbb{Q}$ .

In general, for  $X \rightarrow Y$  a map of spaces and  $R \rightarrow S$  a map of commutative rings, if  $H_*(X; R) \xrightarrow{\cong} H_*(Y; R)$  then  $H_*(X; S) \xrightarrow{\cong} H_*(Y; S)$ ; this implies in turn that an  $S$ -local space is also  $R$ -local, and hence, for a space  $X$ , the initial map  $X \rightarrow X_S$  in  $\text{Ho}(\text{Top})$  to an  $S$ -local space factors through the initial map  $X \rightarrow X_R$  in  $\text{Ho}(\text{Top})$  to an  $R$ -local space. See also Lemma 6.6.12 in the following.

The  $p$ -localization is easier to define than the  $p$ -completion, but the  $p$ -completion is in general more computable and therefore more useful. This is related to the fact that we have more methods for calculating maps into  $p$ -complete spaces — they are “more algebraic” in the sense that  $p$ -completions can be expressed as limits of  $K(\mathbb{F}_p, n)$ ’s for different values of  $n$  (in fact in several different ways). This ultimately gives us ways of understanding the maps into  $p$ -complete spaces in terms of mod  $p$  cohomology.

We will also examine how a space  $X$  can be recovered from these localizations. This is called Sullivan’s arithmetic square or, as Sullivan writes, a Hasse principle for spaces. There is both a Sullivan square for  $p$ -localization and one for  $p$ -completion.

We will now consider some examples of  $R$ -local spaces:

lem:EMploc

LEMMA 6.2.6. *Let  $R$  be a commutative ring. Then the Eilenberg-MacLane space  $K(M, n)$  is  $R$ -local for any  $R$ -module  $M$ .*

PROOF. Let  $f: X \rightarrow Y$  be an  $R$ -local map of spaces, i.e.  $f_*: H_*(X; R) \rightarrow H_*(Y; R)$  is an isomorphism; we want to prove that  $f$  induces a bijection  $[Y, K(M, n)] \xrightarrow{\cong} [X, K(M, n)]$  between homotopy classes of maps into  $K(M, n)$ . We have  $[X, K(M, n)] \cong H^n(X; M)$ , so our goal is to prove that  $f$  induces a bijection on  $H^n(-; M)$ .

Up to replacing  $Y$  by the mapping cylinder of  $f$ , we can assume that  $f$  is a cofibration, and hence we can consider  $(Y, X)$  as a pair of spaces. The hypothesis and the LES in  $H_*(-; R)$ -homology imply that  $H_*(Y, X; R) = 0$  in all degrees.

Now  $H_*(Y, X; R)$  is computed by  $C_*(Y, X; R)$ , i.e. the relative, singular chain complex of  $(Y, X)$  with coefficients in  $R$ : this is a non-negatively graded chain complex of free  $R$ -modules (note also that the boundary maps are  $R$ -linear), and since it is acyclic, it has to be chain null-homotopic.

Applying the additive functor  $\text{Hom}_R(-; M)$  to  $C_*(Y, X; R)$ , we obtain precisely the cochain complex  $C^*(Y, X; M)$ : in this step it is good to notice that  $C_*(Y, X; R)$  is itself  $C_*(Y, X; \mathbb{Z}) \otimes R$ . Hence also  $C^*(Y, X; M)$  is chain null-homotopic, and in particular  $H^*(Y, X; M) = 0$  in all degrees. The LES for  $H^*(-; M)$  implies in degree  $n$  the desired statement.  $\square$

lem:EMpcom

LEMMA 6.2.7. *The spaces  $K(\mathbb{Z}/p^k, n)$  and  $K(\mathbb{Z}_p^\wedge, n)$  are  $\mathbb{F}_p$ -local.*

PROOF. To see this for  $K(\mathbb{Z}/p^k, n)$  we induct on  $k$  (note that the case  $k = 1$  follows from Lemma 6.2.6) and use the long exact sequence in cohomology induced by the short exact sequence of groups

$$0 \rightarrow \mathbb{Z}/p^{k-1} \rightarrow \mathbb{Z}/p^k \rightarrow \mathbb{Z}/p \rightarrow 0$$

plus the 5-Lemma. For  $\mathbb{Z}_p^\wedge$  we use the Milnor  $\lim^1$ -sequence, which is

$$0 \rightarrow \lim_k^1 H^{n-1}(X; \mathbb{Z}/p^k) \rightarrow H^n(X; \mathbb{Z}_p^\wedge) \rightarrow \lim_k H^n(X; \mathbb{Z}/p^k) \rightarrow 0,$$

to see we get an isomorphism on  $\mathbb{Z}_p^\wedge$ -cohomology.  $\square$

### 6.3. $p$ -localization of simply connected spaces

Suppose that  $R \subseteq \mathbb{Q}$  — this implies that  $R = S^{-1}\mathbb{Z}$  for some set  $S$  of primes. Homology with  $R$ -coefficients for such  $R$  is easy to describe: we have  $H_*(-; R) = H_*(-) \otimes R$  by the Universal Coefficient Theorem, since the functor  $- \otimes R$  is exact for  $R$  a torsion-free abelian group. Furthermore, we have an alternative description of  $R$ -equivalences between simply connected spaces:

Whitehead-serre

LEMMA 6.3.1. *Suppose  $R \subseteq \mathbb{Q}$ . Let  $f : X \rightarrow Y$  be a map between simply connected spaces. Then  $f$  is an  $R$ -equivalence if and only if it induces an isomorphism  $\pi_*(X) \otimes R \rightarrow \pi_*(Y) \otimes R$ .*

PROOF. This follows from the Whitehead theorem modulo Serre Classes, cf Theorem 6.3.2 below, taking as Serre class the class of abelian groups  $A$  such that  $A \otimes R = 0$ .  $\square$

Whitehead-serre

THEOREM 6.3.2 (Whitehead theorem modulo Serre classes). *Let  $f : X \rightarrow Y$  be a map between simply connected spaces, and  $\mathcal{C}$  a Serre class. Then  $\pi_*(f)$  is an isomorphism modulo  $\mathcal{C}$  if and only if  $H_*(f)$  is an isomorphism modulo  $\mathcal{C}$ .*

PROOF. This is almost in Hatcher [Hat], (cf. [Hat, Thm. 5.8]) but not stated explicitly.

Recall that a Serre class  $\mathcal{C}$  is a class of abelian groups containing 0 and closed under isomorphisms, taking subgroups, quotients, extensions, arbitrary filtered colimits, tensor products against arbitrary abelian groups and Tor against arbitrary abelian groups. Examples of Serre classes are finite groups, and  $S$ -power torsion groups for a



subset  $S \subset \mathbb{Z}$  of prime numbers (this includes the class of *all* torsion abelian groups). We note that the intersection of a Serre class  $\mathcal{C}$  with finitely generated abelian groups can only be one of the following:

- all f.g. abelian groups;
- for a set of primes  $S \subset \mathbb{Z}$ , all finite abelian groups with only  $S$ -power torsion.

Using the classification of f.g. abelian groups, knowledge about homology of cyclic groups (including  $\mathbb{Z}\dots$ ) and the Künneth theorem, one can prove the following statement (prove it yourself!): if  $M \in \mathcal{C}$  is f.g., then the group homology  $\tilde{H}_*(M; \mathbb{Z}) = \tilde{H}_*(K(M; 1); \mathbb{Z})$  is degree-wise in  $\mathcal{C}$  (we write  $\tilde{H}_*(K(M; 1); \mathbb{Z}) \subset \mathcal{C}$ ).

We then note that a generic  $M \in \mathcal{C}$  can be written as filtered colimit of its f.g. subgroups, which also lie in  $\mathcal{C}$ ; correspondingly  $\tilde{H}_*(K(M; 1); \mathbb{Z})$  is the filtered colimit of groups in  $\mathcal{C}$ , hence is in  $\mathcal{C}$  as well.

We then prove for  $n \geq 2$  and  $M \in \mathcal{C}$  that  $\tilde{H}_*(K(M; n); \mathbb{Z}) \subset \mathcal{C}$ . For this we use the fibration sequence  $K(M; n-1) \simeq \Omega K(M; n) \rightarrow * \simeq PK(M; n) \rightarrow K(M; n)$  and the associated Serre spectral sequence in (unreduced) homology. On the second page we have  $H_p(K(M; n); H_q(K(M; n-1); \mathbb{Z}))$ : this is  $H_p(K(M; n); \mathbb{Z})$  for  $q = 0$ , and is in  $\mathcal{C}$  for  $q \geq 1$  and any  $p$ , using homology universal coefficients, Tor and tensor stability of  $\mathcal{C}$ , and the inductive hypothesis  $\tilde{H}_*(K(M; n-1); \mathbb{Z}) \subset \mathcal{C}$ . The spectral sequence converges to  $0 \in \mathcal{C}$  in all places except  $p = q = 0$  (which is why the final statement only works for reduced homology); using this, one can exhibit each  $H_p(K(M; n); \mathbb{Z})$  (with  $p \geq 1$ ) as an iterated extension of groups in  $\mathcal{C}$ , hence a group in  $\mathcal{C}$  as well.

Once these preliminary steps are done, we are ready to start the actual proof. We say that a map of abelian groups  $\phi : A \rightarrow B$  is an “isomorphism mod  $\mathcal{C}$ ” if  $\ker(\phi)$  and  $\text{coker}(\phi)$  are in  $\mathcal{C}$ . Given a LES of abelian groups of the form

$$\dots \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow A_{n-1} \rightarrow B_{n-1} \rightarrow C_{n-1} \rightarrow \dots$$

we observe that asking all  $A_n \in \mathcal{C}$  is equivalent to asking all maps  $B_n \rightarrow C_n$  be isomorphisms mod  $\mathcal{C}$ .

Let now  $f : X \rightarrow Y$  be a map of simply connected spaces, and assume first that  $\pi_*(f)$  is an isomorphism mod  $\mathcal{C}$ ; let  $Z$  be the homotopy fibre of  $f$  (for a reminder on homotopy fibres, see Definition 6.3.14). The LES of homotopy groups tells us that  $Z$  is connected,  $\pi_1(Z)$  is abelian and  $\pi_*(Z) \subset \mathcal{C}$ . We first want to prove that  $\tilde{H}_*(Z; \mathbb{Z}) \subset \mathcal{C}$ ; to prove that  $H_i(Z; \mathbb{Z}) \in \mathcal{C}$  for a fixed  $i \geq 1$ , we can actually replace  $Z$  by the Postnikov truncation  $P_i Z$  (see Proposition 6.3.5 for a reminder on the Postnikov tower), using that  $Z \rightarrow P_i Z$  is  $(i+1)$ -connected, in particular it is an isomorphism on  $H_i(-; \mathbb{Z})$ ; and  $P_i Z$  enjoys all listed properties of  $Z$ , plus that  $\pi_*(P_i Z) = 0$  for  $* \geq i+1$ .

Now we need the following lemma, left as exercise: if  $F \rightarrow E \rightarrow B$  is a fibre sequence of connected spaces, with  $B$  simply connected, and  $\tilde{H}_*(B), \tilde{H}_*(F) \subset \mathcal{C}$ , then also  $\tilde{H}_*(E) \subset \mathcal{C}$ ; this is an application of the Serre spectral sequence (notice that the very definition of Serre class is designed so that this argument goes through!). Since  $P_i Z$  is a (finite) iterated tower of fibrations with fibres of the form  $K(M; n)$  with  $M \in \mathcal{C}$  and  $n \leq i$ , we obtain that  $H_i(P_i Z; \mathbb{Z}) \cong H_i(Z; \mathbb{Z}) \in \mathcal{C}$  as desired.

Once we have proved that  $\tilde{H}_*(Z; \mathbb{Z}) \subset \mathcal{C}$ , we can run the Serre spectral sequence for  $Z \rightarrow X \rightarrow Y$ : on the  $E^2$ -page we see  $H_*(Y; \mathbb{Z})$  on the bottom line  $E_{0,*}^2$ , and above, on  $E_{p,q}^2$ , there is only some “noise” in  $\mathcal{C}$ ; the limit is  $H_*(X; \mathbb{Z})$ , and the effect of passing to the limit on the bottom row is, modulo noise in  $\mathcal{C}$ , the map  $f_*: H_*(X; \mathbb{Z}) \rightarrow H_*(Y; \mathbb{Z})$ , which is hence an equivalence mod  $\mathcal{C}$ .

Viceversa, suppose  $f: X \rightarrow Y$  is an  $H_*(-; \mathbb{Z})$ -isomorphism mod  $\mathcal{C}$ , replace  $f$  by an inclusion via mapping cylinder, and reinterpret the hypothesis as  $H_*(Y, X) \subset \mathcal{C}$ . Let  $i \geq 2$  be the minimal degree (if any) in which  $\pi_*(f)$  is not an isomorphism.

If  $\pi_i(f)$  is not surjective, let  $M_1 = \text{coker}(\pi_i(f))$ , and note that the relative Hurewicz theorem identifies this with a subgroup of  $\pi_i(Y, X) \cong H_i(Y, X) \in \mathcal{C}$ , so  $M_1 \in \mathcal{C}$ . We can then consider the natural map  $Y \rightarrow K(M_1, i)$  and denote  $Y'$  its homotopy fibre; a Serre spectral sequence argument shows that  $Y' \rightarrow Y$  is a  $H_*$ -isomorphism mod  $\mathcal{C}$ , and it is clearly also a  $\pi_*$ -isomorphism mod  $\mathcal{C}$ ; moreover  $f: X \rightarrow Y$  factors up to homotopy through a map  $f': X \rightarrow Y'$ , which is a  $H_*$ -isomorphism mod  $\mathcal{C}$  by 2-of-3. We can thus just aim at proving that  $f'$  is a  $\pi_*$ -isomorphism mod  $\mathcal{C}$ , and we have basically reduced to the case in which  $\pi_i(f)$  is surjective.

If  $\pi_i(f)$  is not injective (but surjective), denote  $M_2 = \ker(\pi_i(f))$ , and note that relative Hurewicz identifies  $M_2$  with a quotient of  $\pi_{i+1}(Y, X) \cong H_{i+1}(Y, X) \in \mathcal{C}$ . Let  $Y''$  be the fibre product of  $Y$  and  $K(\pi_i(X), i)$  over  $K(\pi_i(Y), i)$ , and note that  $f: X \rightarrow Y$  factors through a map  $f'': X \rightarrow Y''$ . Moreover the homotopy fibre of  $Y'' \rightarrow Y$  is a  $K(M_2, i)$ , which again by a Serre spectral sequence argument implies that  $Y'' \rightarrow Y$  is both a  $H_*$ -isomorphism mod  $\mathcal{C}$  and a  $\pi_*$ -isomorphism mod  $\mathcal{C}$ . This in turn implies that we can ask ourselves whether  $f''$ , which we know is a  $H_*$ -isomorphism mod  $\mathcal{C}$  by 2-of-3, is also a  $\pi_*$ -isomorphism mod  $\mathcal{C}$ . And now we have reduced to the case in which  $\pi_i(f)$  is also injective, hence an isomorphism.

By a sequence of such replacements, for any fixed  $i \geq 2$  we can reduce to the case in which we assume that  $f$  is a  $\pi_*$ -isomorphism for  $* \leq i$  and a  $H_*$ -isomorphism mod  $\mathcal{C}$  in all degrees, and we want to prove that  $\pi_i(f)$  is an isomorphism mod  $\mathcal{C}$ : now this is really obvious! □

Our first main result gives a characterization of  $R$ -local spaces:

thm:plocchar **THEOREM 6.3.3.** *Suppose  $X$  is simply connected and  $R \subseteq \mathbb{Q}$ . Then  $X$  is  $R$ -local if and only if  $\pi_n(X)$  is  $R$ -local for all  $n$ .*

Let us first prove the easy direction:

n:easyplocchar **LEMMA 6.3.4.** *Suppose  $R \subseteq \mathbb{Q}$  and  $X$  is a simply connected  $R$ -local space. Then  $\pi_n(X)$  is  $R$ -local for all  $n$ .*

**PROOF.** The degree- $p$  map  $S^n \xrightarrow{p} S^n$  is in  $W_R$  for all primes  $p$  that are invertible in  $R$ : We have  $H_*(S^n; R) \cong H_*(S^n) \otimes R$ , and on  $H_*(S^n)$  this map is given by multiplication by  $p$ . Thus if  $X$  is  $R$ -local, this map induces an isomorphism on  $[S^n, X] = \pi_n(X)$ , i.e.  $\pi_n(X)$  is  $R$ -local. □

For the less trivial direction, we will use an induction going up the Postnikov tower, so we first briefly review this:

postnikov

PROPOSITION 6.3.5 (Postnikov towers). *Suppose that  $X$  is a simply connected CW complex. Then there exists a tower of principal fibrations*

$$\cdots \rightarrow P_n X \rightarrow P_{n-1} X \rightarrow P_1 X$$

and a map from  $X$  into the tower, satisfying the following:

- $X \rightarrow \lim_n P_n X$  is a homotopy equivalence,
- $\pi_i(X) \rightarrow \pi_i(P_n X)$  is an isomorphism for  $i \leq n$  and  $\pi_i(P_n X) = 0$  for  $i > n$ .

See for example [Hat02, Theorem 4.69] for a proof. Here a fibration  $F \rightarrow E \rightarrow B$  is called *principal* if we have a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \Omega X & \longrightarrow & E' & \longrightarrow & B' \longrightarrow X, \end{array}$$

where  $B' \rightarrow X$  is a fibration, the vertical maps are weak equivalences, and the bottom row is the start of a Puppe sequence. For the principal fibration  $P_n X \rightarrow P_{n-1} X$  it follows from the long exact sequence that the fibre is a  $K(\pi_n X, n)$ , so the “classifying space”  $X$  has to be  $K(\pi_n X, n+1)$ .<sup>3</sup>

Thus we have a homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow k_{n-1} \\ * & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

The maps  $k_{n-1}: P_{n-1} X \rightarrow K(\pi_n X, n+1)$  correspond to cohomology classes  $k_{n-1} \in H^{n+1}(P_{n-1} X; \pi_n X)$  called the  $k$ -invariants of  $X$ .

REMARK 6.3.6. For a general space the Postnikov tower still exists as a tower of fibrations, but in general they do not have to be *principal*. In fact, it can be shown that the fibrations in the Postnikov tower are principal if and only if the space  $X$  is *simple*, which means that  $\pi_1 X$  is abelian and acts trivially on  $\pi_* X$ . (The homotopy fibre of a map of simply connected spaces is always simple; this example will come up later.) More generally, Postnikov towers are also well-behaved for *nilpotent* spaces, meaning spaces for which the fundamental group is nilpotent (in the sense of group theory) and acts nilpotently on the higher homotopy groups; but this is a bit more complicated.

productinvariantW

DEFINITION 6.3.7. Let  $\mathcal{W}$  be a class of maps in  $\text{Ho}(\text{Top})$  as in Remark 6.1.2. We say that  $\mathcal{W}$  is *product invariant* if the following holds: whenever  $f: X \rightarrow Y$  is a map in  $\mathcal{W}$  and  $S$  is a CW complex, then  $S \times f: S \times X \rightarrow S \times Y$  is in  $\mathcal{W}$  as well.

<sup>3</sup>As exercise, prove that the classifying space  $\text{BhAut}(K(A, n))$  for  $K(A, n)$ -fibrations is homotopy equivalent to the homotopy quotient  $K(A, n+1)//\text{Aut}(A)$ ; then *principal*  $K(A, n)$ -fibrations are precisely those whose classifying map lifts to  $K(A, n+1)$ .

For instance, by the Künneth theorem we have that for any commutative ring  $R$  the collection  $W_R$  of  $H_*(-; R)$ -equivalences is product invariant. If you only know the basic Künneth statement saying that  $H_*(X \times Y; R) \cong H_*(X; R) \otimes_R H_*(Y; R)$  if either  $H_*(X; R)$  or  $H_*(Y; R)$  is a free  $R$ -module, this is enough! Just filter  $S$  by skeleta, prove that if  $f : X \rightarrow Y$  is a  $H_*(-; R)$ -isomorphism then also  $H_*(\text{sk}_i S \times X, \text{sk}_{i-1} S \times X; R) \cong H_*(\text{sk}_i S \times Y, \text{sk}_{i-1} Y \times X; R)$ , and conclude via repeated 5-lemma. See also Lemma 6.6.7. And compare the following with Lemma 6.6.8.

ductinvariant $\mathcal{W}$

LEMMA 6.3.8. *Let  $\mathcal{W}$  be a class of maps as in Remark 6.1.2. Then the following are equivalent:*

- (1)  $\mathcal{W}$  is product invariant;
- (2) for any  $f : X \rightarrow Y$  in  $\mathcal{W}$  and any  $\mathcal{W}$ -local space  $Z$ , the induced map between mapping spaces  $\text{map}(f, Z) : \text{map}(Y, Z) \rightarrow \text{map}(X, Z)$  is a weak equivalence.

PROOF. (1)  $\Rightarrow$  (2). Fix  $f : X \rightarrow Y$  in  $\mathcal{W}$  and a  $\mathcal{W}$ -local space  $Z$ . By the Yoneda lemma, it suffices to show that composing with  $f$  induces a natural isomorphism of functors  $[-, \text{map}(Y, Z)]$  and  $[-, \text{map}(X, Z)]$  from CW to Set. However, for  $S$  a CW complex we have  $\text{map}(S, \text{map}(Y, Z)) \cong \text{map}(S \times Y, Z)$ , and similarly for  $X$ , hence, taking components, we have a bijection  $[S, \text{map}(Y, Z)] \cong [S \times Y, Z]$ , and similarly for  $X$ ; we can now use the hypotheses on  $f$  and  $Z$  to get the bijection  $[S \times Y, Z] \cong [S \times X, Z]$ , which translates to the bijection  $[S, \text{map}(Y, Z)] \cong [S, \text{map}(X, Z)]$ .

(2)  $\Rightarrow$  (1). Fix again  $f : X \rightarrow Y$  in  $\mathcal{W}$  and a  $\mathcal{W}$ -local space  $Z$ , and fix  $S \in \text{CW}$ . Then again we can convert the given bijection  $[S, \text{map}(Y, Z)] \cong [S, \text{map}(X, Z)]$  into  $[S \times Y, Z] \cong [S \times X, Z]$ ; since  $Z$  is arbitrary and  $\mathcal{W}$  is assumed to be “saturated” as in Remark 6.1.2, we obtain that  $S \times f$  is also in  $\mathcal{W}$ .  $\square$

Using this we have:

lem:pbloc

LEMMA 6.3.9 (Fibre Lemma). *Suppose  $E \rightarrow B$  is a fibration where  $E$  and  $B$  are  $\mathcal{W}$ -local, and where  $\mathcal{W}$  is a product invariant class of maps. Then the fibre  $F$  is also  $\mathcal{W}$ -local.*

PROOF. By Lemma 6.3.8, given  $f : X \rightarrow Y$  in  $\mathcal{W}$ , we have a commutative diagram whose vertical arrows are weak equivalences

$$\begin{array}{ccc} \text{map}(Y, E) & \xrightarrow{p_Y} & \text{map}(Y, B) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{map}(X, E) & \xrightarrow{p_X} & \text{map}(X, B) \end{array}$$

It follows that, for any  $* \in B$ , the homotopy fibres of  $p_Y$  and  $p_X$  over the constant maps  $Y \rightarrow * \hookrightarrow B$  and  $X \rightarrow * \hookrightarrow B$ , respectively, are homotopy equivalent: these homotopy fibres are however also homotopy equivalent to the actual fibres of the fibrations  $p_Y$  and  $p_X$ , that is  $\text{map}(Y, F)$  and  $\text{map}(X, F)$ , respectively, where  $F$  is the fibre of  $E \rightarrow B$  over  $*$ . Using again Lemma 6.3.8, we conclude that  $F$  is  $\mathcal{W}$ -local.  $\square$

lem:limloc

LEMMA 6.3.10 (Tower Lemma). *Suppose  $\cdots \rightarrow Z_2 \rightarrow Z_1 \rightarrow Z_0$  is a tower of fibrations between  $\mathcal{W}$ -local spaces, where  $\mathcal{W}$  is a product invariant class of maps. Then  $\lim_{n \rightarrow \infty} Z_n$  is also  $\mathcal{W}$ -local.*

PROOF. Using the characterization from Lemma 6.3.8, it suffices to prove that, for  $f: X \rightarrow Y$  in  $\mathcal{W}$ , we have a weak equivalence  $\text{map}(Y, \lim_{n \rightarrow \infty} Z_n) \simeq \text{map}(X, \lim_{n \rightarrow \infty} Z_n)$ . However we can replace  $\text{map}(Y, \lim_{n \rightarrow \infty} Z_n) \cong \lim_{n \rightarrow \infty} \text{map}(Y, Z_n)$ , and similarly for  $X$ . We conclude by noting that  $\lim_{n \rightarrow \infty} \text{map}(Y, Z_n)$  is also the limit of a tower of fibrations, and  $\lim_{n \rightarrow \infty} \text{map}(X, Z_n)$  is the limit of a levelwise weakly equivalent tower: the two limits are therefore weakly equivalent.  $\square$

We can now complete the proof of the Theorem:

PROOF OF THEOREM 6.3.3. It remains to prove that if a simply connected space  $X$  has  $R$ -local homotopy groups, then  $X$  is  $R$ -local. We will first show by induction that the spaces  $P_n X$  in the Postnikov tower of  $X$  are  $R$ -local. Since  $X$  is simply connected, the first of these is  $P_2 X$  which is  $K(\pi_2 X, 2)$ ; this is  $R$ -local by Lemma 6.2.6. Assuming  $P_{n-1} X$  is  $R$ -local, we have a homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow^{k_{n-1}} \\ * & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

Here  $K(\pi_n X, n+1)$  is again  $R$ -local by Lemma 6.2.6, so  $P_n X$  is  $R$ -local by Lemma 6.3.9. Then  $X \simeq \lim_n P_n X$  is  $R$ -local by Lemma 6.3.10.  $\square$

EXAMPLE 6.3.11. Let  $J = \{p_1, p_2, \dots\}$  be the set of all primes except  $p$ . Then the space

$$S_{(p)}^n := \text{hocolim}(S^n \xrightarrow{p_1} S^n \xrightarrow{p_1 p_2} S^n \xrightarrow{p_1 p_2 p_3} \dots)$$

is a  $p$ -localization of  $S^n$ : We can describe this homotopy colimit by replacing the maps by cofibrations and then taking the sequential colimit. Since  $\pi_*$  commutes with sequential colimits along cofibrations (this by a compactness argument), we get

$$\pi_*(S_{(p)}^n) \cong \text{colim}_k \pi_*(S^n) \cong \pi_*(S^n)_{(p)}.$$

Thus  $S_{(p)}^n$  is  $p$ -local by Theorem 6.3.3, and the first-inclusion map  $S^n \rightarrow S_{(p)}^n$  is a  $\mathbb{Z}_{(p)}$ -equivalence by Lemma 6.3.1.

thm:plocexist

THEOREM 6.3.12. Suppose  $X$  is simply connected and  $R \subseteq \mathbb{Q}$ . Then the  $R$ -localization  $X \rightarrow X_R$  exists and is characterized as the unique (up to homotopy) map  $X \rightarrow Y$  to a simply connected  $Y$  that induces an isomorphism  $\pi_*(X) \otimes R \xrightarrow{\sim} \pi_*(Y)$ .

We will again prove this by an induction using the Postnikov tower. The base case is the following:

lem:plocEM

LEMMA 6.3.13. Suppose  $M$  is an abelian group and  $R \subseteq \mathbb{Q}$ . Then the map  $K(M, n) \rightarrow K(M \otimes R, n)$  exhibits  $K(M \otimes R, n)$  as the  $R$ -localization of  $K(M, n)$ .

PROOF. The space  $K(M \otimes R, n)$  is  $R$ -local by Theorem 6.3.3 (or just by Lemma 6.2.6, since  $M \otimes R$  is an  $R$ -module), and the map  $K(M, n) \rightarrow K(M \otimes R, n)$  is an  $R$ -equivalence by Lemma 6.3.1.  $\square$

Since the universal property of  $R$ -localizations only produces homotopy commutative squares, for the induction we will need an observation about homotopy fibres:

homotopyfibre

DEFINITION 6.3.14. The homotopy fiber of a based map  $f : (X, x) \rightarrow (Y, y)$  is defined as the space  $F_f = X \times_Y PY$ , where  $PY$  is the path space consisting of paths  $p: I \rightarrow Y$  such that  $p(0) = y$ ; the pullback is taken along  $f$  and the map  $PY \rightarrow Y$  given by evaluation at 1. Note that the projection map  $F_f \rightarrow X$  is a fibration, since  $PY \rightarrow Y$  is, and that  $F_f$  can be viewed as the actual fiber of the fibration  $X \times_Y Y^I \rightarrow Y$ , given by evaluating  $p$  at 0; moreover the map  $X \rightarrow X \times_Y Y^I$  induced by the constant-path map  $Y \rightarrow Y^I$  is a homotopy equivalence.

(See also the discussion after [Hat02, Prop, 4.64].)

fillermap

LEMMA 6.3.15. *Suppose that we have a homotopy-commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \alpha & & \downarrow \beta \\ Z & \xrightarrow{g} & W \end{array}$$

Then there exists a map  $h : F_f \rightarrow F_g$  such that we have a diagram

$$\begin{array}{ccccccc} \Omega Y & \longrightarrow & F_f & \longrightarrow & X & \xrightarrow{f} & Y \\ \downarrow \Omega \beta & & \downarrow h & & \downarrow \alpha & & \downarrow \beta \\ \Omega W & \longrightarrow & F_g & \longrightarrow & Z & \xrightarrow{g} & W \end{array}$$

where the first and third squares commute up to homotopy and the second square commutes strictly.

For a proof see for example [MP12, Lemma 1.2.3].

PROOF OF THEOREM 6.3.12. Note that it suffices to construct a map of spaces  $X \rightarrow X'$  such that  $\pi_*(X) \otimes R \rightarrow \pi_*(X')$  is an isomorphism — then  $X'$  is  $R$ -local by Theorem 6.3.3, and the map  $X \rightarrow X'$  is an  $R$ -equivalence by Lemma 6.3.1. We construct such a map by going up the Postnikov tower. Since  $X$  is simply connected, the base case is  $P_2X = K(\pi_2X, 2)$  where  $P_2X \rightarrow (P_2X)_R$  is given by  $K(\pi_2X, 2) \rightarrow K(\pi_2X \otimes R, 2)$  by Lemma 6.3.13. Now suppose we have an  $R$ -localization  $P_{n-1}X \rightarrow (P_{n-1}X)_R$ . The composite  $k_{n-1}: P_{n-1}X \rightarrow K(\pi_nX, n+1) \rightarrow K(\pi_nX \otimes R, n+1)$  is a map from  $P_{n-1}X$  to an  $R$ -local space, so by the universal property of  $R$ -localization in  $\text{Ho}(\text{Top})$  there is a unique commutative square

$$\begin{array}{ccc} P_{n-1}X & \longrightarrow & (P_{n-1}X)_R \\ \downarrow k_{n-1} & & \downarrow k_{n-1,R} \\ K(\pi_nX, n+1) & \longrightarrow & K(\pi_nX \otimes R, n+1). \end{array}$$

By Lemma 6.3.15 there is then a diagram

$$\begin{array}{ccccccc} K(\pi_nX, n) & \longrightarrow & F_{k_{n-1}} & \longrightarrow & P_{n-1} & \xrightarrow{k_{n-1}} & K(\pi_nX, n+1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(\pi_nX \otimes R, n) & \longrightarrow & F_{k_{n-1,R}} & \longrightarrow & (P_{n-1}X)_R & \xrightarrow{k_{n-1,R}} & K(\pi_nX \otimes R, n+1), \end{array}$$

where the middle square commutes strictly. The homotopy fibre  $F_{k_{n-1}}$  is homotopy equivalent to  $P_n X$ , and from the fibration sequence we see that  $(P_n X)_R := F_{k_{n-1},R}$  is an  $R$ -localization of  $P_n X$ . Now we define  $X_R := \lim_n (P_n X)_R$ . This is  $R$ -local by Lemma 6.3.10, and  $\pi_i X_R \cong \pi_i((P_n X)_R) \cong \pi_i X \otimes R$  (for  $n \geq i$ ; since the homotopy groups of  $(P_n X)_R$  stabilize we have no  $\lim^1$ -term).  $\square$

### 6.4. Sullivan's arithmetic square for $p$ -localization

In this section we construct Sullivan's arithmetic square for  $p$ -localization, which is easy.

chm:plocsquare

**THEOREM 6.4.1** (Sullivan's arithmetic square for  $p$ -localization). *Suppose  $X$  is simply connected. Then the canonical square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_{(p)} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left( \prod_p X_{(p)} \right)_{\mathbb{Q}} \end{array}$$

*is a homotopy pullback.*

**REMARK 6.4.2.** The bottom horizontal map in the diagram is the  $\mathbb{Q}$ -localization of the top map, so the diagram commutes — strictly if we have a functorial model for  $\mathbb{Q}$ -localization, otherwise just in the homotopy category.

For the proof we need an algebraic lemma:

grouplemma

**LEMMA 6.4.3.** *Let  $M$  be an abelian group. Then the commutative square*

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & \left( \prod_p M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \end{array}$$

*gives isomorphisms on the kernels and cokernels of the rows.*

**PROOF.** We first observe that it suffices to prove that this holds after localizing at all primes  $q$ : A map of abelian groups  $f: A \rightarrow B$  is an isomorphism if and only if the maps  $f \otimes \mathbb{Z}_{(q)}$  are isomorphisms for all primes  $q$ , and since  $- \otimes \mathbb{Z}_{(q)}$  is exact this implies we get an isomorphism on (co)kernels in the square if and only if we get such isomorphisms after tensoring the square with  $\mathbb{Z}_{(q)}$  for any  $q$ .

We can rewrite the square as

$$\begin{array}{ccc} M & \longrightarrow & (M \otimes \mathbb{Z}_{(q)}) \oplus \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & (M \otimes \mathbb{Q}) \oplus \left( \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \end{array}$$

Tensoring this with  $\mathbb{Z}_{(q)}$  now gives

$$\begin{array}{ccc} M \otimes \mathbb{Z}_{(q)} & \longrightarrow & (M \otimes \mathbb{Z}_{(q)}) \oplus \left( \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & (M \otimes \mathbb{Q}) \oplus \left( \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \end{array}$$

(here we use that  $\left( \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Z}_{(q)} \cong \left( \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q}$  as  $q$  is already inverted in this product); this square is clearly both a pushout and a pullback, and so it gives isomorphisms on the kernels and cokernels of the rows.  $\square$

We also make use of the following criterion for homotopy pullbacks:

LEMMA 6.4.4. *A commutative square of spaces*

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow f & & \downarrow g \\ A & \xrightarrow{\phi} & B \end{array}$$

is a homotopy pullback square if and only if the induced map on homotopy fibres  $F_f \rightarrow F_g$  is a weak equivalence for all choices of  $\phi$ -related points in  $A$  and  $B$ .

PROOF. We can replace the maps  $f$  and  $g$  by fibrations. This reduces us to prove that the map  $P \rightarrow \phi^*Q$  of fibrations over  $A$  is a weak equivalence if and only if each map of fibres is a weak equivalence, which is immediate from the long exact sequences on homotopy groups and the 5-Lemma, with all possible choices of basepoints.  $\square$

PROOF OF THEOREM 6.4.1. We assume our square commutes strictly — otherwise we can replace it by a weakly equivalent square that does. Let  $F$  and  $F'$  denote the homotopy fibres of  $\alpha: X \rightarrow \prod_p X_{(p)}$  and  $\beta: X_{\mathbb{Q}} \rightarrow \left( \prod_p X_{(p)} \right)_{\mathbb{Q}}$ , respectively. Then we want to show that the induced map  $F \rightarrow F'$  is a weak equivalence. We consider the map of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n F & \longrightarrow & \pi_n X & \xrightarrow{\pi_n \alpha} & \pi_n \left( \prod_p X_{(p)} \right) & \longrightarrow & \pi_{n-1} F & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_n F' & \longrightarrow & \pi_n X_{\mathbb{Q}} & \xrightarrow{\pi_n \beta} & \pi_n \left( \prod_p X_{(p)} \right)_{\mathbb{Q}} & \longrightarrow & \pi_{n-1} F' & \longrightarrow & \cdots \end{array}$$

By Theorem 6.3.12 we can identify the middle square with

$$\begin{array}{ccc} \pi_n X & \longrightarrow & \prod_p \pi_n X \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \pi_n X \otimes \mathbb{Q} & \longrightarrow & \left( \prod_p \pi_n X \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q}, \end{array}$$



and by Lemma 6.4.3 we have isomorphisms on the kernels and cokernels of the rows, i.e. the induced maps  $\ker \pi_n \alpha \rightarrow \ker \pi_n \beta$  and  $\operatorname{coker} \pi_n \alpha \rightarrow \operatorname{coker} \pi_n \beta$  are isomorphisms.

For each  $n$  we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{coker} \pi_{n+1} \alpha & \longrightarrow & \pi_n F & \longrightarrow & \ker \pi_n \alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{coker} \pi_{n+1} \beta & \longrightarrow & \pi_n F' & \longrightarrow & \ker \pi_n \beta & \longrightarrow & 0. \end{array}$$

The 5-Lemma now implies that the map  $\pi_n F \rightarrow \pi_n F'$  is an isomorphism, which completes the proof.  $\square$

### 6.5. $p$ -completion of simply connected spaces I (the easy case)

We now turn to  $p$ -completion of spaces, i.e. localization with respect to  $\mathbb{F}_p$ . In this section we focus on the case of spaces with finitely generated homotopy groups, which explains where the term “ $p$ -completion” comes from. We will return to the general case below in §6.8 after constructing Sullivan’s arithmetic square in the next section. Our goal in this section is to prove

thm:fgpcom

THEOREM 6.5.1.

- (i) If  $X$  is a simply connected space such that  $\pi_n X$  is a finitely generated  $\mathbb{Z}_p^\wedge$ -module for all  $n$ , then  $X$  is  $p$ -complete.
- (ii) If  $X$  is a simply connected space with finitely generated homotopy groups, then the  $p$ -completion  $X \rightarrow X_p^\wedge$  of  $X$  exists and is characterized by the map  $\pi_* X \otimes \mathbb{Z}_p^\wedge \rightarrow \pi_*(X_p^\wedge)$  being an isomorphism.<sup>4</sup>
- (iii) If  $X$  and  $Y$  are simply connected spaces with finitely generated homotopy groups, then a map  $f: X \rightarrow Y$  is an  $\mathbb{F}_p$ -equivalence if and only if  $\pi_* f \otimes \mathbb{Z}_p^\wedge$  is an isomorphism.

WARNING 6.5.2. The finiteness hypotheses are essential here — for more general homotopy groups simply taking their  $p$ -completion does not give the  $\mathbb{F}_p$ -localization.

We can already prove (i):

PROOF OF THEOREM 6.5.1(I). We saw in Lemma 6.2.7 that the Eilenberg-MacLane spaces  $K(\mathbb{Z}/p^k, n)$  and  $K(\mathbb{Z}_p^\wedge, n)$  are  $p$ -complete. A finitely generated  $\mathbb{Z}_p^\wedge$ -module  $M$  is a finite direct sum of these modules (indeed  $\mathbb{Z}_p^\wedge$  is a PID), so it follows that  $K(M, n)$  is a finite product of  $p$ -complete spaces and hence also  $p$ -complete.

We now consider the Postnikov tower of  $X$ . The space  $P_2 X = K(\pi_2 X, 2)$  is  $p$ -complete since  $\pi_2 X$  is a finitely generated  $\mathbb{Z}_p^\wedge$ -module. And if  $P_{n-1} X$  is  $p$ -complete then we see that  $P_n X$  is  $p$ -complete by applying Lemma 6.3.9 to the homotopy

<sup>4</sup>More precisely: there is an isomorphism  $\pi_* X \otimes \mathbb{Z}_p^\wedge \cong \pi_*(X_p^\wedge)$  under  $\pi_*(X)$ .

pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow k_{n-1} \\ * & \longrightarrow & K(\pi_n X, n+1), \end{array}$$

where  $K(\pi_n X, n+1)$  and  $P_{n-1} X$  are both  $p$ -complete. Since  $X$  is weakly equivalent to  $\lim_n P_n X$  it follows that  $X$  is  $p$ -complete using Lemma 6.3.10.<sup>5</sup>  $\square$

Before we prove part (ii) of the theorem we prove a criterion for a map to be an  $\mathbb{F}_p$ -equivalence in terms of the homotopy groups of the homotopy fibre:

thm:hofibZip

**THEOREM 6.5.3.** *Let  $f: X \rightarrow Y$  be a map between simply connected spaces, and let  $F$  be the homotopy fibre of  $f$ . Then the following are equivalent:*

- (1)  $f$  is an  $\mathbb{F}_p$ -equivalence;
- (2)  $\tilde{H}_*(F; \mathbb{F}_p) = 0$ ;
- (3)  $\tilde{H}_*(F; \mathbb{Z})$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module;
- (4)  $\pi_*(F)$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module.

This will take some work to prove. We start with a 2-of-3 criterion for  $\mathbb{F}_p$ -equivalences:

propthreeSerre

**PROPOSITION 6.5.4.** *Suppose we are given a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow f & & \downarrow g \\ Z & \xrightarrow{\phi} & W \end{array}$$

where  $Z$  and  $W$  are simply connected, and let  $\eta: F_f \rightarrow F_g$  be the induced map on homotopy fibres. If  $R$  is a principal ideal domain and two of the three morphisms  $\phi$ ,  $\psi$ , and  $\eta$  are  $R$ -equivalences, then so is the third.

We will not prove this here. The case where  $\phi$  and  $\eta$  are  $R$ -equivalences is an immediate consequence of naturality for the Serre spectral sequences. The other two cases are also proved using the Serre spectral sequences by more complicated arguments. See [Hat, Proposition 1.12] for an argument in one of these cases.

As a special case, we get:

pullbackcor

**COROLLARY 6.5.5.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow f & & \downarrow g \\ Z & \xrightarrow{\phi} & W \end{array}$$

<sup>5</sup>Notice that the same proof works if we replace the hypothesis “ $X$  simply connected” by “ $X$  simple and connected”: we just have to start from  $P_1 X$ , and use that there is a  $k_1$ -invariant to  $K(\pi_2(X), 3)$ . Similarly for many of the proofs in this chapter.

be a homotopy pullback square of spaces with  $Z$  and  $W$  simply connected. If  $R$  is a principal ideal domain, then  $\phi$  is an  $R$ -equivalence if and only if  $\psi$  is an  $R$ -equivalence.

PROOF. Since the square is a homotopy pullback, the induced map of homotopy fibres  $F_f \rightarrow F_g$  is a weak equivalence by Lemma 6.4.4, so this follows from Proposition 6.5.4.  $\square$

This gives a first approximation to the Theorem:

em:fibhlgyzero

LEMMA 6.5.6. *Let  $f: X \rightarrow Y$  be a map between simply connected spaces, and let  $F$  be the homotopy fibre of  $f$ . Then the following are equivalent:*

- (1)  $f$  is an  $\mathbb{F}_p$ -equivalence,
- (2)  $\tilde{H}_*(F; \mathbb{F}_p) = 0$ ,
- (3)  $\tilde{H}_*(F; \mathbb{Z})$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module.

PROOF. The equivalence of (1) and (2) is a special case of Corollary 6.5.5. The equivalence of (2) and (3) follows from the Bockstein long exact sequence (i.e. the long exact sequence of homology groups induced by the short exact sequence of abelian groups  $0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$ ):

$$\cdots \rightarrow \tilde{H}_i(F; \mathbb{Z}) \xrightarrow{p} \tilde{H}_i(F; \mathbb{Z}) \rightarrow \tilde{H}_i(F; \mathbb{F}_p) \rightarrow \tilde{H}_{i-1}(F; \mathbb{Z}) \rightarrow \cdots .$$

Here we see that  $\tilde{H}_i(F; \mathbb{F}_p) = 0$  for all  $i$ , then multiplication by  $p$  is an isomorphism on  $\tilde{H}_i(F; \mathbb{Z})$  for all  $i$ , and vice versa.  $\square$

The fibre of a map of simply connected spaces is not necessarily simply connected, but it is a *simple* space, meaning its fundamental group is abelian and acts trivially on the higher homotopy groups. Lemma 6.5.6 therefore reduces the proof of Theorem 6.5.3 to showing:

simplehlgyzero

PROPOSITION 6.5.7. *Suppose  $X$  is a simple space. Then  $\tilde{H}_*(X; \mathbb{F}_p) = 0$  if and only if  $\pi_n X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module for all  $n$ .*

We first consider the case of Eilenberg-MacLane spaces:

lipmodhlgyzero

LEMMA 6.5.8. *An abelian group  $A$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module if and only if  $\tilde{H}_*(K(A, n); \mathbb{F}_p) = 0$  (for any  $n \geq 1$ ).*

PROOF. First suppose  $\tilde{H}_*(K(A, n); \mathbb{F}_p) = 0$ . Then as we saw above, the Bockstein long exact sequence implies that  $\tilde{H}_i(K(A, n); \mathbb{Z})$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module for all  $i$ . But since  $A$  is abelian we have  $A \cong H_n(K(A, n); \mathbb{Z})$ , so  $A$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module.

Now suppose  $A$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module. We first consider the case where  $n = 1$ . Since  $\tilde{H}_*(-; \mathbb{F}_p)$  preserves filtered colimits it suffices to consider  $A$  a finitely generated  $\mathbb{Z}[\frac{1}{p}]$ -module. Then  $A$  is a finite direct sum of copies of  $\mathbb{Z}/q^k$ , where  $q^k$  is some prime power with  $q \neq p$ , and copies of  $\mathbb{Z}[\frac{1}{p}]$ . We will not show that  $\tilde{H}_*(K(\mathbb{Z}/q^k, 1); \mathbb{F}_p) = 0$ ; this can for example be done using group cohomology. [Exercise: Check this for  $K(\mathbb{Z}/2, 1) = \mathbb{R}\mathbb{P}^\infty$  using cellular homology.] In the case of  $\mathbb{Z}[\frac{1}{p}]$  we have  $\mathbb{Z}[\frac{1}{p}] =$

$\text{colim}(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \cdots)$  so  $K(\mathbb{Z}[\frac{1}{p}], 1) \cong \text{colim}(K(\mathbb{Z}, 1) \xrightarrow{p} K(\mathbb{Z}, 1) \cdots)$  and so as homology commutes with filtered colimits we have  $\tilde{H}_*(K(\mathbb{Z}[\frac{1}{p}], 1), \mathbb{Z}) \cong \tilde{H}_*(S^1; \mathbb{Z})[\frac{1}{p}]$ , which is a  $\mathbb{Z}[\frac{1}{p}]$ -module, so  $\tilde{H}_*(K(\mathbb{Z}[\frac{1}{p}], 1), \mathbb{F}_p) = 0$ .

We can now inductively extend this to  $n > 1$ : Applying Corollary 6.5.5 to the homotopy pullback square

$$\begin{array}{ccc} K(A, n-1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(A, n). \end{array}$$

gives that  $\tilde{H}_*(K(A, n); \mathbb{F}_p) = 0$  if and only if  $\tilde{H}_*(K(A, n-1); \mathbb{F}_p) = 0$ .  $\square$

**PROOF OF PROPOSITION 6.5.7.** We first consider the case where  $X$  is a simply connected space. For  $n \geq 2$ , by applying Corollary 6.5.5 to the homotopy pullback square

$$\begin{array}{ccc} K(\pi_n X, n) & \longrightarrow & P_n X \\ \downarrow & & \downarrow \\ * & \longrightarrow & P_{n-1} X \end{array}$$

we see that if  $\tilde{H}_*(P_{n-1} X; \mathbb{F}_p) = 0$  then  $\tilde{H}_*(K(\pi_n X, n); \mathbb{F}_p) \xrightarrow{\sim} \tilde{H}_*(P_n X; \mathbb{F}_p)$ . In particular, by Lemma 6.5.8 we have that if  $\tilde{H}_*(P_{n-1} X; \mathbb{F}_p) = 0$  then  $\tilde{H}_*(P_n X; \mathbb{F}_p) = 0$  if and only if  $\pi_n X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module.

The map  $X \rightarrow P_n X$  is  $(n+1)$ -connected (meaning the homotopy fibre  $F_n$  is  $n$ -connected), and since  $P_n X$  is simply connected the Serre spectral sequence for this map has  $E^2$ -term

$$E_{s,t}^2 = H_s(P_n X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_t(F_n; \mathbb{F}_p) \Rightarrow H_{s+t}(X; \mathbb{F}_p).$$

As  $H_t(F_n; \mathbb{F}_p) = 0$  for  $1 \leq t \leq n$  there is no room for differentials in the range  $s+t \leq n$ , and thus  $H_i(X; \mathbb{F}_p) = E_{i,0}^\infty \xrightarrow{\sim} H_i(P_n X; \mathbb{F}_p) = E_{i,0}^2$  for  $i \leq n$ . Moreover, there are no differentials out of  $H_{n+1}(P_n X; \mathbb{F}_p) = E_{n+1,0}^2$  so the map  $H_{n+1}(X; \mathbb{F}_p) \rightarrow H_{n+1}(P_n X; \mathbb{F}_p)$  is surjective.<sup>6</sup>

We can therefore conclude that if  $\pi_i X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module for all  $i \leq n$  then  $\tilde{H}_*(P_n X; \mathbb{F}_p) = 0$  and so  $\tilde{H}_i(X; \mathbb{F}_p) = 0$  for  $i \leq n$ . In particular, if all the homotopy groups of  $X$  are  $\mathbb{Z}[\frac{1}{p}]$ -modules then  $\tilde{H}_*(X; \mathbb{F}_p) = 0$ .

Now suppose  $\tilde{H}_*(X; \mathbb{F}_p) = 0$ . Then for every  $n$  we have  $\tilde{H}_i(P_n X; \mathbb{F}_p) = 0$  for  $i \leq n+1$ . Assume we know that  $\pi_i X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module for all  $i < n$ . Then we get that  $\tilde{H}_i(K(\pi_n X, n); \mathbb{F}_p) = 0$  for  $i \leq n+1$ . But then from the Bockstein long exact sequence we see that multiplication by  $p$  on  $H_n(K(\pi_n X, n); \mathbb{Z}) \cong \pi_n X$  is an isomorphism, i.e.  $\pi_n X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module. By inducting on  $n$  this completes the proof for  $X$  simply connected.

<sup>6</sup>One can of course also use that  $H_*(P_n X, X; \mathbb{F}_p) = 0$  for  $* \leq n+1$  since  $P_n X$  can be constructed from  $X$  by attaching cells of degree  $\geq n+2$ .

If  $X$  is not simply connected, we consider the Serre spectral sequence for the map  $X \rightarrow B\pi_1 X$  whose homotopy fibre is the universal cover  $\tilde{X}$ . Since  $X$  is simple,  $\pi_1(X)$  acts trivially on  $H_*(\tilde{X}; \mathbb{F}_p)$ , i.e. there is no non-trivial local system involved the  $E^2$ -term, so this spectral sequence is of the form

$$E_{s,t}^2 = H_s(\tilde{X}; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_t(B\pi_1 X; \mathbb{F}_p) \Rightarrow H_{s+t}(X; \mathbb{F}_p).$$

If  $\pi_n X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module for all  $n \geq 1$ , then since  $\tilde{X}$  is simply connected we have  $\tilde{H}_*(\tilde{X}; \mathbb{F}_p) = 0$ , and  $\tilde{H}_*(B\pi_1 X; \mathbb{F}_p) = 0$  by Lemma 6.5.8. Thus  $E_{s,t}^2 = 0$  except when  $s = t = 0$  and thus  $\tilde{H}(X; \mathbb{F}_p) = 0$ .

On the other hand, if  $\tilde{H}(X; \mathbb{F}_p) = 0$ , then we know  $\tilde{H}(X; \mathbb{Z})$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module, hence  $\pi_1 X \cong H_1(X; \mathbb{Z})$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module, and so  $\tilde{H}(B\pi_1 X; \mathbb{F}_p) = 0$  by Lemma 6.5.8. Thus  $E_{s,t}^2 = 0$  except when  $t = 0$ , so the spectral sequence collapses and we have  $\tilde{H}_*(\tilde{X}; \mathbb{F}_p) \cong \tilde{H}_*(X; \mathbb{F}_p) = 0$ . Then since  $\tilde{X}$  is simply connected we have that  $\pi_n \tilde{X} = \pi_n X$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module for all  $n \geq 2$ . □

This completes the proof of Theorem 6.5.3, and we are ready to prove the base case of (ii) in Theorem 6.5.1:

lem:fgEMpcom

LEMMA 6.5.9. *Suppose  $A$  is a finitely generated abelian group. Then  $K(A, n) \rightarrow K(A_p, n)$  is a  $p$ -completion, for  $n \geq 1$ .*

PROOF. We know from Theorem 6.5.1(i) that  $K(A_p, n)$  is  $p$ -complete, so it remains to show that the map is a  $\mathbb{F}_p$ -equivalence. Since  $A$  is finitely generated, it suffices to check separately the cases where  $A$  is  $\mathbb{Z}/p^k$ ,  $\mathbb{Z}/q^k$  ( $q$  a prime  $\neq p$ ), and  $\mathbb{Z}$ . In these cases  $A_p$  is  $\mathbb{Z}/p^k$ ,  $0$ , and  $\mathbb{Z}_p$ , respectively. For  $\mathbb{Z}/p^k$  the space  $K(\mathbb{Z}/p^k, n)$  is already  $p$ -complete and there is nothing to prove. The group  $\mathbb{Z}/q^k$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module, so by Theorem 6.5.3 it follows that  $K(\mathbb{Z}/q^k, n)$  is  $\mathbb{F}_p$ -equivalent to a point, as required.

The remaining case  $A = \mathbb{Z}$  is the more interesting one. Here the map  $\phi: \mathbb{Z} \rightarrow \mathbb{Z}_p$  is injective, and we can complete it to a SES  $\mathbb{Z} \rightarrow \mathbb{Z}_p \rightarrow \text{coker}(\phi)$ ; correspondingly, we can exhibit  $K(\mathbb{Z}, n)$  as the homotopy fibre of the map  $K(\mathbb{Z}_p, n) \rightarrow K(\text{coker}(\phi), n)$ . Even when  $n = 1$ , note that the fibre sequence  $K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}_p, n) \rightarrow K(\text{coker}(\phi), n)$  is principal, and therefore we can compute  $H_*(K(\mathbb{Z}_p, n), \mathbb{F}_p)$  as limit of the Serre spectral sequence with  $E_{s,t}^2 = H_s(K(\text{coker}(\phi), n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_t(K(\mathbb{Z}, n); \mathbb{F}_p)$ . If we show that  $\text{coker} \phi$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module we are done: the  $E^2$ -page vanishes for  $s \neq 1$ , and this implies that  $K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}_p, n)$  is an  $H_*(-; \mathbb{F}_p)$ -equivalence.

It thus suffices to show that  $\text{coker} \phi$  is a  $\mathbb{Z}[\frac{1}{p}]$ -module, i.e. multiplication by  $p$  is an isomorphism. An element  $x$  of  $\mathbb{Z}_p$  can be written as  $\sum_{i=0}^{\infty} a_i p^i$ , where  $0 \leq a_i < p$ . The image of  $\mathbb{Z}$  is precisely those sums with only finitely many non-zero  $a_i$ . Modulo  $\mathbb{Z}$  we then see that  $x$  equals  $p(\sum_{i=1}^{\infty} a_i p^{i-1})$ , so multiplication by  $p$  is surjective. On the other hand, if  $p(\sum_{i=0}^{\infty} a_i p^i)$  is  $0$  modulo  $\mathbb{Z}$  then  $a_i = 0$  except for finitely many  $i$ , i.e.  $\sum a_i p^i$  is in the image of  $\mathbb{Z}$ . Hence multiplication by  $p$  is injective. This completes the proof. □

PROOF OF THEOREM 6.5.1(II). We once again work our way up the Postnikov tower. First we can take  $P_2X_p^\wedge := K(\pi_2X \otimes \mathbb{Z}_p^\wedge, 2)$ , then  $P_2X \rightarrow P_2X_p^\wedge$  is a  $p$ -completion by Lemma 6.5.9. Next if we have a  $p$ -completion  $\phi: P_{n-1}X \rightarrow P_{n-1}X_p^\wedge$ , we get from the universal property a homotopy-commutative square

$$\begin{array}{ccc} P_{n-1}X & \xrightarrow{\phi} & P_{n-1}X_p^\wedge \\ \downarrow k_{n-1} & & \downarrow (k_{n-1})_p^\wedge \\ K(\pi_nX, n+1) & \xrightarrow{\psi} & K(\pi_nX \otimes \mathbb{Z}_p^\wedge, n+1). \end{array}$$

On homotopy fibres we get a map  $P_nX \rightarrow P_nX_p^\wedge := F_{(k_{n-1})_p^\wedge}$ , where the space  $P_nX_p^\wedge$  is  $p$ -complete by Lemma 6.3.9. Moreover, since  $\phi$  and  $\psi$  are  $\mathbb{F}_p$ -equivalences it follows from Proposition 6.5.4 that this map is an  $\mathbb{F}_p$ -equivalence, so it is a  $p$ -completion.

Now define  $X_p^\wedge := \text{holim}_n P_nX_p^\wedge$ : here by ‘‘holim’’ (homotopy limit) we mean that we first replace the tower  $\cdots \rightarrow P_3X_p^\wedge \rightarrow P_2X_p^\wedge$  by a tower of fibrations, and then take the limit. The result is  $p$ -complete by Lemma 6.3.10. It remains to show that  $X \rightarrow X_p^\wedge$  is an  $\mathbb{F}_p$ -equivalence. To see this, let  $F_n$  be the homotopy fibre of the map  $P_nX \rightarrow P_nX_p^\wedge$ . Then the homotopy fibre of  $X \rightarrow X_p^\wedge$  is weakly equivalent to  $\text{holim}_n F_n$ . Since the homotopy groups of  $P_nX$  and  $P_nX_p^\wedge$  stabilize (we have in fact  $\pi_i(P_nX) \cong \pi_i(P_{n+1}X)$  and  $\pi_i(P_nX_p^\wedge) \cong \pi_i(P_{n+1}X_p^\wedge)$  for  $i \leq n-1$ ), we see from the long exact sequence that so do those of  $F_n$ . Thus  $\pi_i(\text{lim}_n F_n) \cong \text{lim}_n \pi_i F_n$  with no  $\text{lim}^1$ -term, and the latter is in fact  $\cong \pi_i F_k$  for  $k$  sufficiently large. Since the homotopy groups of the spaces  $F_n$  are  $\mathbb{Z}[\frac{1}{p}]$ -modules, it follows that the same is true for those of  $\text{lim}_n F_n$  (we just use that a limit of abelian groups on which multiplication by  $p$  is invertible has the same property), hence the map  $X \rightarrow X_p^\wedge$  is an  $\mathbb{F}_p$ -equivalence by Theorem 6.5.3.  $\square$

PROOF OF THEOREM 6.5.1(III). Applying the 2-of-3 property for  $\mathbb{F}_p$ -equivalences to the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_p^\wedge & \xrightarrow{f_p^\wedge} & Y_p^\wedge \end{array}$$

we see that  $f$  is an  $\mathbb{F}_p$ -equivalence if and only if  $f_p^\wedge$  is one. But the spaces  $X_p^\wedge$  and  $Y_p^\wedge$  are  $p$ -complete, so  $f_p^\wedge$  is an  $\mathbb{F}_p$ -equivalence if and only if it is an isomorphism in  $\text{Ho}(\text{Top})$ , i.e. if it gives an isomorphism  $\pi_*X_p^\wedge \rightarrow \pi_*Y_p^\wedge$ . Using Theorem 6.5.1(ii) we see from this that  $f$  is an  $\mathbb{F}_p$ -equivalence if and only if  $\pi_*(f) \otimes \mathbb{Z}_p^\wedge$  is an isomorphism.  $\square$

## 6.6. Sullivan’s arithmetic square for $p$ -completion

We now consider Sullivan’s arithmetic square for  $p$ -completion:

n:Sullivanpcom

**THEOREM 6.6.1.** *Suppose  $X$  is simply connected. Then we have a homotopy pullback square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_\mathbb{Q} & \longrightarrow & \left(\prod_p X_p^\wedge\right)_\mathbb{Q}. \end{array}$$

We will prove this in 3 steps:

- (1) Let  $Y$  be the homotopy pullback in the square, then we will show that the induced map  $\phi: X \rightarrow Y$  is a  $\mathbb{Z}$ -equivalence (isomorphism in  $\mathbb{Z}$ -homology).
- (2) Next, we will see that  $Y$  is a  $\mathbb{Z}$ -local space. Combined with (1), this shows that  $Y$  is the  $\mathbb{Z}$ -localization of  $X$ .
- (3) Finally, we will observe that  $X$  is itself  $\mathbb{Z}$ -local, so that (2) implies that  $X \xrightarrow{\sim} Y$ .

**REMARK 6.6.2.** One could also try to prove step (1) and then try to prove that  $Y$  is simply connected, concluding then by Hurewicz. However, in order to prove the vanishing of  $\pi_1(Y)$ , one needs to prove that the map  $\pi_2(X_\mathbb{Q}) \oplus \pi_2(\prod_p X_p^\wedge) \rightarrow \pi_2((\prod_p X_p^\wedge)_\mathbb{Q})$  is surjective, and for this one needs to understand  $\pi_2((\prod_p X_p^\wedge)_\mathbb{Q})$ . This strategy can be pursued effectively when we know that  $\pi_*(X)$  are finitely generated, using the results of the previous section, but is hostic otherwise.

For the proof of the first step we need the following result. With finite generation assumptions this follows from the results of the previous section, and the general case will follow from the results on  $p$ -completion for general simply connected spaces we will prove later.

pipcompZiqmod

**PROPOSITION 6.6.3.** *If  $X$  is a simply connected  $p$ -complete space, then  $\pi_* X$  are  $\mathbb{Z}[\frac{1}{q}]$ -modules for any prime  $q \neq p$ .*

To see that  $\phi$  is a  $\mathbb{Z}$ -equivalence we will use the following criterion:

isolemma

**LEMMA 6.6.4.** *A map of spaces  $f: X \rightarrow Y$  induces an isomorphism on  $H_*(-; \mathbb{Z})$  if and only if it induces an isomorphism on  $H_*(-; \mathbb{Q})$  and on  $H_*(-; \mathbb{F}_p)$  for all primes  $p$ .*

**PROOF.** The forward direction follows from the universal coefficient theorem. For the backward direction, we can assume that  $f$  is an inclusion, and the result follows if we can show that  $H_*(Y, X) = 0$  under the stated assumptions. The universal coefficient sequence gives us for any abelian group  $A$  an exact sequence

$$0 \rightarrow H_n(Y, X) \otimes A \rightarrow H_n(Y, X; A) \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(Y, X), A) \rightarrow 0.$$

If  $H_i(Y, X; \mathbb{F}_p) = 0$  for  $i = n-1, n$  then it follows that  $H := H_n(Y, X)$  satisfies  $H \otimes \mathbb{F}_p = 0$  and  $\mathrm{Tor}_1^{\mathbb{Z}}(H, \mathbb{F}_p) = 0$ . This implies that  $H$  is uniquely  $p$ -divisible for all primes  $p$ , i.e. it is a  $\mathbb{Q}$ -vector space. But then  $H \cong H \otimes \mathbb{Q}$ , and if  $H_n(Y, X; \mathbb{Q}) = 0$  the exact sequence implies that  $H \otimes \mathbb{Q} = 0$ .  $\square$

propn:phiZeq

**PROPOSITION 6.6.5.** *The map  $\phi: X \rightarrow Y$  is a  $\mathbb{Z}$ -equivalence.*

PROOF. By Lemma 6.6.4 it is enough to prove that  $\phi$  gives isomorphisms on  $H_*(-; \mathbb{F}_p)$  for all primes  $p$  and on  $H_*(-; \mathbb{Q})$ .

Consider the following diagram, where the isomorphisms are ensured by the universal property of  $\mathbb{Q}$ -localization:

$$\begin{array}{ccccc} H_*(X; \mathbb{Q}) & \longrightarrow & H_*(Y; \mathbb{Q}) & \longrightarrow & H_*\left(\prod_p X_p^\wedge; \mathbb{Q}\right) \\ & \searrow \cong & \downarrow & & \downarrow \cong \\ & & H_*(X_\mathbb{Q}; \mathbb{Q}) & \longrightarrow & H_*\left(\left(\prod_p X_p^\wedge\right)_\mathbb{Q}; \mathbb{Q}\right), \end{array}$$

Corollary 6.5.5 implies that the left vertical map is also an isomorphism, which shows that  $X \rightarrow Y$  is an isomorphism on  $H_*(-; \mathbb{Q})$ .

Now consider the following diagram:

$$\begin{array}{ccccc} & & \cong & & \\ & \searrow & & \searrow & \\ H_*(X; \mathbb{F}_q) & \longrightarrow & H_*(Y; \mathbb{F}_q) & \longrightarrow & H_*\left(\prod_p X_p^\wedge; \mathbb{F}_q\right) \\ & & \downarrow & & \downarrow \\ & & H_*(X_\mathbb{Q}; \mathbb{F}_q) & \xrightarrow{\cong} & H_*\left(\left(\prod_p X_p^\wedge\right)_\mathbb{Q}; \mathbb{F}_q\right), \end{array}$$

Here the top map is an isomorphism since  $\prod_p X_p^\wedge = X_q^\wedge \times \prod_{p \neq q} X_p^\wedge$  where the homotopy groups of the second space are  $\mathbb{Z}[\frac{1}{q}]$ -modules by Proposition 6.6.3, and so its reduced  $\mathbb{F}_q$ -homology is trivial by Theorem 6.5.3. By the Künneth theorem this means

$$H_*\left(\prod_p X_p^\wedge; \mathbb{F}_q\right) \cong H_*(X_q^\wedge; \mathbb{F}_q) \otimes_{\mathbb{F}_q} H_*\left(\prod_{p \neq q} X_p^\wedge; \mathbb{F}_q\right) \cong H_*(X_q^\wedge; \mathbb{F}_q),$$

and so the map  $H_*(X; \mathbb{F}_q) \rightarrow H_*(\prod_p X_p^\wedge; \mathbb{F}_q)$  is an isomorphism. The bottom horizontal map is also an isomorphism: since the homotopy groups of the spaces at the bottom are  $\mathbb{Q}$ -vector spaces, again by Theorem 6.5.3 the  $\mathbb{F}_q$ -homology in the bottom row is trivial, so the bottom arrow is trivially an isomorphism.

Now using Corollary 6.5.5 again the top horizontal map is also an isomorphism, which shows that  $X \rightarrow Y$  gives an isomorphism on  $H_*(-; \mathbb{F}_q)$  for all  $q$ . Thus  $X \rightarrow Y$  gives an isomorphism on  $H_*(-; \mathbb{Z})$ , as required.  $\square$

For step (2), we need the following result:

PROPOSITION 6.6.6. *If*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$



is a homotopy pullback square and  $Y$ ,  $Z$ , and  $W$  are  $R$ -local for some ring  $R$ , then  $X$  is  $R$ -local.

To see this we need some observations about  $R$ -equivalences and weak equivalences:

**lem:Reqprod**

LEMMA 6.6.7. *If  $f: X \rightarrow Y$  is an  $R$ -equivalence, then so is  $f \times T: X \times T \rightarrow Y \times T$  for any space  $T$ .*

PROOF. We have a (convergent) Künneth spectral sequence

$$E_{p,q}^2 = \bigoplus_{i+j=p} \mathrm{Tor}_R^q(H_i(X; R), H_j(T; R)) \Rightarrow H_{i+j}(X \times T; R),$$

and similarly for  $Y$ . This is natural, and the map induced by  $f$  gives an isomorphism on the  $E^2$ -terms. Therefore we get an isomorphism  $H_*(X \times T; R) \xrightarrow{\cong} H_*(Y \times T; R)$ , as required. See also the discussion after Definition 6.3.7.  $\square$

**lem:ReqMap**

LEMMA 6.6.8. *A space  $Z$  is  $R$ -local if and only if for every  $R$ -equivalence  $f: X \rightarrow Y$  the induced map between mapping spaces*

$$\mathrm{map}(Y, Z) \rightarrow \mathrm{map}(X, Z)$$

*is a weak equivalence.*

To show this, we will use the following fact:

FACT 6.6.9. *A map of spaces  $f: X \rightarrow Y$  is a weak equivalence if and only if the induced map on homotopy classes  $[T, X] \rightarrow [T, Y]$  is an isomorphism of sets for all CW complexes  $T$ . (Equivalently,  $f$  induces an isomorphism in  $\mathrm{Ho}(\mathrm{Top})$ .)*

This is a special case of a standard fact about model categories, for example.

WARNING 6.6.10. These are unpointed homotopy classes. To show that  $X \rightarrow Y$  is a weak equivalence it is *not* enough to know that  $[S^n, X] \rightarrow [S^n, Y]$  is an isomorphism for all spheres. Think of the infinite symmetric group  $G = \bigcup_{n \geq 1} \mathfrak{S}_n$ , obtained as union of all finite symmetric groups, and construct a “shift” map  $BG \rightarrow BG$  giving a counterexample.

PROOF OF LEMMA 6.6.8. If  $\mathrm{map}(Y, Z) \rightarrow \mathrm{map}(X, Z)$  is a weak equivalence for all  $R$ -equivalences  $f$ , then on  $\pi_0$  we get an isomorphism  $[Y, Z] \rightarrow [X, Z]$  for every  $R$ -equivalence, i.e.  $Z$  is  $R$ -local.

Conversely, suppose  $Z$  is  $R$ -local and  $f: X \rightarrow Y$  is an  $R$ -equivalence. The map  $\mathrm{map}(Y, Z) \rightarrow \mathrm{map}(X, Z)$  is a weak equivalence if and only if for all spaces  $T$  the induced map

$$[T, \mathrm{map}(Y, Z)] \rightarrow [T, \mathrm{map}(X, Z)]$$

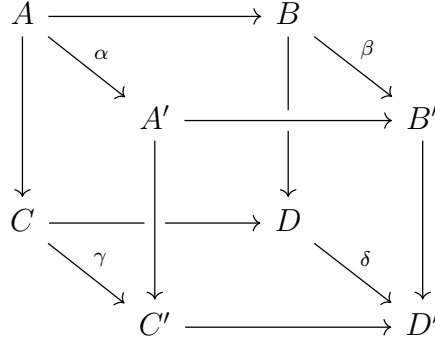
is an isomorphism. But this is isomorphic to the map

$$[T \times Y, Z] \rightarrow [T \times X, Z]$$

induced by  $T \times f$ , which is an  $R$ -equivalence by Lemma 6.6.7, and so this map is an isomorphism since  $Z$  is  $R$ -local.  $\square$

lem:cubehpb

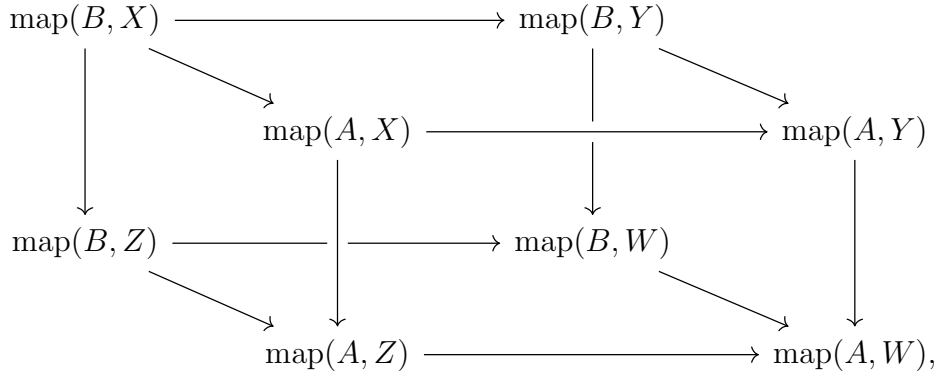
LEMMA 6.6.11. *Given a commutative cube*



where the front and back faces are homotopy pullbacks, if the maps  $\beta$ ,  $\gamma$ , and  $\delta$  are weak equivalences, so is  $\alpha$ .

PROOF. This is a special case of homotopy invariance for homotopy limits.  $\square$

PROOF OF PROPOSITION 6.6.6. If  $f: A \rightarrow B$  is an  $R$ -equivalence, by Lemma 6.6.8 we need to prove that  $\text{map}(B, X) \rightarrow \text{map}(A, X)$  is a weak equivalence. This follows by applying Lemma 6.6.11 to the cube



where the front and back faces are homotopy pullbacks since  $\text{map}(A, -)$  preserves homotopy limits.  $\square$

We need one more observation before we can prove Theorem 6.6.1:

LEMMA 6.6.12. *If a space  $X$  is  $R$ -local for some ring  $R$ , then  $X$  is  $\mathbb{Z}$ -local.*

PROOF. It suffices to show that if  $f: Y \rightarrow Z$  is a  $\mathbb{Z}$ -equivalence then it is an  $R$ -equivalence for any ring  $R$ . From the universal coefficient theorem we have a map of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_n(Y) \otimes R & \longrightarrow & H_n(Y; R) & \longrightarrow & \text{Tor}^1(H_{n-1}(Y), R) \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H_n(Z) \otimes R & \longrightarrow & H_n(Z; R) & \longrightarrow & \text{Tor}^1(H_{n-1}(Z), R) \longrightarrow 0,
 \end{array}$$

so the 5-Lemma implies that  $f$  is an  $R$ -equivalence if it is a  $\mathbb{Z}$ -equivalence.  $\square$

lem:RlocisZloc

PROOF OF THEOREM 6.6.1. The spaces  $X_{\mathbb{Q}}$ ,  $X_p^{\wedge}$  and  $(\prod_p X_p^{\wedge})_{\mathbb{Q}}$  are  $\mathbb{Z}$ -local by Lemma 6.6.12, and the space  $\prod_p X_p^{\wedge}$  is also  $\mathbb{Z}$ -local since an arbitrary product of  $R$ -local spaces is always  $R$ -local (just consider that mapping spaces into a product are product of mapping spaces). Thus the homotopy pullback  $Y$  is  $\mathbb{Z}$ -local by Proposition 6.6.6. Since  $X \rightarrow Y$  is also a  $\mathbb{Z}$ -equivalence by Proposition 6.6.5, this means that  $Y$  is the  $\mathbb{Z}$ -localization of  $X$ . But we know from Theorem 6.3.3 that a simply connected space is  $R$ -local for  $R \subseteq \mathbb{Q}$  if and only if its homotopy groups are  $R$ -modules — taking  $R = \mathbb{Z}$  this says that every simply connected space is  $\mathbb{Z}$ -local. Thus the  $\mathbb{Z}$ -localization of  $X$  is just  $X$ , and so  $X \rightarrow Y$  is a weak equivalence, as required.  $\square$

REMARK 6.6.13. Theorem 6.6.1 says that we can recover a (simply connected) space  $X$  from its rationalization  $X_{\mathbb{Q}}$ , its  $p$ -completions  $X_p^{\wedge}$  at all primes, and the map  $X_{\mathbb{Q}} \rightarrow (\prod_p X_p^{\wedge})_{\mathbb{Q}}$ . In fact, we can say a bit more: If we have  $p$ -complete spaces  $Y(p)$  for all primes  $p$  and a rational space  $Q$ , all simply connected, together with a map  $Q \rightarrow (\prod_p Y(p))_{\mathbb{Q}}$ , let  $P$  be the homotopy pullback in

$$\begin{array}{ccc} P & \longrightarrow & \prod_p Y(p) \\ \downarrow & & \downarrow \\ Q & \longrightarrow & (\prod_p Y(p))_{\mathbb{Q}}. \end{array}$$

Then the same proof shows:  $Q$  is the rationalization of  $P$  and  $Y(p)$  is the  $p$ -completion of  $P$ . Thus we can build a space with arbitrary rationalization and  $p$ -completions, provided we have the bottom map — this is the only “interaction” between the rational and  $p$ -complete “parts” of the space. Note also that  $\pi_1(P)$  may be non-trivial (but it is a quotient of  $\pi_2((\prod_p Y(p))_{\mathbb{Q}})$ , so for instance it is surely a divisible abelian group).

### 6.7. An algebraic interlude: derived $p$ -completion

We saw above that  $p$ -completion of spaces is closely related to  $p$ -completion of abelian groups — but only for finitely generated abelian groups. The reason is that the naïve definition of the  $p$ -completion of an abelian group  $A$  as  $A_p^{\wedge} := \lim_k A/p^k$  must be replaced by a “derived”  $p$ -completion in chain complexes of abelian groups. This involves using derived versions of both parts of the construction, i.e. quotienting by  $p^k$  and taking the limit:

- Instead of just taking the quotient  $A/p^k$ , i.e. the cokernel of the map  $A \xrightarrow{p^k} A$  we take the “derived cokernel”, meaning the *mapping cone* of this map. This is the chain complex

$$\cdots \rightarrow 0 \rightarrow A \xrightarrow{p^k} A \rightarrow 0 \rightarrow \cdots$$

with the non-zero groups in degrees 0 and 1. We will denote this  $A//p^k$ . Note that  $H_0(A//p^k)$  recovers  $A/p^k$ , whereas  $H_1(A//p^k)$  is the kernel of multiplication by  $p^k$ , i.e. the group of  $p^k$ -torsion elements in  $A$ .

- Now we want to take the limit of the chain complexes  $A//p^k$  over  $k$  — but since  $\lim$  is not an exact functor (though it is left exact) we must take the *right derived* limit. This means we first replace the diagram  $\mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$  sending  $k \mapsto A//p^k$  by a *fibrant* object in the category  $\text{Ch}(\mathbb{Z})^{\mathbb{N}^{\text{op}}}$  of diagrams of chain complexes, and then take the limit of that. This, of course, requires fixing a model structure; alternatively, we just need a right deformation  $Q$  of the identity functor of  $\text{Ch}(\mathbb{Z})^{\mathbb{N}^{\text{op}}}$  as in Definition 4.1.7, such that  $\lim$  is homotopy invariant on the essential image of  $Q$ . We will denote this derived limit of an inverse system of chain complexes  $C(k)_{k \geq 0}$  by  $\mathbb{R}\lim_k C(k)$ .

REMARK 6.7.1. The derived limit  $\mathbb{R}\lim_k C(k)$  of an inverse system  $C: \mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$ ,  $k \mapsto C(k)$ , is well-defined up to quasi-isomorphism.

To compute it, we can for instance fix a model structure on  $\mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$ : one possible model structure has weak equivalences given by natural transformations  $C(-) \rightarrow D(-)$  that for each  $k$  evaluate to a quasi-isomorphism  $C(k) \xrightarrow{\sim} D(k)$ ; and it has cofibrations given by natural transformations  $C(-) \rightarrow D(-)$  that for each  $k$  evaluate to an inclusion of chain complexes  $C(k) \hookrightarrow D(k)$ . A fibrant replacement of  $C \in \mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$  can be then obtained by replacing each  $C(k)$  by a fibrant object<sup>7</sup> in  $\text{Ch}(\mathbb{Z})$ , in such a way that all maps  $C(k) \rightarrow C(k-1)$  are fibrations in  $\text{Ch}(\mathbb{Z})$ . Examples of fibrant objects in  $\text{Ch}(\mathbb{Z})$  are bounded above chain complexes of injective abelian groups (and if you want, this can be a motivation for the very definition of “injective abelian group”), and examples of fibrations are mapping cylinder projections  $\text{Cyl}(A \rightarrow B) \rightarrow B$ , if  $A \rightarrow B$  is a chain map of fibrant chain complexes.

Note also that since we are working with chain complexes rather than cochain complexes the  $\lim^1$ -term will contribute in degree  $-1$  when we consider an inverse system  $\cdots \rightarrow M_1 \rightarrow M_0$  of abelian groups (considered as chain complexes concentrated in degree 0): we have

$$H_i(\mathbb{R}\lim_k M_k) = \begin{cases} \lim_k M_k, & i = 0, \\ \lim_k^1 M_k, & i = -1, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, for a sequence  $C(k)$  of chain complexes we get short exact sequences

$$0 \rightarrow \lim_k^1 H_{i+1} C(k) \rightarrow H_i \mathbb{R}\lim_k C(k) \rightarrow \lim_k H_i C(k) \rightarrow 0.$$

(This is a degenerate special case of Grothendieck’s hyperhomology spectral sequence, for example.)

DEFINITION 6.7.2. The *derived  $p$ -completion* of an abelian group  $A$  is  $\mathbb{R}\lim_k A//p^k$ . We will denote this  $\mathbb{D}A_p^\wedge$ .

REMARK 6.7.3. From the short exact sequence for  $\mathbb{R}\lim_k$  above, we see that

$$H_1 \mathbb{D}A_p^\wedge \cong \lim_k \text{Tor}(\mathbb{Z}/p^k, A)$$

<sup>7</sup>Again, we are considering a particular model structure on  $\text{Ch}(\mathbb{Z})$ , namely the one in which weak equivalences are quasi-isomorphisms and cofibrations are inclusions.

and there is a short exact sequence

$$0 \rightarrow \lim_k^1 \operatorname{Tor}(\mathbb{Z}/p^k, A) \rightarrow H_0 \mathbb{D}A_p^\wedge \rightarrow A_p^\wedge \rightarrow 0.$$

The group  $\operatorname{Tor}(\mathbb{Z}/p^k, A)$  is naturally isomorphic to  $A[p^k]$ , i.e. the subgroup of  $A$  of  $p^k$ -torsion elements: just compute  $\operatorname{Tor}$  by considering the resolution  $\dots 0 \rightarrow \mathbb{Z} \xrightarrow{p^k} \mathbb{Z}$  of  $\mathbb{Z}/p^k$  and tensoring it with  $A$ . Moreover the natural map  $\operatorname{Tor}(\mathbb{Z}/p^{k+1}, A) \rightarrow \operatorname{Tor}(\mathbb{Z}/p^k, A)$ , induced by the quotient projection  $\mathbb{Z}/p^{k+1} \rightarrow \mathbb{Z}/p^k$ , corresponds to the map  $A[p^{k+1}] \rightarrow A[p^k]$  given by multiplication by  $p$  inside  $A$ : just lift the quotient projection to the chain map

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{p^{k+1}} & \mathbb{Z} \\ \downarrow p & & \downarrow 1 \\ \mathbb{Z} & \xrightarrow{p^k} & \mathbb{Z} \end{array}$$

and again tensor with  $A$  and compute first homology. Thus if  $A[p^k] = A[p^{k+1}]$  for  $k$  sufficiently large, then  $H_1 \mathbb{D}A_p^\wedge = 0$  and  $H_0 \mathbb{D}A_p^\wedge \cong A_p^\wedge$ ,<sup>8</sup> so  $\mathbb{D}A_p^\wedge$  is (up to quasi-isomorphism) just  $A_p^\wedge$ . This is, for example, always true if  $A$  is finitely generated, or if multiplication by  $p$  is injective on  $A$ .

DEFINITION 6.7.4. An abelian group  $A$  is *derived  $p$ -complete* (or *Ext- $p$ -complete*) if the natural map  $A \rightarrow \mathbb{D}A_p^\wedge$  is a quasi-isomorphism, i.e. if  $A \xrightarrow{\sim} H_0 \mathbb{D}A_p^\wedge$  and  $H_1 \mathbb{D}A_p^\wedge = 0$ .

DEFINITION 6.7.5. Let  $\mathbb{Z}/p^\infty$  denote the abelian group obtained as the colimit of the sequence

$$\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \xrightarrow{p} \dots$$

REMARK 6.7.6. Our description of  $H_1 \mathbb{D}A_p^\wedge$  above can be interpreted as

$$\lim_k \operatorname{Hom}(\mathbb{Z}/p^k, A) \cong \operatorname{Hom}(\operatorname{colim}_k \mathbb{Z}/p^k, A) \cong \operatorname{Hom}(\mathbb{Z}/p^\infty, A).$$

We will now see that  $H_0 \mathbb{D}A_p^\wedge$  can similarly be interpreted as  $\operatorname{Ext}(\mathbb{Z}/p^\infty, A)$ . In fact, we will see that the chain complex  $\mathbb{D}A_p^\wedge$  can be described as a shift of a derived  $\operatorname{Hom}$  in the following sense:

DEFINITION 6.7.7. If  $C$  and  $D$  are chain complexes, we write  $\mathbb{R}\operatorname{Hom}(C, D)$  for the chain complex obtained as either  $\operatorname{Hom}(C', D)$  where  $C'$  is a cofibrant replacement of  $C$ , or as  $\operatorname{Hom}(C, D')$  where  $D'$  is a fibrant replacement of  $D$ , after fixing a suitable model structure on  $\operatorname{Ch}(\mathbb{Z})$ .

Concretely, if  $C$  is bounded below we can replace it with  $C'$ , where  $C' \rightarrow C$  is a quasi-isomorphism and  $C'$  is bounded below and levelwise a projective (free) abelian group. Or if  $D$  is bounded above, we can replace it with  $D'$ , where  $D \rightarrow D'$  is a quasi-isomorphism and  $D'$  is bounded above and levelwise an injective abelian group.

<sup>8</sup>The fact that in the given hypothesis  $\lim_k^1 \operatorname{Tor}(\mathbb{Z}/p^k, A)$  vanishes is an instructive exercise.

LEMMA 6.7.8. *The chain complexes  $\mathbb{D}A_p^\wedge$  and  $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A)[1]$  are quasi-isomorphic. In particular we have*

$$\begin{aligned} H_0\mathbb{D}A_p^\wedge &\cong \mathrm{Ext}(\mathbb{Z}/p^\infty, A), \\ H_1\mathbb{D}A_p^\wedge &\cong \mathrm{Hom}(\mathbb{Z}/p^\infty, A). \end{aligned}$$

SKETCH PROOF. We will use the fact that  $\mathbb{R}\mathrm{Hom}(-, A)$  takes derived colimits (in the first variable) to derived limits.<sup>9</sup> This is just because we can compute  $\mathbb{R}\mathrm{Hom}(-, A)$  by using a fibrant replacement (aka injective resolution) of  $A$ , and if  $A$  is already a fibrant chain complex, then  $\mathbb{R}\mathrm{Hom}(-, A): \mathrm{Ch}(\mathbb{Z}) \rightarrow \mathrm{Ch}(\mathbb{Z})$  sends cofibrations to fibrations.

The colimit defining  $\mathbb{Z}/p^\infty$  is already derived (all maps involved are already cofibrations, i.e. injective maps of abelian groups, seen as chain complexes concentrated in degree 0), so  $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathbb{R}\mathrm{lim}_k \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^k, A)$ . Now the chain complex  $\mathbb{Z}/p^k$  is a projective resolution of  $\mathbb{Z}/p^k$ , so the chain complex  $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^k, A)$  is quasi-isomorphic to  $\mathrm{Hom}(\mathbb{Z}/p^k, A)$ , which can be identified with  $A/p^k[-1]$ . Thus  $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathbb{R}\mathrm{lim}_k A/p^k[-1] \simeq \mathbb{D}A_p^\wedge[-1]$ .  $\square$

LEMMA 6.7.9. *There is a short exact sequence of abelian groups*

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}/p^\infty \rightarrow 0.$$

PROOF. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^k & \longrightarrow & 0 \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow p & & \downarrow p & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^{k+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^{k+1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

Since  $\mathbb{Z}[\frac{1}{p}] = \mathrm{colim}_k (\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \dots)$ , taking colimits in this diagram gives the desired short exact sequences (as taking the colimit of a sequence is an exact functor).  $\square$

As a consequence, if  $A'$  is an injective resolution of an abelian group  $A$ , we get a short exact sequence of chain complexes

$$0 \rightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, A') \rightarrow \mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A') \rightarrow A' \rightarrow 0,$$

or

$$0 \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}, A) \rightarrow 0.$$

<sup>9</sup>Here we use the model structure on  $\mathrm{Ch}(\mathbb{Z})$  for which weak equivalences are quasi-isomorphisms and cofibrations are inclusions.

The associated long exact sequence in homology is

$$0 \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \text{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow A \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, A) \rightarrow \text{Ext}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow 0.$$

From this we immediately see:

LEMMA 6.7.10. *An abelian group  $A$  is derived  $p$ -complete if and only if  $\text{Hom}(\mathbb{Z}[\frac{1}{p}], A) = 0$  and  $\text{Ext}(\mathbb{Z}[\frac{1}{p}], A) = 0$ , i.e. if and only if  $\mathbb{R}\text{Hom}(\mathbb{Z}[\frac{1}{p}], A) \simeq 0$ .*

We will also need the following characterization of derived  $p$ -complete groups:

l:pcompextonly

PROPOSITION 6.7.11. *If  $A \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, A)$  is an isomorphism, then all groups  $\text{Hom}(\mathbb{Z}/p^\infty, A)$ ,  $\text{Hom}(\mathbb{Z}[\frac{1}{p}], A)$  and  $\text{Ext}(\mathbb{Z}[\frac{1}{p}], A)$  vanish. In other words,  $A$  is derived  $p$ -complete if and only if  $A \xrightarrow{\sim} \text{Ext}(\mathbb{Z}/p^\infty, A)$ .*

PROOF. From the long exact sequence

$$0 \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \text{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow A \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, A) \rightarrow \text{Ext}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow 0$$

we see that if  $A \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, A)$  is an isomorphism, then the map  $\text{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow A$  is zero. A homomorphism  $\mathbb{Z}[\frac{1}{p}] \rightarrow A$  is determined by a sequence  $(a_0, a_1, \dots)$  such that  $a_i = pa_{i+1}$ . If the map to  $A$  is zero, then this means that for any such sequence we must have  $a_0 = 0$ . But  $(a_k, a_{k+1}, \dots)$  is a sequence in  $A$  of the same form, so we must have  $a_k = 0$  for all  $k$ . Thus  $\text{Hom}(\mathbb{Z}[\frac{1}{p}], A) = 0$ , and then from the exact sequence we get that  $\text{Hom}(\mathbb{Z}/p^\infty, A) = 0$  as well.  $\square$

nk:derpcomploc

REMARK 6.7.12. The derived  $p$ -complete abelian groups are precisely the ones that are  $\mathcal{W}$ -local where  $\mathcal{W}$  is the class of maps  $A \rightarrow B$  such that  $\text{Ext}(\mathbb{Z}/p^\infty, A) \xrightarrow{\sim} \text{Ext}(\mathbb{Z}/p^\infty, B)$  is an isomorphism. The localization of an abelian group  $A$  with respect to  $\mathcal{W}$  is  $LA = \text{Ext}(\mathbb{Z}/p^\infty, A)$ , with  $A \rightarrow LA$  induced by the boundary map from the sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}/p^\infty \rightarrow 0$ .

### 6.8. $p$ -completion of simply connected spaces II (the general case)

sec:pcompgen

We now want to consider  $p$ -completion for general simply connected spaces. The theorem we want to prove is:

thm:pcompgen

THEOREM 6.8.1.

- (i) *A simply connected space  $X$  is  $p$ -complete if and only if  $\pi_n X$  is derived  $p$ -complete for all  $n$ .*
- (ii) *If  $X$  is a simply connected space then its  $p$ -completion  $X \rightarrow X_p^\wedge$  exists, and for every  $n$  there is a short exact sequence*

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \rightarrow \pi_n X_p^\wedge \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \rightarrow 0.$$

The key step is understanding the  $p$ -completion of Eilenberg-MacLane spaces in terms of the algebraic derived  $p$ -completion we defined in the previous section. This requires introducing some notation:

DEFINITION 6.8.2. Suppose  $C$  is a non-negatively graded chain complex of abelian groups, with finitely many non-zero homology groups. For  $n \geq 0$  we can define an Eilenberg-MacLane space from  $C$  by

$$K(C, n) := \prod_i K(H_i C, n + i).$$

REMARK 6.8.3. Using a more functorial version of this definition, it can be shown that this construction is compatible with derived limits. Moreover, if we have a short exact sequence of chain complexes

$$0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$$

then  $K(C, n)$  is the homotopy fibre of the induced map<sup>10</sup>  $K(C', n) \rightarrow K(C'', n)$ . We will not prove this, but it should be plausible since the long exact sequence in homology from the short exact sequence looks the same as the long exact sequence on homotopy groups.

DEFINITION 6.8.4. Let  $A$  be an abelian group. We define  $K(A, n)_p^\wedge$  to be  $K(\mathbb{D}A_p^\wedge, n)$ .

More concretely, we can first define  $K(A, n)/p^k$  to be  $K(A//p^k, n)$ , which is equivalently the homotopy fibre of  $K(A, n + 1) \xrightarrow{p^k} K(A, n + 1)$ . Then we can define  $K(A, n)_p^\wedge$  as the homotopy limit of the maps  $K(A, n)/p^{k+1} \rightarrow K(A, n)/p^k$  (where the homotopy limit is given by replacing these maps by fibrations and then taking the usual limit).

REMARK 6.8.5. From our description of  $\mathbb{D}A_p^\wedge$  above, we see that

$$\pi_* K(A, n)_p^\wedge \cong \begin{cases} 0, & * \neq n, n + 1, \\ \text{Ext}(\mathbb{Z}/p^\infty, A), & * = n, \\ \text{Hom}(\mathbb{Z}/p^\infty, A), & * = n + 1. \end{cases}$$

PROPOSITION 6.8.6. *If  $A$  is an abelian group, then  $K(A, n) \rightarrow K(A, n)_p^\wedge$  is a  $p$ -completion.*

Before we give the proof we need to make a simple observation:

LEMMA 6.8.7. *Suppose  $M$  is a  $\mathbb{Z}/p^k$ -module. Then  $K(M, n)$  is  $p$ -complete.*

PROOF. We prove this by induction on  $k$ . For  $k = 1$  the universal coefficient sequence gives (as there is no Ext term over the field  $\mathbb{F}_p$ )  $H^*(X, M) \cong \text{Hom}_{\mathbb{F}_p}(H_*(X, \mathbb{F}_p), M)$ , so an  $\mathbb{F}_p$ -equivalence induces an isomorphism on  $H^n(-, M) \cong [-, K(M, n)]$ .

Now suppose  $M$  is a  $\mathbb{Z}/p^k$ -module. Then there is a short exact sequence

$$0 \rightarrow pM \rightarrow M \rightarrow M/pM \rightarrow 0,$$

where  $pM$  and  $M/pM$  are  $\mathbb{Z}/p^{k-1}$ -modules. This induces a long exact sequence

$$\dots \rightarrow H^n(X, pM) \rightarrow H^n(X, M) \rightarrow H^n(X, M/pM) \rightarrow H^{n+1}(X, pM) \rightarrow \dots$$

<sup>10</sup>This induced map is *not* just the product of the maps  $K(H_i C', n + i) \rightarrow K(H_i C'', n + i)$  induced by the group homomorphisms  $H_i C' \rightarrow H_i C''$ , but something more complicated, built into the functorial construction

pn:EMpcompgen

Zpkmodispcomp



Using the 5-Lemma this gives inductively that an  $\mathbb{F}_p$ -equivalence gives an isomorphism on  $H^n(-, M) \cong [-, K(M, n)]$ .  $\square$

**PROOF OF PROPOSITION 6.8.6.** To see that  $K(A, n)_p^\wedge$  is  $p$ -complete it suffices by Lemma 6.3.10 to see that  $K(A, n)/p^k$  is  $p$ -complete for each  $k$ . But this space is weakly equivalent to  $K(\mathrm{Tor}(\mathbb{Z}/p^k, A), n+1) \times K(A/p^k, n)$ . Here  $\mathrm{Tor}(\mathbb{Z}/p^k, A)$  and  $A/p^k$  are both  $\mathbb{Z}/p^k$ -modules, hence this space is  $p$ -complete by Lemma 6.8.7.

We saw in the previous section that we have a short exact sequence of chain complexes

$$0 \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}, A) \rightarrow 0,$$

which as  $\mathbb{D}A_p^\wedge \simeq \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A)[1]$  gives a fibre sequence

$$K(A, n)_p^\wedge \rightarrow K(\mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A), n+1) \rightarrow K(A, n+1).$$

Continuing this (as a Puppe sequence) we get a fibre sequence

$$K(\mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A), n) \rightarrow K(A, n) \rightarrow K(A, n)_p^\wedge.$$

Thus the homotopy groups of the fibre are  $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A)$  in degree  $n$  and  $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], A)$  in degree  $n+1$ . These are in particular both  $\mathbb{Z}[\frac{1}{p}]$ -modules, and so the map  $K(A, n) \rightarrow K(A, n)_p^\wedge$  is an  $\mathbb{F}_p$ -equivalence by Theorem 6.5.3.  $\square$

**COROLLARY 6.8.8.** *An Eilenberg-MacLane space  $K(A, n)$  is  $p$ -complete if and only if  $A$  is derived  $p$ -complete.*

**PROOF.** Since  $K(A, n)_p^\wedge$  is the  $p$ -completion of  $K(A, n)$ , the space  $K(A, n)$  is  $p$ -complete if and only if the natural map  $K(A, n) \rightarrow K(A, n)_p^\wedge$  is a weak equivalence. From the computation of  $\pi_* K(A, n)_p^\wedge$  we see this is equivalent to  $\mathrm{Hom}(\mathbb{Z}/p^\infty, A) = 0$  and  $A \xrightarrow{\sim} \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ .  $\square$

Now a Postnikov tower argument gives one direction of (i) in Theorem 6.8.1:

**PROPOSITION 6.8.9.** *If  $X$  is a simply connected space such that the groups  $\pi_* X$  are derived  $p$ -complete, then  $X$  is  $p$ -complete.*

**PROOF.** Exactly as the proof of Theorem 6.5.1(i).  $\square$

Using this we can prove part (ii) of the theorem:

**PROOF OF THEOREM 6.8.1(ii).** We construct the  $p$ -completion exactly as in the Proof of Theorem 6.5.1(ii): We take  $P_2 X_p^\wedge := K(\pi_2 X, 2)_p^\wedge$ , then  $P_2 X \rightarrow P_2 X_p^\wedge$  is a  $p$ -completion by Proposition 6.8.6. Then if we have a  $p$ -completion  $P_{n-1} X \rightarrow P_{n-1} X_p^\wedge$ , we get from the universal property a homotopy-commutative square

$$\begin{array}{ccc} P_{n-1} X & \xrightarrow{\phi} & P_{n-1} X_p^\wedge \\ \downarrow k_{n-1} & & \downarrow (k_{n-1})_p^\wedge \\ K(\pi_n X, n+1) & \xrightarrow{\psi} & K(\pi_n X, n+1)_p^\wedge \end{array}$$

which gives on homotopy fibres a map  $P_n X \rightarrow P_n X_p^\wedge := F_{(k_{n-1})_p^\wedge}$ , where the space  $P_n X_p^\wedge$  is  $p$ -complete by Lemma 6.3.9. Finally, we take  $X_p^\wedge := \lim_n P_n X_p^\wedge$ , which

is  $p$ -complete by Lemma 6.3.10. The maps  $P_n X \rightarrow P_n X_p^\wedge$  and  $X \rightarrow X_p^\wedge$  are  $\mathbb{F}_p$ -equivalences by the same arguments as in the finitely generated case.

It remains to show that we have the stated description of  $\pi_n X_p^\wedge$ . From the long exact sequence from the fibration  $P_n X_p^\wedge \rightarrow P_{n-1} X_p^\wedge \rightarrow K(\pi_n X, n+1)_p^\wedge$  we see that there are isomorphisms

$$\begin{aligned}\pi_i P_n X_p^\wedge &\cong \pi_i P_{n-1} X_p^\wedge, \quad i < n, \\ \text{Hom}(\mathbb{Z}/p^\infty, \pi_n X) &\cong \pi_{n+1} P_n X_p^\wedge,\end{aligned}$$

and (using this isomorphism for  $\pi_n P_{n-1} X_p^\wedge$ ) there is a short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \rightarrow \pi_n P_n X_p^\wedge \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \rightarrow 0.$$

The homotopy group  $\pi_i P_n X_p^\wedge$  thus stabilizes for  $n \geq i$  so there is no  $\lim^1$  and we get  $\pi_n X_p^\wedge \cong \pi_n P_n X_p^\wedge$ , and so we have the desired description of this group.  $\square$

REMARK 6.8.10. In fact, it can be shown that the short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \rightarrow \pi_n X_p^\wedge \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \rightarrow 0$$

always splits.

Finally, we end by proving the other direction in Theorem 6.8.1(i):

PROOF OF THEOREM 6.8.1(I). It remains to show that if  $X$  is simply connected and  $p$ -completed, then the abelian groups  $\pi_* X$  are derived  $p$ -complete. By the description of  $X_p^\wedge$  in Theorem 6.8.1(ii) we see if  $X$  is  $p$ -complete then there is a short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \rightarrow \pi_n X \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \rightarrow 0.$$

For  $n = 2$  this says  $\text{Ext}(\mathbb{Z}/p^\infty, \pi_2 X) \xrightarrow{\sim} \pi_2 X$ . It follows that  $\pi_2 X$  is derived  $p$ -complete, for example by Remark 6.7.12, or by using that the short exact sequence splits and then applying Proposition 6.7.11. Now if  $\pi_{n-1} X$  is derived  $p$ -complete, then  $\text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) = 0$  so the short exact sequence for  $\pi_n X$  gives  $\pi_n X \cong \text{Ext}(\mathbb{Z}/p^\infty, \pi_n X)$ . Thus we see by induction that all the homotopy groups of  $X$  are derived  $p$ -complete, as required.  $\square$

## 6.9. Cosimplicial spaces, and the Bousfield–Kan model for $p$ -completion

Notes from 2019.

### 6.10. More exercises

## Smith theory and Sullivan’s fixed point conjecture

[ MORE HANDWRITTEN NOTES!!]

### 7.1. Equivariant homotopy theory and the theory of diagrams

Ref: [Die87; Die79; May96]

**7.1.1. Assumptions and setup.** Throughout the chapter,  $G$  will be a finite group. By a “collection” of subgroups  $\mathcal{C}$  of  $G$  we will mean a set of subgroups of  $G$  closed under conjugation by elements of  $G$ . By a “family”  $\mathcal{F}$  of subgroups we mean a set of subgroups  $G$  closed under conjugation and also closed under taking subgroups.

DEFINITION 7.1.1. A  $G$ -space is a topological space  $X$  with an action of  $G$ . A  $G$ -map (or  $G$ -equivariant map)  $f : X \rightarrow Y$  between  $G$ -spaces is a map of spaces which respect the  $G$ -action, i.e.,  $f(g \cdot x) = g \cdot f(x)$  for all  $x \in X$ .

Two maps  $f, f' : X \rightarrow Y$  are called  $G$ -homotopic, if there exists a  $G$ -map  $F : X \times I \rightarrow Y$  such that  $F(\cdot, 0) = f$  and  $F(\cdot, 1) = g$ . Here  $X \times I$  is endowed with the  $G$ -action which is the product of the  $G$ -action on  $X$  and the trivial action on  $I$ .

defn:GCW

DEFINITION 7.1.2. A  $G$ -CW complex is a space  $X$  with a  $G$ -action, which is built out of equivariant cells  $G/H \times D^n$  for  $H \leq G$  and  $n \in \mathbb{N}_0$ . More precisely,  $X$  is an increasing union  $\bigcup_{n \geq 0} \text{sk}_n X$  of  $G$ -spaces  $\text{sk}_n$  (and is endowed with the weak topology of the union), by convention  $\text{sk}_{-1} X = \emptyset$ , and each  $\text{sk}_n X$  is obtained from the previous  $\text{sk}_{n-1} X$  by a pushout of the form

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod \phi_i} & \text{sk}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & \text{sk}_n, \end{array}$$

where  $I_n$  is a set (parametrising “equivariant  $n$ -cells”), with each  $i \in I_n$  we associate a subgroup  $H_i \leq G$ , the space  $G/H_i \times D^n$  and its “boundary” subspace  $G/H_i \times S^{n-1}$  are endowed with the  $G$ -action coming from the trivial<sup>1</sup> action of  $G$  on  $D^n$  (resp.  $S^{n-1}$ ) and the coset action of  $G$  on  $G/H_i$ , and with each  $i \in I_n$  we associate a  $G$ -equivariant map  $\phi_i : G/H_i \times S^{n-1} \rightarrow \text{sk}_{n-1} X$ .

<sup>1</sup>Some authors consider generic  $G$ -actions on  $(D^n, S^{n-1})$  coming from  $n$ -dimensional representations of  $G$ ; this can have the advantage of reducing the number of needed cells, but prevents the cellular approximation theorem to hold.

In these notes we will mainly consider  $G$ -spaces  $X$  which admit the structure of a  $G$ -CW complex. (Since  $G$  is finite this essentially amounts to assuming that  $X$  admits the structure of a CW complex on which  $G$  acts cellularly, i.e. permuting cells).

There are (at least) two different, natural notions of “ $G$ -equivalence”:

DEFINITION 7.1.3. A  $G$ -map  $f : X \rightarrow Y$  is called a  $G$ -equivalence (or sometimes strong  $G$ -equivalence) if there exists a  $G$ -map  $g : Y \rightarrow X$ , such that  $f \circ g$  and  $g \circ f$  are  $G$ -homotopic to the identity on  $Y$  and  $X$  respectively.

A  $G$ -map  $f : X \rightarrow Y$  is called an  $hG$ -equivalence (or sometimes weak  $G$ -equivalence) if  $f$  is a homotopy equivalence when considered as a non-equivariant map, i.e. as a plain map of spaces.

Note that “being  $G$ -equivalent” is an equivalence relation on  $G$ -spaces, by the  $G$ -equivariant analogue of the usual argument for spaces. Instead, being “ $hG$ -equivalent” is not even a symmetric relation: for instance, if we consider a point  $*$  with trivial  $G$ -action and a contractible space  $EG$  with free  $G$ -action, then the map  $EG \rightarrow *$  is a  $hG$ -equivalence, but there exists no  $G$ -equivariant map  $* \rightarrow EG$ , even less a  $hG$ -equivalence.

DEFINITION 7.1.4. Two  $G$ -spaces  $X$  and  $Y$  are said to be  $hG$ -equivalent if there is a zig-zag of spaces and  $hG$ -equivalences connecting them, of the form

$$X \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \leftarrow Z_{2n-1} \rightarrow Y.$$

equivequiv

THEOREM 7.1.5 (Equivariant Whitehead theorem). A  $G$ -map  $f : X \rightarrow Y$  between  $G$ -CW complexes is a (strong)  $G$ -equivalence if and only if the restricted map  $f^H : X^H \rightarrow Y^H$  is a (non-equivariant) homotopy equivalence for all subgroups  $H \leq G$ . Here  $X^H \subseteq X$  denotes the subspace of  $H$ -fixed points (points that are fixed by all elements of  $H$ ).

SKETCH OF PROOF. That  $G$ -equivalences induces equivalences on fixed-points follows directly from the definition: just observe that a  $G$ -homotopy  $X \times I \rightarrow Y$  restricts for all  $H \leq G$  to a homotopy  $X^H \times I \rightarrow Y^H$ .

Conversely, given a  $G$ -map  $f : X \rightarrow Y$  between  $G$ -CW complexes, we can approximate it up to  $G$ -homotopy by a *cellular*  $G$ -map (prove this! You will see that you need that  $G$  acts trivially in the  $D^n$ -direction of an equivariant cell). If  $f$  is cellular, the mapping cylinder  $\text{Cyl}(f)$  deformation retracts  $G$ -equivariantly onto  $Y$ ; so we can replace  $Y$  by  $\text{Cyl}(f)$ , and assume that  $f$  is an inclusion of  $G$ -CW complexes.

Proceeding inductively on skeleta, we can now construct a  $G$ -equivariant deformation retraction of  $Y$  onto  $X$ : for this we use repeatedly that if  $H : Y \times I \rightarrow Y$  is a  $G$ -homotopy rel  $X$  from the identity of  $Y$  to a map  $g_{n-1} : Y \rightarrow Y$  sending  $X \cup \text{sk}_{n-1} Y$  inside  $X$ , for every attaching map  $\phi_i : G/H_i \times S^{n-1} \rightarrow Y$  of a  $G$ -equivariant cell in  $Y \setminus X$ , the composite  $g_{n-1} \circ \phi_i$  restricts to a map  $D^n \cong H_i/H_i \times D^n \rightarrow Y^{H_i}$  which, by hypothesis, is homotopic rel  $H_i/H_i \times S^{n-1}$  to a  $G$ -equivariant map  $H_i/H_i \times D^n \rightarrow X^{H_i}$ ; this homotopy can be uniquely extended  $G$ -equivariantly over the entire  $G$ -equivariant

cell, and we thus get a  $G$ -equivariant homotopy  $X \cup \text{sk}_n Y \times I \text{ rel } X$  starting from  $g_{n-1}|_{\text{sk}_n Y}$  and ending with a map  $X \cup \text{sk}_n Y \rightarrow X$ .

We can then prove a  $G$ -equivariant version of the homotopy extension property for  $G$ -CW complexes (relative to their skeleta), and thus extend the previous to a homotopy  $Y \times I \rightarrow Y$ , ending with  $g_n : Y \rightarrow Y$  which sends  $X \cup \text{sk}_n Y$  inside  $X$ . The details are left as exercise; see also [Ben91, Thm. 6.4.2].  $\square$

nofixedpoints

REMARK 7.1.6. Note that, contrary to the above, a map being an  $hG$ -equivalence tells us nothing on what happens on the fixed-points for non-trivial subgroups  $H$ . E.g., for any  $G$ -space  $X$ , the projection  $X \times EG \rightarrow X$  is an  $hG$ -equivalence, but  $(X \times EG)^H = \emptyset$  for all  $1 \subsetneq H \leq G$ . (Where  $EG$  as usual denotes a free contractible  $G$ -space.)

In the next subsection, we will introduce a sort of derived version of the functor “taking fixed-points”, the homotopy fixed-points, which we can still work with in absence of meaningful actual fixed-points, and which by deep theorems agree with actual fixed-points in good cases (e.g.  $X$  finite, and “at a prime  $p$ ”).

hGequiv

PROPOSITION 7.1.7. *A  $G$ -map  $f : X \rightarrow Y$  is an  $hG$ -equivalence if and only if  $f \times EG : X \times EG \rightarrow Y \times EG$  is a  $G$ -equivalence.*

*In particular two  $G$ -spaces  $X$  and  $Y$  are  $hG$ -equivalent if and only if  $X \times EG$  and  $Y \times EG$  are  $G$ -equivalent.*

PROOF. The diagram

$$\begin{array}{ccc} X \times EG & \xrightarrow{f \times EG} & Y \times EG \\ \simeq \downarrow hG & & \simeq \downarrow hG \\ X & \xrightarrow{f} & Y \end{array}$$

shows that  $f$  is a homotopy equivalence of CW complexes if and only if  $f \times EG$  is. Furthermore, if  $H \leq G$  is a nontrivial subgroup then map  $(f \times EG)^H$  is a map between empty spaces, so it is also a homotopy equivalence. Therefore, by Lemma 7.1.5, the map  $f \times EG$  is a homotopy equivalence if and only if it is a  $G$ -equivalence.

For the “in particular”: If  $X \times EG$  and  $Y \times EG$  are  $G$ -equivalent, then  $X$  and  $Y$  are  $hG$ -equivalent via the zig-zag of  $G$ -equivariant homotopy equivalences

$$X \leftarrow X \times EG \simeq Y \times EG \rightarrow Y.$$

Conversely, if  $X$  and  $Y$  are  $hG$ -equivalent they are connected by a zig-zag of  $hG$ -equivalences.

$$X = X_0 \rightarrow X_1 \leftarrow X_2 \rightarrow \cdots \leftarrow X_n = Y$$

so applying  $EG \times -$  shows that  $X \times EG$  and  $Y \times EG$  are  $G$ -equivalent, by the first part of the proposition and by the fact that  $G$ -equivalence is an actual equivalence relation.  $\square$

REMARK 7.1.8. The localization of  $G\text{Top}$  at the class of  $hG$ -equivalences, denoted  $G\text{Top}[hG^{-1}]$  or briefly  $hG - \text{Top}$ , is thus equivalent to the  $G$ -equivariant homotopy category of spaces with a free  $G$ -action. The functor  $EG \times -$  is a left deformation of  $G\text{Top}$  at  $hG$ -equivalences, and its essential image is contained in free  $G$ -spaces.

REMARK 7.1.9. The theorem has a generalization for all families  $\mathcal{C}$  of subgroups of  $G$ . We will prove this later; for the moment we just anticipate that  $EG$  has to be replaced by the space  $E\mathbf{O}_{\mathcal{C}}$  constructed as follows: we consider the “orbit category”  $\mathbf{O}_{\mathcal{C}}$ , with objects the cosets  $G/H$  for  $H \in \mathcal{C}$  and morphisms  $G$ -equivariant maps; we consider the functor  $\Theta: \mathbf{O}_{\mathcal{C}} \rightarrow \text{Top}$ , sending  $\mathbf{O}_{\mathcal{C}} \ni G/H \mapsto G/H \in \text{Set} \subset \text{Top}$ ; and  $E\mathbf{O}_{\mathcal{C}} := \text{hocolim } \Theta = B(*, \mathbf{O}_{\mathcal{C}}, \Theta)$ . There is an action of  $G$  on  $E\mathbf{O}_{\mathcal{C}}$  induced by an action of  $G$  on  $\Theta$  by natural isomorphisms, and we immediately notice that for  $H \leq G$  the fix-points  $(E\mathbf{O}_{\mathcal{C}})^H$  are empty if  $H \notin \mathcal{C}$ , whereas  $(E\mathbf{O}_{\mathcal{C}})^H \cong B(\mathbf{O}_{\mathcal{C}})_{(G/H)/} \simeq *$  if  $H \in \mathcal{C}$ .

Note that for  $\mathcal{C}$  containing only the trivial group we have that  $E\mathbf{O}_{\mathcal{C}}$  is a contractible, free  $G$ -space.

### 7.1.2. Equivariant homotopy theory, and the diagram categories: Elmendorf’s theorem. Ref: [Elm83].

Proposition 7.1.5 shows that equivariant equivalences can be described as non-equivariant equivalences on fixed-points.

Note that the category of  $G$ -spaces and  $hG$ -equivalences can be described as a subcategory of a functor category, namely the category of functors from  $G$  (viewed as a category with one object, and  $G$  as morphisms) to spaces: the allowed morphisms are the natural equivalences which object-wise are weak equivalences.

In this subsection we will address two natural questions:

- Can the the category of  $G$ -spaces and  $G$ -equivalences be understood via non-equivariant homotopy theory as a certain functor category? (yes, Prop )
- Is there a relationship between the homotopy type of  $X$  and that of, say,  $X^{\mathbb{Z}/p}$ ? (In general little by Remark 7.1.6, but under finiteness and “at  $p$ ” assumptions a lot, cf later).

To warm up to Elmendorf’s theorem, we do the following baby case first.

DEFINITION 7.1.10 (The orbit category and the  $p$ -orbit category). Let  $\mathbf{O}(G)$  denote the orbit category of  $G$ , i.e., the category with objects  $G/P$ , where  $P$  runs through the subgroups of  $G$ , and morphisms  $G$ -maps.

The  $p$ -orbit category  $\mathbf{O}_p(G)$  is the full subcategory of  $\mathbf{O}(G)$ , where  $P$  is assumed to be a  $p$ -subgroup.

Note that

$$\text{Hom}_{\mathbf{O}(G)}(G/P, G/Q) = \{g \in G \mid g^{-1}Pg \leq Q\}/Q = (G/Q)^P$$

and in particular  $\text{Hom}_{\mathbf{O}(G)}(G/P, G/P) = N_G(P)/P$ .

We will, in fact, most often be interested in the *opposite* orbit and  $p$ -orbit categories. Let us do a couple of examples:

EXAMPLE 7.1.11. (1) Suppose that  $G = \mathbb{Z}/p$ , then the  $p$ -orbit category  $\mathbf{O}_p(G)$  coincides with the entire category  $\mathbf{O}(G)$ : it has two objects  $G/G$  and  $G/e$ ; moreover  $\text{Hom}_{\mathbf{O}(G)}(G/e, G/e) \cong G$ , there is precisely one morphism in each hom set  $\text{Hom}_{\mathbf{O}(G)}(G/e, G/G)$  and  $\text{Hom}_{\mathbf{O}(G)}(G/G, G/G)$  (i.e.  $G/G$  is terminal), and  $\text{Hom}_{\mathbf{O}(G)}(G/G, G/e) \cong \emptyset$ . Diagrammatically, the opposite  $p$ -orbit category  $\mathbf{O}_p(G)^{\text{op}}$  looks like:

$$\bullet_{G/G} \longrightarrow \bullet_{G/e} \begin{array}{c} \xrightarrow{G^{\text{op}}} \\ \downarrow \\ \bullet_{G/e} \end{array}$$

(2) Suppose  $G = \mathfrak{S}_3$ , the 3<sup>rd</sup> symmetric group. A skeleton of the opposite 2-orbit category  $\mathbf{O}_2(G)^{\text{op}}$ , accounting for only one of the three (isomorphic) objects of the form  $G/P$  with  $|P| = 2$ , looks like

$$\bullet_{G/P} \xrightarrow{G/P} \bullet_{G/e} \begin{array}{c} \xrightarrow{G^{\text{op}}} \\ \downarrow \\ \bullet_{G/e} \end{array}$$

that is only the identity self-map of  $G/P$ ,  $G$  worth of self-maps of  $G/e$ , and 3 maps from  $G/P$  to  $G$ , which naturally identify with  $G/P$ .

THEOREM 7.1.12 (Elmendorf [Elm83]). *The functor*

$$\Phi : G\text{Top} \rightarrow \text{Top}^{\mathbf{O}(G)^{\text{op}}}, \quad X \mapsto \{X^H\}_{G/H \in \mathbf{O}(G)^{\text{op}}}$$

*induces a 1-to-1-correspondence between classes of objects*

$$\{G\text{-spaces}\}/G\text{-equivalence} \longleftrightarrow \{\text{Functors } \mathbf{O}(G)^{\text{op}} \rightarrow \text{Top}\}/\text{objectwise h.e.}$$

*In fact, with suitable definition of model category structures on the left- and right-hand side, this induces a Quillen equivalence of model categories.*

*In particular  $\Phi$  induces an equivalence between localizations of categories*

$$G\text{Top}[\{G\text{-equivalences}\}^{-1}] \xrightarrow{\cong} \text{Top}^{\mathbf{O}(G)^{\text{op}}}[\{\text{objectwise homotopy equivalences}\}^{-1}].$$

//

Before giving a sketch of proof of this theorem, let us do the special case  $G = \mathbb{Z}/p$ , which is already interesting.

EXAMPLE 7.1.13 (Elmendorf’s theorem for  $G = \mathbb{Z}/p$ ). Elmendorf’s theorem for  $\mathbb{Z}/p$  claims that giving a  $G$ -space  $X$ , up to  $G$ -equivalence, is equivalent to giving a space  $Y_0$ , a  $G$ -space  $Y_1$  and a (non-equivariant) map  $f : Y_0 \rightarrow Y_1^G$ , up to natural equivalence of diagrams. We want to give an inverse functor. Given  $Y_0$  and  $Y_1$ , we can define a space  $X$  as the homotopy pushout of the diagram

$$\begin{array}{ccc} EG \times Y_0 & \xrightarrow{\text{proj}} & Y_0 \\ \downarrow EG \times f & & \\ EG \times Y_1 & & \end{array}$$

The action of  $G$  on  $X$  is given by constructing the homotopy pushout as a double mapping cylinder and by noticing that, considering  $Y_0$  as a  $G$ -space with trivial action, the previous is a diagram of  $G$ -spaces.

To show well-definedness up to equivalence of data, suppose that  $Y'_0, Y'_1$  and  $f : Y'_0 \rightarrow (Y'_1)^G$  is another set of data as  $Y_0, Y_1$  and  $f : Y_0 \rightarrow Y_1^G$ , and suppose that there are maps  $g_0 : Y_0 \rightarrow Y'_0$  and  $g_1 : Y_1 \rightarrow Y'_1$  such that  $f_1$  is  $G$ -equivariant, the composite  $f' \circ g_0 : Y_0 \rightarrow (Y'_1)^G \subset Y'_1$  coincides with the composite  $g_1 \circ f : Y_0 \rightarrow Y_1^G \subset Y_1 \rightarrow Y'_1$ ; finally, suppose that  $g_0$  and  $g_1$  are weak equivalences of spaces. Then we can use  $g_0$  and  $g_1$  to construct a natural transformation between the two pushout diagrams  $EG \times Y_1 \leftarrow EG \times Y_0 \rightarrow Y_0$  and  $EG \times Y'_1 \leftarrow EG \times Y'_0 \rightarrow Y'_0$ ; notice that this is a natural transformation of diagrams of  $G$ -spaces. Taking homotopy pushouts/double mapping cylinders, we obtain a weak equivalence  $X \rightarrow X'$  which is also  $G$ -equivariant, showing that  $X$  and  $X'$  are  $hG$ -equivalent. Moreover  $X^G \cong Y_0$  and  $(X')^G \cong Y'_0$ , and the restricted map  $X^G \rightarrow (X')^G$  coincides with  $g_0$ , so it is a weak equivalence. It follows from Theorem 7.1.5 that  $X$  is  $G$ -equivalent to  $X'$ .

Finally, one checks that these procedures are each other inverses. Taking fixed-points on the homotopy pushout one easily sees that one recovers the diagram  $Y_0 \rightarrow Y_1$ , up to homotopy. For this, note also that  $X$  is homotopy equivalent (non- $G$ -equivariantly) to the homotopy pushout of  $Y_1 \leftarrow Y_0 \rightarrow Y_0$ , i.e. to  $Y_1$ .

Likewise, if one starts with a  $G$ -space, the homotopy colimit of  $EG \times X \leftarrow EG \times X^G \rightarrow X^G$  maps to  $X$ , and this map is a  $G$ -equivalence, since it induces an equivalence on all fixed-points (on  $e$ -fixed-points and on  $G$ -fixed points).

SKETCH OF PROOF OF ELMENDORF'S THEOREM; GENERAL CASE. The inverse  $\Psi : \text{Top}^{\mathbf{O}(G)^{\text{op}}} \rightarrow G\text{Top}$  is given by sending a functor  $F : \mathbf{O}(G)^{\text{op}} \rightarrow \text{Top}$  to the geometric realization of the simplicial space whose space of  $n$ -simplices contains sequences

$$(G/e \rightarrow G/P_0 \rightarrow G/P_1 \rightarrow \cdots \rightarrow G/P_n, x \in F(G/P_n)),$$

and the topology and the simplicial maps are the obvious ones.

This is by definition the two-sided bar construction  $B(F, \mathbf{O}(G), \Theta)$ , where  $\Theta : \mathbf{O}(G) \rightarrow G\text{Top}$  is the "identity functor", sending the object  $G/H \in \mathbf{O}(G)$  to the  $G$ -space (which is in fact a  $G$ -set)  $G/H$ ; another description of  $\Psi(F)$ , equivalent to  $B(F, \mathbf{O}(G), \Theta)$ , is  $\text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/e} F$ .

We can consider  $\Theta$  also as a functor  $\mathbf{O}(G) \rightarrow \text{Top}$  together with an action of  $G$  on  $\Theta$  by natural isomorphisms; in this light we immediately obtain an action of  $G$  on  $B(F, \mathbf{O}(G), \Theta)$ , by naturality of the bar construction in the second functor.

Alternatively, we can let  $G$  act on the overcategory  $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$  by letting it act on  $G/e$ , using naturality of the overcategory construction in the object "over which" the overcategory is taken. The functor  $F : (\mathbf{O}(G)^{\text{op}}) \downarrow G/e$  is really  $F$  composed with the "projection-on-the-source" functor  $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e \rightarrow (\mathbf{O}(G)^{\text{op}})$ , so  $F$  is invariant under the action of  $G$  on  $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$ . We then have an action of  $G$  on  $\text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/e} F$ . This gives another description of the action of  $G$  on  $\Psi(F)$ .

We now want to construct a natural transformation  $\eta : \Phi\Psi \Rightarrow \text{Id}_{\text{Top}^{\mathbf{O}(G)^{\text{op}}}}$  which is objectwise a levelwise homotopy equivalence: that is, for all  $F$  and all  $G/H$ ,  $(\eta_F)_{G/H} : (\Phi\Psi(F))(G/H) \rightarrow F(G/H)$  is a homotopy equivalence of spaces.



For this, note that since the functor  $F: (\mathbf{O}(G)^{\text{op}}) \downarrow G/e \rightarrow \text{Top}$  is  $G$ -invariant (and in particular  $H$ -invariant for all  $H \leq G$ ), we have a natural identification

$$(\Phi\Psi(F))(G/H) = \left( \text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/e} F \right)^H \cong \text{hocolim}_{((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H} F.$$

Moreover note that a morphism in  $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$  is fixed by  $H$  if and only if its source and target objects are fixed by  $H$ : this is again a consequence of the fact that the action of  $G$  on  $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$  is given by postcomposing with automorphisms of  $G/e$ ; so  $((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H$  is a full subcategory of  $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$ , and its objects are precisely those arrows  $G/e \xrightarrow{f} G/P$  in  $\mathbf{O}(G)$  that are invariant under precomposition by elements of  $H \leq \text{Aut}_{\mathbf{O}(G)}(G/e)$ ; this translates into the requirement that  $f$  factors through the projection  $G/e \rightarrow G/H$ ; but then we just have  $((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H \cong (\mathbf{O}(G)^{\text{op}}) \downarrow G/H$ , which is a category with terminal object  $G/H \xrightarrow{\text{id}_{G/H}} G/H$ , hence we have a natural weak equivalence

$$\text{hocolim}_{((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H} F \cong \text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/H} F \xrightarrow{\simeq} F(G/H).$$

We declare  $(\eta_F)_{G/H}$  to be the composite weak equivalence

Viceversa, we have a natural transformation  $\sigma: \Psi\Phi \Rightarrow \text{Id}_{G\text{Top}}$ : given a  $G$ -space  $X$ , we consider the composite map

$$\sigma_X: \Psi\Phi(X) = \text{hocolim}_{(f:G/e \rightarrow G/H) \in (\mathbf{O}(G)^{\text{op}}) \downarrow G/e} X^H \rightarrow \text{colim}_{(f:G/e \rightarrow G/H) \in (\mathbf{O}(G)^{\text{op}}) \downarrow G/e} X^H \rightarrow X$$

given by first mapping the homotopy colimit to the actual colimit, and then by mapping the copy of  $X^H$  corresponding to  $f$  to  $X$  along multiplication by  $f$ : here we use that a  $G$ -equivariant map  $G/e \cong G \rightarrow G/H$  is given by first applying a *right* multiplication by an element  $f \in G$ , getting a  $G$ -equivariant map  $\cdot \cdot f: G \rightarrow G$ , and then projecting onto  $G/H$ ; the final map only depends on the coset  $fH \in G/H$  (and similarly the map  $f: X^H \rightarrow X$  only depends on the coset  $fH \in G/H$ ).

One checks that  $\sigma_X$  is  $G$ -equivariant, and that it induces for all  $H \leq G$  a weak equivalence on  $H$ -fixed-points, by an argument analogue to the one above.  $\square$

### 7.2. Homotopy limits and homotopy fixed-points

- Model for homotopy fixed points as space of sections.
- examples.

Notes from CatTop 2009.

————— For a subgroup  $H \leq G$  we consider the functors  $(-)^H: G\text{Top} \rightarrow \text{Top}$  and  $(-)/H: G\text{Top} \rightarrow \text{Top}$ , called “ $H$ -fixed-points” and “ $H$ -orbits” respectively. Both functors are homotopy invariant under  $G$ -equivalences, but they are not homotopy invariant under  $hG$ -equivalences when  $H$  is not the trivial group.

In this section we will introduce the functor “ $H$ -homotopy-fixed-points”, denoted  $(-)^{hH}: G\text{Top} \rightarrow \text{Top}$ , and show that this functor is homotopy invariant under  $hG$ -equivalence. We will furthermore discuss how to calculate fixed-points and homotopy fixed-points, and how to relate them. More precisely we want to see:

- When  $V$  is an *elementary abelian group* and  $X$  is a finite  $V$ -CW complex, the cohomology  $H^*(X^V; \mathbb{F}_p)$  is determinable from cohomological information which only depends on the  $hV$ -homotopy type of  $X$  (i.e. we can replace  $X$  up to  $hV$ -equivalence and still be able to compute correctly  $H^*(X^V; \mathbb{F}_p)$ ).
- More generally, when  $p$  is a prime,  $P$  is a  $p$ -group and  $X$  is a finite  $P$ -CW complex, there is a close relationship between the  $hP$ -homotopy type of  $X$ , and the homotopy type of the space  $X^P$  “at the prime  $p$ ” (read: after  $p$ -completion).
- Replacing fixed-points by homotopy fixed-points, we find a natural home for these theorems, and we can significantly weaken the finiteness assumption on  $X$ .

This will be used in the next section to explore how we can get strong invariants of  $X$  (at a prime  $p$ ) by just considering all the spaces  $X^{hP}$  together (assembled as a functor out of the  $p$ -orbit category); these will be clearly invariants of  $X$  up to  $hG$ -equivalence.

### 7.2.1. Homotopy orbit space and homotopy fixed points. Ref: [DW94]

Denote by  $BG$  a classifying space of  $G$ , i.e.,  $BG \simeq K(G, 1)$ , and let  $EG$  denote a contractible  $G$ -space on which  $G$  acts freely. We have  $BG \simeq EG/G$ ; moreover two different choices of  $BG$  as above are homotopy equivalent, and two choices of  $EG$  are  $G$ -equivalent.

DEFINITION 7.2.1. We define the Borel construction, or “ $G$ -homotopy-orbits” as the functor  $(-)_hG : G\text{Top} \rightarrow \text{Top}$ , sending

$$X \mapsto X_{hG} := (X \times EG)/G,$$

where  $G$  acts diagonally on the product.

Dually we define the “ $G$ -homotopy-fixed-points” as the functor  $(-)^{hG} : G\text{Top} \rightarrow \text{Top}$ , sending

$$X \mapsto X^{hG} := \text{map}_G(EG, X)$$

i.e., the space of  $G$ -equivariant maps, or equivalently the  $G$ -fixed-points of the mapping space  $\text{map}(EG, X)$ , which is a  $G$ -space by letting  $g \in G$  act on  $f : EG \rightarrow X$  by  $(g \cdot f)(x) = gf(g^{-1}x)$ .

Note that  $*_{hG} = BG$ . Note also that we have a map  $X^G \rightarrow X^{hG}$  induced by the  $G$ -equivariant map  $EG \rightarrow *$  (which is a  $hG$ -equivalence!). The homotopy fixed-points should be thought of as (right) derived fixed-points.

sectionslemma

LEMMA 7.2.2. *Let  $X$  be a  $G$ -space; then the equivariant map  $X \rightarrow *$  induces a map  $\lambda : X_{hG} \rightarrow BG$ , which is a fibre bundle with fibre  $X$ . Then  $X^{hG}$  can be identified with the space of sections of  $\lambda : X_{hG} \rightarrow BG$ .*

PROOF. The map  $EG \times X \rightarrow EG \times *$  is a fibre bundle, and  $G$  acts freely (and properly discontinuously, up to choosing a good model for  $EG$ ...) on both spaces; it follows that  $EG \rightarrow BG$  is a covering map, and a product trivialisation of  $\lambda : X_{hG} \rightarrow BG$  can be found over any open set of  $BG$  for which  $EG \rightarrow BG$  can be trivialised, pushing down the (global) trivialisation of  $EG \times X \rightarrow EG \times *$ .

We now describe the maps in the two directions: Given a  $G$ -equivariant map  $f : EG \rightarrow X$ , we get the commutative diagram of  $G$ -maps

$$\begin{array}{ccc} EG & \xrightarrow{(f, \text{id}_{EG})} & X \times EG \\ & \searrow & \downarrow \text{proj} \\ & & EG \end{array}$$

which upon quotienting out by  $G$  produces the section.

In the other direction, given a section  $s : BG \rightarrow X_{hG}$  of  $\lambda$  we get a  $G$ -equivariant map  $\tilde{s} : EG \rightarrow X \times EG$  by using that the trivial bundle  $X \times EG \rightarrow EG$  is the pullback of the bundle  $\lambda$  along  $EG \rightarrow BG$ ; we then consider the composite  $EG \xrightarrow{\tilde{s}} X \times EG \xrightarrow{\text{proj}} X$ . □

Invarianceprop

**PROPOSITION 7.2.3.** *If  $f : X \rightarrow Y$  is an  $hG$ -equivalence, then it induces homotopy equivalences  $f_{hG} : X_{hG} \rightarrow Y_{hG}$  and  $f^{hG} : X^{hG} \rightarrow Y^{hG}$ .*

**PROOF.** If  $X \rightarrow Y$  is an  $hG$ -equivalence, then  $X \times EG \rightarrow Y \times EG$  is a  $G$ -equivalence, by Lemma 7.1.5, and hence  $X_{hG} \rightarrow Y_{hG}$  is an equivalence (over  $BG$ ) by the invariance property of orbit spaces under  $G$ -equivalence: in general, if  $A$  and  $B$  are  $G$ -equivalent  $G$ -spaces, then the  $G$ -maps and the  $G$ -homotopies witnessing this can be quotiented by  $G$  to get maps and homotopies witnessing that  $A/G$  is homotopy equivalent to  $B/G$ . This shows the first claim.

To see the second claim, note that by Lemma 7.2.2 the map  $X^{hG} \rightarrow Y^{hG}$  identifies with the map from the space of sections of  $\lambda_X : X_{hG} \rightarrow BG$  to the space of sections of  $\lambda_Y : Y_{hG} \rightarrow BG$ , given by composition with  $f_{hG}$  (here we use that  $f_{hG}$  is a map of bundles over  $BG$ ). By elementary homotopy theory, we can choose an inverse equivalence *over*  $BG$  to the map  $X_{hG} \rightarrow Y_{hG}$ : for example we can first choose any homotopy inverse  $i : Y_{hG} \rightarrow X_{hG}$  to  $f_{hG}$ ; we then have that  $\lambda_Y$  is homotopic to  $\lambda_X \circ i$ , and lifting such a homotopy along  $\lambda_X$ , starting with  $i$ , gives the desired homotopy inverse *over*  $BG$ . Similarly one can achieve that also the homotopies witnessing the homotopy equivalence of  $X_{hG}$  and  $Y_{hG}$  are *over*  $BG$ . But then we can use these maps and homotopies over  $BG$  to get maps and homotopies between the spaces of sections. □

**PROPOSITION 7.2.4.** *The functors  $EG \times -$  and  $\text{map}(EG, -)$  turn an  $hG$ -equivalence  $f : X \rightarrow Y$  into a  $G$ -equivalence.*

*In particular two  $G$ -spaces  $X$  and  $Y$  are  $hG$ -equivalent if and only if  $X \times EG$  and  $Y \times EG$  are  $G$ -equivalent, and if and only if  $\text{map}(EG, X)$  and  $\text{map}(EG, Y)$  are  $G$ -equivalent.*

**PROOF.** We have already seen the claim about  $EG \times -$  in Proposition 7.1.7. The proof about  $\text{map}(EG, -)$  is similar: The map  $\text{map}(EG, f)$  is an  $hG$ -equivalence, since it is  $G$ -equivariant and a homotopy equivalence of spaces. Furthermore, if  $H \leq G$  is a nontrivial subgroup then  $\text{map}(EG, f)^H : \text{map}(EG, X)^H \rightarrow \text{map}(EG, Y)^H$  can be identified, up to using  $EG$  also as a model for  $EH$  (and indeed  $EG$  is contractible and admits a free  $H$ -action), with  $\text{map}_H(EH, f) = f^{hG}$ , which is a homotopy equivalence

by Proposition 7.2.3. It follows from Lemma 7.1.5 that  $\text{map}(EG, f)$  is in fact  $G$ -equivalences.

Likewise if  $X$  and  $Y$  are  $hG$ -equivalent they are connected by a zig-zag of  $hG$ -equivalences.

$$X = X_0 \rightarrow X_1 \leftarrow X_2 \rightarrow \cdots \leftarrow X_n = Y$$

Applying  $\text{map}(EG, -)$  to this shows the one direction of the 'in particular'. The other direction is immediate, noting that  $X = \text{map}(*, X) \rightarrow \text{map}(EG, X)$  is an  $hG$ -equivalence.  $\square$

REMARK 7.2.5. Examining this a bit more closely, one can prove that the category of  $G$ -spaces localised at  $hG$ -equivalences, is equivalent to the category of spaces over  $BG$  localised at homotopy equivalences; in symbols

$$G\text{Top}[\{hG\text{-equivalences}\}^{-1}] \cong \text{Top}_{/BG}[\{\text{homotopy equivalences (over } BG)\}^{-1}].$$

One can in fact construct two model structures on  $hG$ -spaces, where  $EG \times -$  and  $\text{map}(EG, -)$  are cofibrant respectively fibrant replacements.

### 7.3. Smith theory and the localization theorem

**7.3.1. Smith theory: The fixed-points  $X^{\mathbb{Z}/p}$  from  $X$ .** We now want to discuss to which extent  $X^G$  can be determined from the  $hG$ -homotopy type of  $X$ .

#### 7.3.2. The localization theorem.

relcohomology

DEFINITION 7.3.1. For a  $G$ -space  $X$ , we define the Borel equivariant cohomology of  $X$  as the ordinary cohomology of the Borel construction, i.e.,  $H_{hG}^*(X) = H^*(X_{hG})$ . Similarly we define, for a commutative ring  $R$ ,  $H_{hG}^*(X; R) = H^*(X_{hG}; R)$ ; and if  $Y \subset X$  is an inclusion of  $G$ -spaces, we get an inclusion  $Y_{hG} \subset X_{hG}$  and define  $H_{hG}^*(X, Y; R) = H^*(X_{hG}, Y_{hG}; R)$ .

Borel cohomology is a  $G$ -equivariant cohomology theory (it is in particular a contravariant functor from  $G\text{Top}$  to graded abelian groups), and obviously it only depends on  $X$  up to  $hG$ -equivalence. Note that  $H_{hG}^*(*)$  is by definition  $H^*(BG)$ , which coincides with the group cohomology of  $G$ .

Notice also that  $H_{hG}^*(-)$  is the composition of two functors, namely  $(-)_hG: G\text{Top} \rightarrow \text{Top}$  and  $H^*(-)$ , which is in fact a contravariant functor from  $\text{Top}$  to graded rings; hence  $H_{hG}^*(X)$  is naturally a graded ring; similarly in the relative case and with coefficients in  $R$ .

To understand the statement of the next theorem, note that the  $G$ -equivariant map  $X \rightarrow *$  always gives rise to a map of rings  $H^*(BG) \rightarrow H_{hG}^*(X)$ . Moreover, for any couple of  $G$ -spaces  $(X, Y)$  there is a relative version of the cup product

$$H_{hG}^*(X, Y) \otimes H_{hG}^*(X) \rightarrow H_{hG}^*(X, Y)$$

which essentially is the standard, relative cup product

$$H^*(X_{hG}, Y_{hG}) \otimes H^*(X_{hG}) \rightarrow H^*(X_{hG}, Y_{hG}).$$

This makes  $H_{hG}^*(X, Y)$  into a graded module over the graded ring  $H_{hG}^*(X)$ . In particular  $H_{hG}^*(X, Y)$  is a  $H^*(BG)$ -module.

n:BorelQuillen

**THEOREM 7.3.2** (Localization theorem, Borel, Quillen,...). *Let  $V = \mathbb{Z}/p$  and consider the cohomology ring  $H^*(V; \mathbb{F}_p)$ ; let  $S$  denote the set of non-trivial elements in  $H^1(V; \mathbb{F}_p)$  if  $p = 2$ , and the set of non-nilpotent elements in  $H^2(V; \mathbb{F}_p)$  if  $p$  is odd.*

*For a finite  $V$ -CW complex  $X$  and a  $V$ -subcomplex  $Y$  we have an isomorphism of localised modules, induced by the inclusion of pairs  $(X^V, Y^V) \hookrightarrow (X, Y)$ :*

$$S^{-1}H_{hV}^*(X, Y; \mathbb{F}_p) \xrightarrow{\cong} S^{-1}H_{hV}^*(X^V, Y^V; \mathbb{F}_p) = S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(X^V, Y^V; \mathbb{F}_p)$$

*There is also a more general version involving two elementary abelian  $p$ -groups  $W \leq V$ , which we leave it to the reader to state (and prove).<sup>2</sup>*

**SKETCH OF PROOF.** The second equality is just the Künneth isomorphism together with localisation by  $S$ : the action of  $V$  on the pair  $(X^V, Y^V)$  is trivial, hence the pair  $((X^V)_{hV}, (Y^V)_{hV})$  can be identified with the pair  $(X^V \times BV, Y^V \times BV)$ .

For the first isomorphism, by the long exact sequence in Borel cohomology, exactness of  $S$ -localisation and induction on cells, it is enough to prove the statement for relative cells, i.e. when  $(X, Y)$  is of the form  $(V/V \times D^n, V/V \times S^{n-1})$  or  $(V/e \times D^n, V/e \times S^{n-1})$ .

For  $V/V$  there is nothing to prove, as  $(X^V, Y^V) = (X, Y)$ . For  $V/e$  instead we have  $(X^V, Y^V) = (\emptyset, \emptyset)$ , so that  $S^{-1}H_{hV}^*(X^V, Y^V; \mathbb{F}_p)$  vanishes. However in this case  $H_{hV}^*(X, Y; \mathbb{F}_p)$  can be identified with the ordinary, relative cohomology  $H^*(D^n, S^{n-1}; \mathbb{F}_p)$ , which is concentrated in the single degree  $n$ . Since  $S$  is non-empty and contains positive-degree elements, multiplication by an element in  $S$  is the zero map, and localising at  $S$  gives  $S^{-1}H_{hV}^*(X, Y; \mathbb{F}_p) = 0$  as desired.  $\square$

In the previous proof it is crucial that  $X$  is a finite  $V$ -CW complex, or at least it is finite-dimensional. Otherwise taking  $X = EV$  and  $Y = \emptyset$  gives an easy counterexample to the statement (at least if we know that  $S^{-1}H^*(BV; \mathbb{F}_p)$  is non-trivial).

**COROLLARY 7.3.3** (P.A.Smith). *Let  $X$  be a finite  $V$ -CW complex. If  $X$  is  $\mathbb{F}_p$ -acyclic, then so is  $X^V$ . If  $X$  is an  $\mathbb{F}_p$ -homology sphere, then so is  $X^V$ .*

**PROOF.** The map  $X_{hG} \rightarrow BG$  can be regarded as a map of fibre bundles over  $BG$ , with map between fibres being  $X \rightarrow *$ . If  $X$  is  $\mathbb{F}_p$ -acyclic, the comparison map between second pages of Serre spectral sequences in  $\mathbb{F}_p$ -cohomology is an isomorphism, hence also the  $\infty$ -pages are isomorphic and therefore we have  $H_{hV}^*(X; \mathbb{F}_p) \cong H_{hV}^*(*; \mathbb{F}_p)$ . This implies by Theorem 7.3.2 that  $S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(X^V; \mathbb{F}_p)$  is isomorphic (along the natural map) to  $S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(*; \mathbb{F}_p)$ , and using that tensoring over  $\mathbb{F}_p$  with a non-zero graded  $\mathbb{F}_p$ -vector space such as  $S^{-1}H^*(V; \mathbb{F}_p)$  detects isomorphisms, we obtain the first statement.

For the second statement we use similarly that  $X$  is a  $\mathbb{F}_p$ -homology sphere if and only if the pair  $(C, X)$  has relative  $\mathbb{F}_p$ -cohomology concentrated in a single degree, where  $C = \text{Cone}(X)$ ; we then run a similar argument with spectral sequences to compute  $H^*(C_{hV}, X_{hV}; \mathbb{F}_p)$  and identify it (as a graded  $\mathbb{F}_p$ -vector space) with  $H^*(C, X; \mathbb{F}_p) \otimes H^*(V; \mathbb{F}_p)$ : here it is crucial that the action of  $V$  on  $H^*(C, X)$  is trivial (this is because a  $p$ -group cannot act non-trivially on  $\mathbb{F}_p$ , as  $\text{Aut}(\mathbb{F}_p) = \mathbb{F}_p^* \cong C_{p-1}$ ),

<sup>2</sup>An “elementary abelian  $p$ -group is a (finite-dimensional)” vector space over  $\mathbb{F}_p$ .

and that there is a single non-trivial row in the  $E_2$ -page so the spectral sequence collapses. One concludes again using Theorem 7.3.2, using that  $H^*(V; \mathbb{F}_p)$  is finite-dimensional in each degree, and noticing that  $(C^V, X^V) \cong (\text{Cone}(X^V), X^V)$ .  $\square$

UNTIL HERE

**7.3.3. Smith theory, Dwyer-Wilkerson and Lannes style.** The formula for the homology of the fixed-points, due to Dwyer-Wilkerson, where we'll explain the notation after the theorem.

dwttheorem

**THEOREM 7.3.4** (Dwyer-Wilkerson [DW88; DW91; LZ95]). *Let  $X$  be a finite  $V$ -CW complex. Then*

$$H_{hV}^*(X^V; \mathbb{F}_p) \xrightarrow{\cong} \text{Un}(S^{-1}H_{hV}^*(X^V; \mathbb{F}_p)) \xleftarrow{\cong} \text{Un}(S^{-1}(H_{hV}^*(X; \mathbb{F}_p)))$$

and in particular

$$H^*(X^V; \mathbb{F}_p) \cong \mathbb{F}_p \otimes_{H^*(V; \mathbb{F}_p)} \text{Un}(S^{-1}H_{hV}^*(X; \mathbb{F}_p)).$$

Here  $\text{Un}$  is the largest unstable module, i.e.,

**DEFINITION 7.3.5.** For a module  $M$  over the Steenrod algebra  $\mathcal{A}_2$  define

$$\text{Un}(M) = \{x \in M \mid Sq^I(x) = 0 \text{ if } \text{excess}(I) > |x|\}$$

i.e., the largest submodule which satisfies the instability condition. Similarly for odd primes.

We also need to say how we get a Steenrod action after inverting  $S$ . This is described as follows.

**OBSERVATION 7.3.6.** Let  $R$  be an  $\mathbb{F}_p$ -algebra with an action of the Steenrod algebra  $\mathcal{A}_p$ , let  $M$  an unstable  $R$ -module over the Steenrod algebra (i.e.,  $R \otimes M \rightarrow M$  satisfies the Cartan formula), and let  $S$  a multiplicative subset of  $R$ ; then  $S^{-1}M$  has a Steenrod algebra action given by the following formula

$$P_\xi(x/s) = P_\xi(x)(P_\xi(s))^{-1} = P_\xi(x)(s + Sq^1(s)\xi + Sq^2(s)\xi^2 + \cdots)^{-1}$$

**SKETCH OF PROOF OF THEOREM 7.3.4.** By the localization theorem we just need to see that

$$\text{Un}(S^{-1}H^*(V) \otimes H^*(X^V)) = H^*(V) \otimes H^*(X^V)$$

By a small calculation this in fact holds for any unstable module  $M$

$$H^*(V) \otimes M \xrightarrow{\cong} \text{Un}(S^{-1}H^*(V) \otimes M)$$

This is a small calculation. The key reason is that, by the formula above, elements in the denominator have quite long (infinite?) Steenrod squares on them, and presence of these in hence not compatible with instability.... The short paper by Dwyer-Wilkerson well is worth reading!  $\square$

REMARK 7.3.7. Note how Theorem 7.3.4 should seem surprising, in that, a priori, there would be no reason to believe that the cohomology  $H^*(X^V; \mathbb{F}_p)$  should be determinable from the  $hV$ -homotopy type of  $X$ . (Historically, the first display of this was through the Sullivan conjecture; see below.)

EXERCISE 7.3.8. Use the Dwyer-Wilkerson theorem to prove the following:  $X$  finite  $\mathbb{Z}/2$ -CW complex with  $H^*(X; \mathbb{F}_2) \cong H^*(\mathbf{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/x^{n+1}$  then  $H^*(X^G; \mathbb{F}_2) = H^*(\mathbf{RP}^{i-1}) \oplus H^*(\mathbf{RP}^{j-1})$ ,  $i + j = n + 1$ . (These are obviously realized by flipping  $i$  of the axes.)

#### 7.4. The Sullivan conjecture and homotopy Smith theory

**lannes**

THEOREM 7.4.1 (Lannes, see [Lan92, §4.9]). *There exists a functor  $Fix$  (which is in fact the functor  $\mathrm{Un}(S^{-1}(-))$ , derived from the so-called Lannes  $T$ -functor, such that for any space  $X$  (subject to some mild technical conditions)*

$$H^*(X^{hV}; \mathbb{F}_p) = \mathbb{F}_p \otimes_{H^*(V)} Fix(H_{hV}^*(X))$$

**sullivan**

COROLLARY 7.4.2 (Sullivan Conjecture, Miller, Carlsson, Lannes). *Let  $X$  be a finite  $P$ -CW complex,  $P$  a finite group, then*

$$(X^P)_p \xrightarrow{\cong} (X_p)^{hP}$$

*In particular, under mild assumptions e.g., fixed-points simply connected,*

$$H^*(X^P; \mathbb{F}_p) \cong H^*(X^{hP}; \mathbb{F}_p)$$

GUIDE TO VARIOUS PROOFS OF COR 7.4.2: Before you go, yeah, well, sure, note that in the case the action is trivial, the statement reads that  $\mathrm{map}(BP, X) \xrightarrow{\cong} X$ , in particular  $\mathrm{map}_*(\mathbf{RP}^\infty, X) \cong *$ , when  $X$  is a finite complex, which are not at all obvious—Sullivan considered it a test case for the conjecture. There are several proofs of the general statement, none of them easy!

There's a (technical) proof by Dwyer-Miller-Neisendorfer [DMN89], building on the original work by Miller in the trivial action case [Mil84]. There is a technical proof by Carlsson [Car91], deriving in from the Segal conjecture which he proved.

Finally there is Lannes proof which deduces it from Theorem 7.4.1. It is more "robust" than the others in that it builds up a lot of machinery of independent interest along the way, and then deduces the statement as a special case.  $\square$

SKETCH OF PROOF OF THEOREM 7.4.1. Lannes proof (as simplified by Farjoun-Smith [DFS90]) proceeds in a number of steps, see [Lan92]: The steps in the proof:

- Observe that the functor  $H^*(V; \mathbb{F}_p) \otimes -$  has a left adjoint  $T_V$  on the category of unstable modules and unstable algebras. i.e.,

$$\mathrm{Hom}_{\mathcal{K}}(T_V A, B) \cong \mathrm{Hom}_{\mathcal{K}}(A, H^*(V; \mathbb{F}_p) \otimes B)$$

- Prove that  $T_V$  is exact and commutes with tensor products. [Note that exactness of  $T_V$  in degree 0 means that  $H^*(V) \otimes -$  is exact, i.e. that  $H^*(V)$  is injective in the category of unstable modules over the Steenrod algebra.

- Prove under mild conditions on  $X$

$$T_V H^*(X; \mathbb{F}_p) \xrightarrow{\cong} \text{map}(BV, X)$$

by first verifying this for Eilenberg-MacLane spaces and do an induction on the Postnikov tower.

- Observe that  $BV \times X^{hV} = BV \times \text{map}(BV, X_{hV})_1 = \text{map}(BV, X_{hV})_{(1)}$  Furthermore  $\text{map}(BV, X_{hV})_{(1)}$  is a component of  $\text{map}(BV, X_{hV})$

$$H^*(\text{map}(BV, X_{hV})) = T_V(H_{hV}^*).$$

This can be given a componentwise version

$$H^*(\text{map}(BV, X_{hV})_{(1)}) = \mathbb{F}_p \otimes_{T_V^0(H_{hV}^*(X))} T_V(H_{hV}^*(X))$$

and we set  $Fix = \mathbb{F}_p \otimes_{T_V^0(-)} T_V(-)$ .

□

REMARK 7.4.3. Note how the Sullivan conjecture relates to the Dwyer-Wilkerson formula: The formula of Dwyer-Wilkerson tells us that that the homology of  $X^V$  can be extracted from the  $hV$ -homotopy type of  $X$ . The Sullivan conjecture then says that the homotopy type of  $X^V$  can be recovered from the  $hV$ -homotopy of  $X$ , up to  $p$ -completion, and even gives a computable formula for it as  $X^{hV}$ .

### 7.5. Loop structures on $S^3$

We will apply what we have learnt so far to prove the following, very concrete statement:

thm:loopS3

THEOREM 7.5.1. *There exist uncountably many connected spaces  $Y$  such that  $\Omega Y$  is homotopy equivalent to  $S^3$  (the 3-sphere), that are pairwise not homotopy equivalent to each other.*

A good start towards Theorem 7.5.1 is to show that there exists a space whose loop space is  $\simeq S^3$ .

EXAMPLE 7.5.2. For  $n \geq 0$ , let  $\mathbb{H}P^n$ , the  $n^{\text{th}}$  quaternionic projective space, be the space of  $\mathbb{H}$ -lines in  $\mathbb{H}^{n+1}$ , where an  $\mathbb{H}$ -line is an  $\mathbb{H}$ -linear subspace of  $\mathbb{H}^{n+1}$  of  $\mathbb{H}$ -dimension 1. The space  $\mathbb{H}P^n$  can also be described as the quotient  $(\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$ , or also as  $S^{4n+3}/\mathbb{S}^3$ , if we consider the group  $\mathbb{S}^3 \cong SU(2) \cong \text{Spin}(3)$  of norm-1 quaternions under multiplication, and we consider the unit sphere  $S^{4n+3} \subset \mathbb{H}^{n+1}$ .<sup>3</sup> Here the blackboard notation “ $\mathbb{S}^3$ ” is to emphasize that this is a Lie group, and not just a space. We have a fibre bundle whose base, fibre and total space are orientable manifolds  $S^3 \hookrightarrow S^{4n+3} \rightarrow \mathbb{H}P^n$ .

The inclusion  $\mathbb{H}^{n+1} \subset \mathbb{H}^{n+2}$  induces an inclusion  $\mathbb{H}P^n \subset \mathbb{H}P^{n+1}$ , with complement a  $(4n+4)$ -dimensional cell. It follows that  $\mathbb{H}P^n$  has a cell decomposition with one cell in each even dimension between 0 and  $4n$ , and in particular its cohomology  $H^*(\mathbb{H}P^n)$  is isomorphic as graded abelian group to the quotient ring  $\mathbb{Z}[x]/x^{n+1}$ , where

<sup>3</sup>You can also say: we first quotient  $\mathbb{H}^{n+1}$  by the action of the (central) subgroup  $\mathbb{R}_+ \subset \mathbb{H}^*$ , obtaining  $S^{4n+3}$ , and then quotient the latter by the residual action of  $\mathbb{S}^3 \cong \mathbb{H}^*/\mathbb{R}_+$ .



$x$  is a variable in degree 4. Poincaré duality implies moreover that this must be an isomorphism of rings (exercise for you!).

We then consider  $\mathbb{H}P^\infty = \bigcup_{n \geq 0} \mathbb{H}P^n$ , which is a CW complex having one cell in each dimension multiple of 4. The above considerations imply that  $\mathbb{H}P^\infty$  has cohomology ring isomorphic to  $\mathbb{Z}[x]$ , the polynomial ring in one variable of degree 4; moreover we have a fibre bundle  $S^3 \hookrightarrow S^\infty \rightarrow \mathbb{H}P^\infty$ , and since the infinite-dimensional sphere  $S^\infty$  is contractible we obtain that  $S^3$  is weakly equivalent to  $\Omega\mathbb{H}P^\infty$ . So  $\mathbb{H}P^\infty$  is one of the spaces  $X$  in Theorem 7.5.1.

Compare this example with the probably more familiar one of  $\mathbb{C}P^\infty$ , a space whose loop space is homotopy equivalent to the Lie group  $S^1$ .

In fact we can also think of  $S^\infty$  as an example of a contractible space  $ES^3$  with a free action of the compact Lie group  $S^3$ ; we can then have an equivalence  $\mathbb{H}P^\infty \simeq BS^3$ , meaning that  $\mathbb{H}P^\infty$  is a classifying space for principal  $S^3$ -bundles. And for any Lie group  $G$  we have indeed  $\Omega BG \simeq G$ .

We observe that if  $Y$  is a (connected) space with  $\Omega Y \simeq S^3$ , then  $Y$  must be simply connected (in fact,  $Y$  must be 3-connected); moreover we have an arithmetic square

$$\begin{array}{ccc} Y & \longrightarrow & \prod_p Y_p^\wedge \\ \downarrow & & \downarrow \\ Y_\mathbb{Q} & \longrightarrow & \left(\prod_p Y_p^\wedge\right)_\mathbb{Q}, \end{array}$$

which is a homotopy pullback whose vertical maps are rationalisations.

**exercise:YQ**

EXERCISE 7.5.3. Check that if  $\Omega Y \simeq S^3$ , then  $Y_\mathbb{Q} \simeq K(\mathbb{Q}, 4)$ : you can for instance prove that  $S_\mathbb{Q}^3 \simeq \Omega(Y_\mathbb{Q})$ , invoking Lemma 6.3.1 and Theorem 6.3.3, and then prove aside that  $S_\mathbb{Q}^3 \simeq K(\mathbb{Q}, 3)$ .

LEMMA 7.5.4. *Let  $Y$  be a connected space with  $\Omega Y \simeq S^3$ , and let  $p$  be a prime. Then  $\Omega(Y_p^\wedge) \simeq (S^3)_p^\wedge$ .*

PROOF. By Theorem 6.8.1 we have that  $Y_p^\wedge$  is also 3-connected as  $Y$ , and its homotopy groups are derived  $p$ -complete. It follows that  $\Omega(Y_p^\wedge)$  also has derived  $p$ -complete homotopy groups and is also simply connected, hence it is  $p$ -complete. Finally, by Proposition 6.5.4 we have that the map  $\Omega Y \rightarrow \Omega(Y_p^\wedge)$  is a  $\mathbb{F}_p$ -equivalence.  $\square$

The homotopy groups of  $(S^3)_p^\wedge$  can be “computed” (i.e., understood to a sufficient level for our purposes) thanks to Theorem 6.5.1, together with Serre’s classical result that  $\pi_*(S^3)$  are finitely generated: we have  $\pi_3((S^3)_p^\wedge) \cong \mathbb{Z}_p$ , whereas  $\pi_{* \geq 4}((S^3)_p^\wedge)$  is the  $p$ -power torsion part of  $\pi_{* \geq 4}(S^3)$ . Therefore, if  $Y$  is a space with  $\Omega Y \simeq S^3$ , we

have isomorphisms

$$\begin{aligned} \pi_4 \left( \prod_p Y_p^\wedge \right) &\cong \prod_p \pi_4(Y_p^\wedge) \cong \prod_p \pi_3((S^3)_p^\wedge) \cong \prod_p \mathbb{Z}_p; \\ \pi_i \left( \prod_p Y_p^\wedge \right) &\cong \pi_{i-1}(S^3) \quad \text{for } i \neq 4 \quad (\text{a finite group}). \end{aligned}$$

We can now rationalise  $\prod_p Y_p^\wedge$  to get a space  $\left(\prod_p Y_p^\wedge\right)_\mathbb{Q}$  whose only non-zero homotopy group is  $\pi_4$ , equal to  $(\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$ .

DEFINITION 7.5.5. We denote by  $\mathbb{A}_f$  the abelian group  $(\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$ , called the group of “finite adèles”. It has a ring structure, coming from tensor product of  $\mathbb{Q}$  and the product ring  $\prod_p \mathbb{Z}_p$ . Note also that  $\mathbb{A}_f$  is an infinite-dimensional vector space over  $\mathbb{Q}$ .

The previous discussion shows that whenever  $Y$  is connected and  $\Omega Y \simeq S^3$ , the space occurring in the arithmetic square for  $Y$  must be (homotopy equivalent to)  $K(\mathbb{A}_f, 4)$ .

At this point we restrict our attention to spaces  $Y$  such that  $\Omega Y \simeq S^3$ , and such that, moreover, there are equivalences  $Y_\mathbb{Q} \simeq \mathbb{H}P_\mathbb{Q}^\infty$  and, for each prime  $p$ ,  $Y_p^\wedge \simeq (\mathbb{H}P^\infty)_p^\wedge$ . Exercise 7.5.3 shows that assuming  $\Omega Y \simeq S^3$  we immediately have  $Y_\mathbb{Q} \simeq \mathbb{H}P_\mathbb{Q}^\infty \simeq K(\mathbb{Q}, 4)$ ; it is instead a more difficult fact (that we will not prove) that also the equivalence  $Y_p^\wedge \simeq (\mathbb{H}P^\infty)_p^\wedge$  follows just from the assumption  $\Omega Y \simeq S^3$ .

To simplify notation, we denote in the following  $X_p = (\mathbb{H}P^\infty)_p^\wedge$ . The cohomology ring  $H^*(X_p; \mathbb{F}_p)$  is isomorphic to  $\mathbb{F}_p[x]$ , with  $x$  in degree 4.

n: squaredegree

LEMMA 7.5.6. *Let  $p$  be an odd prime and let  $f: X_p \rightarrow X_p$  be a self-map. Then the map  $f^*: H^4(X_p; \mathbb{F}_p) \rightarrow H^4(X_p; \mathbb{F}_p)$  is the self-map of  $\mathbb{F}_p$  induced by multiplication by an element  $d \in \mathbb{F}_p$  which is a square, i.e.  $d = a^2$  for some  $a \in \mathbb{F}_p$ .*

PROOF. The argument follows closely another argument to be found in [Hat02, Example 4L.4]. There is a collapse map  $\phi: \mathbb{C}P^\infty \rightarrow \mathbb{H}P^\infty$  inducing an isomorphism on  $H^4(-; \mathbb{F}_p)$  (in fact, also on  $H^4(-; \mathbb{Z})$ ); after  $p$ -completion we obtain a map  $\phi_p^\wedge: (\mathbb{C}P^\infty)_p^\wedge \rightarrow X_p$  inducing again an isomorphism on  $H^4(-; \mathbb{F}_p)$ . If  $\gamma$  is a generator of  $H^4(X_p, \mathbb{F}_p)$ , then also  $(\phi_p^\wedge)^*\gamma$  is a generator of  $H^4((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p)$ . The  $p$ -completion map  $\iota_\mathbb{C}: \mathbb{C}P^\infty \rightarrow (\mathbb{C}P^\infty)_p^\wedge$  induces an isomorphism  $\iota_\mathbb{C}^*: H^*((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p) \rightarrow H^*(\mathbb{C}P^\infty; \mathbb{F}_p)$  which is compatible with the action of the mod  $p$  Steenrod algebra  $\mathcal{A}_p$ ; in particular  $P^1(\phi^*\gamma) = 2(\phi^*\gamma)^{(p+1)/2}$  since this is true (read Hatcher) after pulling back along  $\iota_\mathbb{C}^*$ .

Now the naturality of the Steenrod operations implies that  $\phi^*(P^1(\gamma)) = P^1(\phi^*\gamma)$ ; on the other hand  $\phi^*: H^{2p+2}(X_p; \mathbb{F}_p) \xrightarrow{\cong} H^{2p+2}((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p)$  is an isomorphism, implying  $P^1(\gamma) = 2\gamma^{(p+1)/2}$ .

And now, again by naturality, we have a chain of equalities

$$2d\gamma^{(p+1)/2} = dP^1(\gamma) = P^1(d\gamma) = P^1(f^*\gamma) = f^*(P^1(\gamma)) = f^*(2\gamma^{(p+1)/2}) = 2d^{(p+1)/2}\gamma^{(p+1)/2},$$

implying that  $d$  is a square mod  $p$ . □

In particular, for  $p$  odd, if  $f: X_p \rightarrow X_p$  is a self-homotopy equivalence, it induces on  $\pi_4(X_p) \cong \mathbb{Z}_p$  an automorphism of the abelian group  $\mathbb{Z}_p$ ; such endomorphism is given by multiplication by an *invertible* element  $\lambda \in \mathbb{Z}_p^\times$ ; Lemma 7.5.6 ensures that  $\lambda$  reduces to a square in  $\mathbb{F}_p$ , and this suffices to ensure that  $\lambda$  is the square of an element in  $\mathbb{Z}_p^\times$ , using that  $p$  is odd and Hensel's lemma. In fact the same holds also for the prime 2: a self-homotopy equivalence  $f: X_2 \rightarrow X_2$  induces on  $\pi_4(X_2) \cong \mathbb{Z}_2$  multiplication by a perfect square  $\lambda \in (\mathbb{Z}_2^\times)^\times$ , but we will not prove this.

If now  $Y$  is a space such that  $Y_p^\wedge \simeq X_p$  and  $Y_\mathbb{Q} \simeq X_\mathbb{Q} := K(\mathbb{Q}, 4)$ , then we have an arithmetic square, which is a homotopy pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \prod_p X_p \\ \downarrow & & \downarrow \phi \\ X_\mathbb{Q} & \xrightarrow{\psi} & \left( \prod_p X_p \right)_\mathbb{Q}, \end{array}$$

where  $\phi$  is a  $\mathbb{Q}$ -equivalence and  $\psi$  is some map; viceversa, for any  $\mathbb{Q}$ -equivalence  $\phi$  and any map  $\psi$  we can define  $Y$  as the above homotopy pullback, and then  $Y$  is an example of a space with  $\Omega Y$  fitting in a pullback square with vertical maps being  $\mathbb{Q}$ -equivalences

$$\begin{array}{ccc} \Omega Y & \longrightarrow & \prod_p (S^3)_p^\wedge \\ \downarrow & & \downarrow \Omega \phi \\ S_\mathbb{Q}^3 & \xrightarrow{\Omega \psi} & \left( \prod_p (S^3)_p^\wedge \right)_\mathbb{Q}, \end{array}$$

and this implies that  $\Omega Y$  is simply connected and has the same homology (rationally and mod  $p$ , hence integrally) as  $S^3$ : it follows from Hurewicz that  $\Omega Y \simeq S^3$ .

Now we should avoid overcounting: how many essentially distinct choices of  $\phi$  and  $\psi$  as above do we have? Clearly, if we fix homotopy automorphisms  $\alpha$ ,  $\beta$  and  $\gamma$  of  $\prod_p X_p$ ,  $X_\mathbb{Q}$  and  $(\prod_p X_p)_\mathbb{Q}$  respectively, we can replace  $\phi$  by  $\phi' = \gamma \circ \phi \circ \alpha^{-1}$  and  $\psi$  by  $\psi' = \gamma \circ \psi \circ \beta^{-1}$ , and the new homotopy pullback  $Y'$  (obtained using  $\phi'$  and  $\psi'$ ) is weakly equivalent to  $Y$ . Conversely, if  $Y$  and  $Y'$  are two spaces with rationalisation  $X_\mathbb{Q}$  and  $p$ -completion  $X_p$ , and if  $f: Y \rightarrow Y'$  is a weak equivalence, then the composites  $X_\mathbb{Q} \simeq Y_\mathbb{Q} \xrightarrow{f_\mathbb{Q}} Y'_\mathbb{Q} \simeq X_\mathbb{Q}$ ,  $\prod_p X_p \simeq \prod_p Y_p^\wedge \xrightarrow{\prod_p f_p^\wedge} \prod_p Y'_p^\wedge \simeq \prod_p X_p$  and  $(\prod_p X_p)_\mathbb{Q} \simeq (\prod_p Y_p^\wedge)_\mathbb{Q} \xrightarrow{(\prod_p f_p^\wedge)_\mathbb{Q}} (\prod_p Y'_p^\wedge)_\mathbb{Q} \simeq (\prod_p X_p)_\mathbb{Q}$  are homotopy automorphisms of  $X_\mathbb{Q}$ ,  $\prod_p X_p$  and  $(\prod_p X_p)_\mathbb{Q}$  respectively, and the maps  $\psi'$  and  $\phi'$  for  $Y'$  are up to homotopy obtained from the maps  $\psi$  and  $\phi$  for  $Y$  as described above.

We observe moreover that if  $V$  is a  $\mathbb{Q}$ -vector space, then the group  $\pi_0(\text{hAut}(K(V, 4)))$  of homotopy automorphisms of  $K(V, 4)$  up to homotopy is isomorphic to the group  $GL_\mathbb{Q}(V)$  of  $\mathbb{Q}$ -linear automorphisms of  $V$ .

All in all, if we set  $\mathcal{G} = \pi_0(\text{hAut}(\prod_p X_p))$ , there is an action of  $GL_\mathbb{Q}(\mathbb{A}_f) \times GL_\mathbb{Q}(\mathbb{Q}) \times \mathcal{G}$  on the set  $\mathcal{S}$  of choices of  $(\phi, \psi)$  as above; moreover the action of the subgroup  $GL_\mathbb{Q}(\mathbb{A}_f)$  on the set of choices of  $\phi$  is simply transitive, since  $\phi$  is itself required

to be a rational equivalence, and hence can be defined uniquely by passing to the rationalisation of  $\prod_p X_p$  and then taking an automorphism of  $(\prod_p X_p)_{\mathbb{Q}}$ . We can thus fix our favourite rational equivalence  $\bar{\phi}: \prod_p X_p \rightarrow (\prod_p X_p)_{\mathbb{Q}}$  and i) restrict ourselves to the set  $\bar{\mathcal{S}}$  of choices of  $(\phi, \psi)$  with  $\phi = \bar{\psi}$  and ii) restrict ourselves to the stabiliser of the set  $\bar{\mathcal{S}}$  along the previous action: this is the subgroup of  $GL_{\mathbb{Q}}(\mathbb{A}_f) \times GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$  of triples  $(\alpha, \beta, \gamma)$  such that  $\bar{\phi}$  is homotopic to  $\gamma \circ \phi \circ \alpha^{-1}$ , and this is equivalent to requiring that  $\gamma$  is the rationalisation of  $\alpha$ .

We can thus consider the residual action of  $GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$  on  $\bar{\mathcal{S}}$ , where  $(\alpha, \beta) \in \mathcal{G}$  sends  $(\bar{\phi}, \psi) \mapsto (\bar{\phi}, \alpha_{\mathbb{Q}} \circ \psi \circ \beta^{-1})$ . We can now identify the homotopy classes of maps  $\psi: K(\mathbb{Q}, 4) \rightarrow K(\mathbb{A}_f, 4)$  with the set  $\text{Hom}(\mathbb{Q}, \mathbb{A}_f)$  of additive maps  $\mathbb{Q} \rightarrow \mathbb{A}_f$ . This set can be identified with  $\mathbb{A}_f$  itself, in such a way that the action of  $GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$  acquires the following description:

- $GL_{\mathbb{Q}}(\mathbb{Q})$  can be identified with  $\mathbb{Q}^{\times}$ , and acts on  $\mathbb{A}_f$  by scalar multiplication (using the ring structure of  $\mathbb{A}_f$ );
- if an element  $\alpha \in \mathcal{G}$  acts on each  $\pi_4(X_p)$  by multiplication by an element  $\lambda_p \in (\mathbb{Z}_p^{\times})^2$ , then it acts on  $\mathbb{A}_f$  by scalar multiplication by  $\prod_p \lambda_p$ .

We thus have a bijection of sets

$$\{\text{spaces } Y \text{ with } \Omega Y \simeq S^3, Y_p \hat{\simeq} X_p \text{ and } Y_{\mathbb{Q}} \simeq X_{\mathbb{Q}}\} / \text{w.e.} \leftrightarrow \mathbb{A}_f / \left( \mathbb{Q}^{\times} \times \prod_p (\mathbb{Z}_p^{\times})^2 \right).$$

**THEOREM 7.5.7** (Dwyer-Miller-Wilkerson '84). *There is in fact a bijection*

$$\{\text{spaces } Y \text{ with } \Omega Y \simeq S^3\} / \text{w.e.} \leftrightarrow \mathbb{A}_f / \left( \mathbb{Q}^{\times} \times \prod_p (\mathbb{Z}_p^{\times})^2 \right).$$

We now notice that there is a surjection

$$\mathbb{A}_f \twoheadrightarrow \left( \prod_p \mathbb{F}_p \right) \otimes \mathbb{Q},$$

inducing a surjection

$$\mathbb{A}_f / \left( \mathbb{Q}^{\times} \times \prod_p (\mathbb{Z}_p^{\times})^2 \right) \twoheadrightarrow \left( \left( \prod_p \mathbb{F}_p \right) \otimes \mathbb{Q} \right) / \left( \mathbb{Q}^{\times} \times \prod_p (\mathbb{F}_p^{\times})^2 \right),$$

and we leave to the reader the pleasure of proving that the last set is more than countable!

## 7.6. $p$ -compact groups

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