LECTURE NOTES FOR HOMOLOGICAL ALGEBRA, NOVEMBER 2021 - JANUARY 2022

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ABSTRACT. These are the lecture notes for the course "Homological Algebra" held at the University of Copenhagen between November 2021 and January 2022 (blok 2). We refer by [Rot] to the book "J.J.Rotman, *An introduction to Homological Algebra*, Second edition, *Springer*, 2009". If you spot any mistakes, I would be glad to correct them; so please let me know!

CONTENTS

Date: March 22, 2023.

1. Modules over commutative rings

In this and the next lecture we introduce the notion of R-modules, where R is an associative ring with unit $1_R \in R$ (often denoted just $1 = 1_R \in R$). In this lecture we focus on commutative rings, for which module theory is slightly easier, as one does not have to take too much care of the distinction between left and right modules. Most of the content of this lecture will be generalised to the case of a non-commutative ring in the next lecture. For this lecture and the next one, refer to [Rot, 2.1-2.2].

1.1. **Definition of modules and first examples.**

Definition 1.1. Let R be a *commutative*, unital ring. An R-module is an abelian group M endowed with an operation, called "scalar multiplication" and denoted

$$R \times M \to M, \qquad (r, m) \mapsto r \cdot m,$$

satisfying the following properties, for all $r, r' \in R$ and all $m, m' \in M$:

(1) $r \cdot (m + m') = r \cdot m + r \cdot m'$;
(2) $(r + r') \cdot m = r \cdot m + r' \cdot m$;
(3) $r \cdot (r' \cdot m) = (r \cdot r') \cdot m$;
(4) $1_R \cdot m = m$.

Example 1.2. The trivial group 0 can be upgraded to an R-module by setting $r \cdot 0 = 0$. This is the most boring example.

Example 1.3. If $R = \mathbb{F}$ is a field, then an R-module is precisely a $\mathbb{F}$-vector space. Hence the notion of module generalises the notion of vector space.

Example 1.4. By definition, every R-module is in particular an abelian group. If $R = \mathbb{Z}$ are the integers, then the viceversa holds: every abelian group M can be given a *unique* structure of $\mathbb{Z}$-module, by declaring:

- $0 \cdot m = 0 \in M$ for all $m \in M$: check that this is forced, using that $0 \cdot m = (0 + 0) \cdot m = 0 \cdot m + 0 \cdot m$;
- $r \cdot m = m + \cdots + m$, where the sum contains $r \geq 1$ equal summands, for all $r \geq 1$: check by induction that this assignment is also forced;

- $r \cdot m = -((-r) \cdot m)$, for $r < 0$, where $-r \in \mathbb{Z}$ is positive and $-((-r) \cdot m)$ is the opposite of $((-r) \cdot m)$ in the abelian group M: check again that this assignment is forced.

Example 1.5. The ring R is an R-module, by using the ring multiplication $R \times R \to R$ as scalar multiplication. In the commutative case, it does not matter which copy of R in $R \times R$ plays the role of ring and which of module.
More generally, if I is any set, consider the set R^I of families $(r_i)_{i \in I}$ of elements of R indexed by I; then R^I is an R-module by *pointwise/coordinatewise operations*. For example, we set $r \cdot (r_i)_{i \in I} = (r \cdot r_i)_{i \in I}$, and $(r_i)_{i \in I} + (r'_i)_{i \in I} = (r_i + r'_i)_{i \in I}$.

For abelian groups, we have a notion of subgroup and quotient group. Something similar occurs for R-modules.

Definition 1.6. Let M be an R-module. A subset N of M is a *submodule* (more precisely, a sub-R-module) if it is closed under all operations; in particular N is a sub-abelian group of N, and for all $r \in R$ and $m \in N$, the scalar multiplication $r \cdot m$ yields again an element in N.
Given M and a submodule N, the *quotient module* M/N is the following module: the underlying abelian group of M/N is the quotient abelian group M/N; the scalar multiplication is defined by setting $r \cdot [m]_N = [r \cdot m]_N$ for all $r \in R$ and $m \in M$, where $[m]_N$ denotes the class of m in the quotient abelian group M/N.

By taking submodules and quotient modules we can create many more examples of R-modules.

Example 1.7. Recall Example 1.5. A submodule N of R is what is usually called an *ideal* of the commutative ring R. The quotient R/N is then an R-module as well. This can be seen as a particular case of Example 1.9, recalling that R/N has a natural structure of commutative ring.

Definition 1.8. Let M be an R-module and $\mathcal{S} \subset M$ any subset. We denote by $\mathrm{Span}_R(\mathcal{S}) \subset M$ the *submodule generated by* $\mathcal{S}$, which can be alternatively described by either of the following (check that they are equivalent!):

- $\mathrm{Span}_R(\mathcal{S})$ is the subset of M containing all elements m that can be written as a finite linear combination $\sum_{i=1}^{k} r_i \cdot m_i$, with $r_i \in R$ and $m_i \in \mathcal{S}$;
- $\mathrm{Span}_R(\mathcal{S})$ is the intersection of all submodules N of M containing $\mathcal{S}$.

We say that a subset $\mathcal{S} \subset M$ *generates* M if $\mathrm{Span}_R(\mathcal{S}) = M$ (and is thus not a proper submodule); we also say that $\mathcal{S}$ is a *generating set* for M.

For example $1 \in R$ generates R as an R-module (see Example 1.5); but for example $\left\{x, x^2\right\} \in \mathbb{Q}[x]$ only generates a proper sub-$\mathbb{Q}[x]$-module of $\mathbb{Q}[x]$, the ideal (x): in fact this submodule is also generated by the subset $\{x\}$ of $\left\{x, x^2\right\}$. Another example is the following: if $\mathbb{F}$ is a field and V is a $\mathbb{F}$-vector space, then any basis of V is a generating set for V.

Example 1.9. Let $f\colon R \to S$ be a map of commutative rings. Then S is naturally an R-module by keeping the abelian group structure and by setting $r \cdot s = f(r) \cdot s$, where the second $\cdot$ is the product of the ring S.

Example 1.10. In fact much more than what discussed in Example 1.9: if $f\colon R \to S$ is a map of rings and M is an S-module, then M becomes automatically an R-module by setting $r \cdot m = f(r) \cdot m$. We say that M is an R-module by *restriction of*

scalars, though this terminology might suggest that we want only to consider the case in which f is injective (but there is no need for this constraint).
As a special case, consider, for a commutative ring R, the ring of polynomials $R[x]$ in one variable; then the inclusion $R \hookrightarrow R[x]$ makes every $R[x]$-module into an R-module. Viceversa, every choice of $\lambda \in R$ gives a (unique) map of rings $R[x] \to R$ sending $x \mapsto \lambda$ and such that the composite $R \hookrightarrow R[x] \to R$ is the identity of R. Thus, for every choice of λ, we have a way to transform an R-module into a $R[x]$-module.

Example 1.11. For a fixed commutative ring R and abelian group M, it is not always possible to upgrade M to an R-module. Here are some examples (more refined examples could be given after seeing more material in the course).

- If $R = \mathbb{Q}$ is the field of rational numbers (or any infinite field), then an R-module M must be either 0 or infinite. If M is instead a finite, non-trivial group, then it cannot be given a structure of $\mathbb{Q}$-module.
- In a similar way, if $R = \mathbb{Z}/k$ is the ring of integers modulo $k \geq 2$, then a necessary (and actually sufficient) condition on an abelian group M to admit a structure of R-module is the following: for every $m \in M$, the k-fold sum $m + \cdots + m$ is equal to 0.

For the following example, we need some notation.

Notation 1.12. Let M be an R-module and $r \in R$. We denote by $r \cdot - \colon M \to M$ the map $m \mapsto r \cdot m$, and call it the "scalar multiplication by r".

Example 1.13. For a fixed commutative ring R and abelian group M, it can happen that M can be upgraded to an R-module in several different ways. For example, recall Example 1.10, let $R = \mathbb{Q}$ and consider two different values $\lambda, \lambda' \in \mathbb{Q}$. Then $\mathbb{Q}$ becomes a $\mathbb{Q}[x]$-module in two ways, which we denote by $\mathbb{Q}^{\lambda}$ and $\mathbb{Q}^{\lambda'}$, depending on whether the scalar multiplication by x coincides with multiplication (in $\mathbb{Q}$) by λ or by λ'. These two modules have a quite different behaviour with respect to scalar multiplication by elements in $\mathbb{Q}[x]$:

- consider the element $(x - \lambda) \in \mathbb{Q}[x]$: the associated map $((x - \lambda) \cdot -)$ sends all of $\mathbb{Q}^{\lambda}$ to 0, whereas it is a bijection $\mathbb{Q}^{\lambda'} \to \mathbb{Q}^{\lambda'}$;
- viceversa, the element $(x - \lambda') \in \mathbb{Q}[x]$ acts as a bijection on $\mathbb{Q}^{\lambda}$ and as the constant map to 0 on $\mathbb{Q}^{\lambda'}$.

In example 1.5 we have considered families of elements of the *same* R-module, namely R. We can generalise this idea as follows.

Definition 1.14. Let I be a set and let $(M_i)_{i \in I}$ be a family of R-modules. The product of sets $\prod_{i \in I} M_i$ has a natural structure of R-module by defining operations coordinatewise. For example, we set $r \cdot (m_i)_{i \in I} = (r \cdot m_i)_{i \in I}$. We call this the *product* of R-modules.
We also define the *direct sum* $\bigoplus_{i \in I} M_i$ as the submodule of $\prod_{i \in I} M_i$ of families $(m_i)_{i \in I}$ with the following property: there are at most finitely many indices $i \in I$ for which $m_i \neq 0$ in M_i.

If I is finite, note that in fact $\bigoplus_{i \in I} M_i$ is equal to the whole $\prod_{i \in I} M_i$.

Notation 1.15. In the case of I of cardinality 2, we also write $M \oplus M' = M \times M'$ for the direct sum/product of two modules. Similarly for any finite set $I = \{i_1, \ldots, i_n\}$, we may write $M_1 \oplus \cdots \oplus M_n$ for the direct sum of the n modules $M_1, \ldots, M_n$.

1.2. **Homomorphisms of modules.** We have now a good arsenal of operations to construct many different R-modules. The next important step is to define what *morphisms* of R-modules are, in order to be able to compare different R-modules.

Definition 1.16. Let M and N be R-modules. A map of sets $f\colon M \to N$ is an *R-linear map*, or a *homomorphism of R-modules*, if f is a map of abelian groups and f is compatible with the two scalar multiplications, i.e. $f(r \cdot m) = r \cdot f(m)$ for all $m \in M$ and $r \in R$. The set of all R-linear maps $M \to N$ is denoted $\mathrm{Hom}_R(M, N)$. A map $f\colon M \to N$ is an *isomorphism of R-modules* if it is R-linear and bijective.

Example 1.17. The identity map $\mathrm{Id}_M\colon M \to M$ of an R-module M is R-linear. More generally, since we assume R commutative, for all $r \in R$ the map $r \cdot -$ from Notation 1.12 is R-linear. Let us check in particular the compatibility with scalar multiplication: for all $s \in R$ and $m \in M$ we have

$$(r \cdot -)(s \cdot m) = r \cdot (s \cdot m) = (r \cdot s) \cdot m = (s \cdot r) \cdot m = s \cdot (r \cdot m) = s \cdot \big((r \cdot -)(m)\big).$$

Example 1.18. If $R = \mathbb{F}$ is a field, then an $\mathbb{F}$-linear map is precisely a $\mathbb{F}$-linear map of vector spaces in the usual sense from linear algebra.

Example 1.19. If $R = \mathbb{Z}$, then a $\mathbb{Z}$-linear map of $\mathbb{Z}$-modules is the same as a homomorphism of abelian groups.

Note also that if $f\colon M \to M'$ and $g\colon M' \to M''$ are R-linear maps between R-modules, then the map of sets $g \circ f\colon M \to M''$ also satisfies all properties to be R-linear. This allows us to compose R-linear maps and stay in the realm of R-linear maps.

Example 1.20. For all R-modules M, N there is always the 0 map $0\colon M \to N$, sending all elements of M to $0 \in N$. It can happen that this is the unique R-linear map, even if neither M nor N is the zero module. Here are two examples.

- Let $R = \mathbb{Z}$ and let $M = \mathbb{Z}/2$ and $N = \mathbb{Z}/3$. Then there is no non-trivial homomorphism of abelian groups $\mathbb{Z}/2 \to \mathbb{Z}/3$.
- Let $R = \mathbb{Q}[x]$ and consider the two modules $\mathbb{Q}^\lambda$ and $\mathbb{Q}^{\lambda'}$ from Example 1.13. Then every $\mathbb{Q}[x]$-linear map $f\colon \mathbb{Q}^\lambda \to \mathbb{Q}^{\lambda'}$ is the zero map. To see this, let $m \in \mathbb{Q}^\lambda$; then
$$0 = f(0) = f((x - \lambda) \cdot m) = (x - \lambda) \cdot f(m) = (\lambda' - \lambda) \cdot f(m) \in \mathbb{Q}^{\lambda'}.$$
Now recall that $(\lambda' - \lambda) \in \mathbb{Q} \subset \mathbb{Q}[x]$ acts on $\mathbb{Q}^{\lambda'} \cong \mathbb{Q}$ just by $m' \mapsto (\lambda' - \lambda) \cdot m' \in \mathbb{Q}$. Since $(\lambda' - \lambda) \cdot -$ is an invertible map $\mathbb{Q} \to \mathbb{Q}$ (it is an automorphism of $\mathbb{Q}$-vector spaces), we can "divide by $(\lambda' - \lambda)$" and conclude that $f(m) = 0$. This holds for all $m \in \mathbb{Q}^\lambda$.

The previous example is in net contrast with what happens with vector spaces over a field: given two non-trivial vector spaces, there is always a non-trivial linear map between them.

Example 1.21. Let $(M_i)_{i \in I}$ be a family of R-modules. Then each projection

$$\pi_i\colon \prod_{i \in I} M_i \to M_i,$$

which is a priori a map of sets, is in fact an R-linear map. Moreover, let N be another R-module; then a map of sets $f\colon N \to \prod_{i \in I} M_i$ carries the same information of (can be retrieved from, and determines) a family of maps $f_i\colon N \to M_i$, one for each $i \in I$: just take $f_i = \pi_i \circ f$. Check that f is R-linear if and only if all maps f_i, for $i \in I$, are R-linear. We have in fact a natural bijection of sets

$$\mathrm{Hom}_R(N, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Hom}_R(N, M_i).$$

This makes it easy to construct R-linear maps with $\prod_{i\in I} M_i$ *as target*: one just has to specify the "coordinates" of a map f, i.e. its composition with the projections π_i.

Notation 1.22. For a direct sum $\bigoplus_{i\in I} M_i$ of R-modules, and for an index $j \in I$, we denote by $\iota_j : M_j \to \bigoplus_{i\in I} M_i$ the map sending $m \in M_j$ to the family $(m_i)_{i\in I}$ with $m_j = m$ and $m_i = 0$ for all $i \neq j$. Note that it is an R-linear, injective map. By slight abuse of notation, we often regard M_j as a submodule of $\bigoplus_{i\in I} M_i$, and just write $M_j \subset \bigoplus_{i\in I} M_i$.

Each element $m \in \bigoplus_{i\in I}$ can be written uniquely as a finite sum $\sum_{j\in I} \iota_j(m_j)$, for a suitable family $(m_j)_{j\in I}$ of elements $m_j \in M_j$, with all but finitely many m_j vanishing: check that the only option is to take m_j equal to $\pi_j(m)$, after considering m as an element in $\prod_{i\in I} M_i$. Use this remark to solve the following exercise.

Exercise 1.23. Let $(M_i)_{i\in I}$ be a family of R-modules, and for all $i \in I$ let $\mathcal{S}_i \subset M_i$ be a generating set (see Definition 1.8). Then $\bigcup_{i\in I} \iota_i(\mathcal{S}_i) \subset \bigoplus_{i\in I} M_i$ is a generating set for the direct sum.

Example 1.24. Let N be an R-module, and suppose we are given R-linear maps $f_i : M_i \to N$ for all $i \in I$; we can then construct an R-linear map $f : \bigoplus_{i\in I} M_i \to N$ by setting

$$f(m) = \sum_{j\in I} f_j(m_j),$$

where we use, for each element $m \in \bigoplus_{i\in I} M_i$, the decomposition $m = \sum_{j\in I} \iota_j(m_j)$ described above. Note that the sum for $f(m)$ is a finite sum of elements of N (maybe up to a lot of vanishing summands, that we can neglect), so it is well-defined. The map $f : \bigoplus_{i\in I} M_i \to N$ is R-linear, and moreover it is the *unique* R-linear map satisfying the following property: for all $i \in I$, the composition $f \circ \iota_i : M_i \to N$ is precisely the map f_i.
We have in fact a natural bijection of sets

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, N).$$

This makes it easy to construct R-linear maps with $\bigoplus_{i\in I} M_i$ *as source*: one just has to specify the "restrictions" f_j of a map f on each summand M_j, i.e. its compositions with the inclusions ι_j. As soon as each f_j is R-linear, we get an R-linear map f.

The next definition upgrades the set $\mathrm{Hom}_R(M, N)$ to a new R-module. The construction crucially relies on R being commutative, and it generalises the well-known fact that, over a field $\mathbb{F}$, if V and W are $\mathbb{F}$-vector spaces then $\mathrm{Hom}_{\mathbb{F}}(V, W)$ carries a natural structure of $\mathbb{F}$-vector space.

Definition 1.25. Let M, N be R-modules. For $f, g \in \mathrm{Hom}_R(M, N)$ we define $f + g \in \mathrm{Hom}_R(M, N)$ by the formula

$$(f + g)(m) = f(m) + g(m) \qquad \forall m \in M.$$

For $r \in R$ we moreover define $r \cdot f \in \mathrm{Hom}_R(M, N)$ by the formula

$$(r \cdot f)(m) = r \cdot (f(m)).$$

Let us check that, indeed, the defined function of sets $g := r \cdot f$ is again R-linear; in particular, let us check that for all $s \in R$ and $m \in M$ we have $g(s \cdot m) = s \cdot g(m)$. We have indeed

$$g(s \cdot m) = (r \cdot f)(s \cdot m) = r \cdot (f(s \cdot m)) = r \cdot (s \cdot f(m)) = (r \cdot s) \cdot f(m) = \\ = (s \cdot r) \cdot f(m) = s \cdot (r \cdot f(m)) = s \cdot g(m).$$

As you see, we have used the axioms of R-modules, the fact that f is known to be R-linear, and most crucially the fact that R is a commutative ring. Compare this last computation with Example 1.17.

Example 1.26. The module $\mathrm{Hom}_R(R, M)$ is canonically isomorphic to M. The bijection is given by pairing $f\colon R \to M$ with $f(1) \in M$. Check that this bijection is R-linear.
More generally, the R-module $\mathrm{Hom}_R(\bigoplus_{i \in I} R, M)$ is canonically isomorphic to the product $\prod_{i \in I} M$. Here, by $\prod_{i \in I} M$ we mean the product of the family of modules $(M_i)_{i \in I}$, with all M_i equal to M. It is common to say that one takes the product of "I copies of the R-module M". Similarly for $\prod_{i \in I} R$.

It is then *extremely easy* (even in comparison with Example 1.24) to construct R-linear maps with $\bigoplus_{i \in I} R$ as source: one just has to specify a family of elements $(m_i)_{i \in I}$ in the target M, i.e. $m_i \in M$. This justifies the following definition.

Definition 1.27. Let I be a set. The *free R-module with basis indexed by I* is the direct sum $\bigoplus_{i \in I} R$.
Abstractly, an R-module M is *free* if it is isomorphic to one of the form $\bigoplus_{i \in I} R$, i.e. there exists a suitable set I and an R-linear isomorphism $\phi\colon \bigoplus_{i \in I} R \cong M$. The elements $\phi(\iota_i(1)) \in M$ are said to form a *basis* of M.

Note that in the case of vector spaces over a field we precisely recover the notion of basis. In fact, over a field, every vector space admits a basis, i.e. it is free. This is not the case over a generic ring R.

Example 1.28. A free module over $\mathbb{Z}$ is either 0 or infinite. Hence, for instance, $\mathbb{Z}/k$ is not free for $k \geq 2$.

1.3. **Bilinear maps.** In the last part of the lecture we address the following question: *Is there a meaningful way to compute the product of elements of R-modules?* A priori, the answer is NO: by definition, if M is an R-module, the only defined operations are sum and multiplication of an element of M with an element of the ring R; but given two R-modules M and M' and elements $m \in M$ and $m' \in M'$, the product $m \cdot m'$ just does not make sense. However, there are situations, such as the following, in which a meaningful product is indeed defined.

Example 1.29. Let R be a ring and consider $M = R[x]$ and $M' = R[y]$ as R-modules. Given polynomials $f(x) \in M$ and $g(y) \in M'$, the product $f(x) \cdot g(y)$ makes perfectly sense in the bigger polynomial ring $R[x, y]$. We have in fact a multiplication map

$$\mu\colon R[x] \times R[y] \to R[x, y].$$

Note that the target is a new R-module, different from both M and M'.

Example 1.30. Let $R = \mathbb{Z}$ and consider $M = \mathbb{Z}/10$ and $M' = \mathbb{Z}/20$ as $\mathbb{Z}$-modules. Then we may define a map

$$\mu\colon M \times M' \to \mathbb{Z}/2, \qquad ([m]_{10}, [m']_{20}) \mapsto [mm']_2.$$

The previous map is essentially given by projecting both M and M' onto $\mathbb{Z}/2$, and then taking the product in the ring $\mathbb{Z}/2$.
Of course, we could also have projected onto $\mathbb{Z}/5$ instead, or even better onto $\mathbb{Z}/10$ (or even worse, onto the 0 module!).

The previous examples are instances of the following definition.

Definition 1.31. Let R be a commutative ring and let M, M' and P be R-modules. An *R-bilinear map*

$$\mu\colon M \times M' \to P$$

is a map of sets satisfying the following properties, for all $m_1, m_2 \in M$, $m'_1, m'_2 \in M'$ and $r \in R$:

(1) $\mu(m_1 + m_2, m'_1) = \mu(m_1, m'_1) + \mu(m_2, m'_1) \in P$;
(2) $\mu(m_1, m'_1 + m'_2) = \mu(m_1, m'_1) + \mu(m_1, m'_2) \in P$;
(3) $\mu(r \cdot m_1, m'_1) = \mu(m_1, r \cdot m'_1) = r \cdot \mu(m_1, m'_1) \in P$.

The three properties in the previous definition extrapolate what one usually whishes from a multiplication: the first two are a form of distributive law with respect to the addition; the third is a form of compatibility of the multiplication μ with the scalar multiplication (multiplication by elements in R).
Given M and M', the question becomes: *what are the possible choices of P and of an R-bilinear map $\mu\colon M \times M' \to P$? Is there a choice which is better than the other ones?* The second part of the question is justified by the trivial example in which we take $P = 0$ and μ the constant, zero map: in this case we do get an R-bilinear map, but it is a quite boring and useless one!
In general, if we want to construct an R-bilinear map $\mu\colon M \times M' \to P$, we need the following: for all $(m, m') \in M \times M'$ we need to identify an element $\mu(m, m') \in P$; up to replacing P with a submodule, it does not harm to assume that P is in fact generated by the set of elements $(\mu(m, m'))_{(m,m') \in M \times M'}$. Moreover the relations (1)-(3) from Definition 1.31 must hold between these elements.
In few words, the tensor product $M \otimes_R M'$ will be constructed in the most direct way to have all the previous properties: it is obtained from a free module with basis the elements (m, m') of the set $M \times M$, by quotienting the suitable submodule that guarantees that (1)-(3) hold. We will see this in the next lecture.

2. Tensor product, and modules over non-commutative rings

In the first part of the lecture we still work over a commutative ring R.

2.1. **Kernel and cokernel.**

Kernel and cokernel of R-linear maps are defined as for abelian groups in the setting of R-modules, and have a natural structure of R-modules.

Definition 2.1. Given an R-linear map $f\colon M \to N$, the *kernel* $\ker(f) \subset M$ is the subset of elements $m \in M$ such that $f(m) = 0$. Let us check that it is a sub-R-module:

- if $m, m' \in \ker(f)$, then $f(m + m') = f(m) + f(m') = 0 + 0 = 0 \in N$, so also $m + m' \in \ker(f)$; this is the same argument used for abelian groups and vector spaces;
- if $r \in R$ and $m \in \ker(f)$, then $f(r \cdot m) = r \cdot f(m) = r \cdot 0 = 0 \in N$, so also $r \cdot m \in \ker(f)$; this is also the same argument used for vector spaces.

The *image* $\mathrm{Im}(f)$ is the subset of N containing elements of the form $f(m)$ for some $m \in M$. Check that it is a submodule of N.
The *cokernel* $\mathrm{coker}(f)$ is defined as the quotient $N/\mathrm{Im}(f)$.

As for abelian groups, an R-linear map is injective if and only if it has trivial kernel, and it is surjective if and only if it has trivial cokernel.

2.2. **Maps from a quotient module and from a cokernel.** Let M, N and P be R-modules, with $N \subset M$ a submodule. Let $\pi \colon M \to M/N$ be the projection to the quotient, which is R-linear. Then any R-linear map $f \colon M/N \to P$ gives rise to a map $g = f \circ \pi \colon M \to P$, with the property that g restricts on N to the zero map of R-modules: $g|_N \equiv 0 \colon N \to P$. Viceversa, given a map $g \colon M \to P$ such that $g|_N$ is the zero map, we can define $f \colon M/N \to P$ by setting $f([m]_N) = g(m)$, and this f is well-defined and also R-linear (check this).
In few words: an R-linear map with M/N as source is the same amount of information as an R-linear map with M as source, whose restriction on N vanishes.
The situation with a cokernel is very similar. Let $h \colon N \to M$ be an R-linear map (possibly, a non-injective one), and let P be an R-module; let $\pi_h \colon M \to \mathrm{coker}(h)$ be the projection to the cokernel, which is by definition a quotient of M. Then there is a bijective correspondence between the following:

- R-linear maps $f \colon \mathrm{coker}(h) \to P$;
- R-linear maps $g \colon M \to P$ such that the composition $g \circ h \colon N \to P$ is the zero map.

The bijection associates $f \colon \mathrm{coker}(h) \to P$ with $f \circ \pi_h \colon M \to P$.

2.3. **Presenting a module by generators and relations.** Let now M be an R-module, and let $\mathcal{S} \subset M$ be a generating set (e.g. the entire set M, but often something smaller suffices!). Let $F_0 = \bigoplus_{s \in \mathcal{S}} R$ [1]; then there is a surjective map

$$g_0 \colon F_0 \to M, \qquad g(\iota_s(1)) = s.$$

Here we use that an R-linear map out of a free R-module is uniquely determined by its values on the basis of the free module. The image of g contains $\mathcal{S}$, hence it contains $\mathrm{Span}_R(\mathcal{S}) = M$, and that's why we know that g is surjective.
The kernel $\ker(g_0)$ is then some submodule of F_0; we can fix a set $\mathcal{S}' \subset \ker(g_0)$ of generators and repeat the trick: we let $F_1 = \bigoplus_{s' \in \mathcal{S}'} R$ and consider the map $g_1 \colon F_1 \to F_0$ given by $\iota_{s'}(1) \mapsto s'$. The image of this map is the submodule of F_0 generated by $\mathcal{S}'$, i.e. $\mathrm{Im}(g_1) = \ker(g_0)$.
The map $g_0 \colon F_0 \to M$ is surjective, and vanishes on $\ker(g_0)$, hence it induces an R-linear map $f_0 \colon F_0/\ker(g_0) \to M$: check that this map is both surjective and injective, hence an R-linear isomorphism. Since $\ker(g_0) = \mathrm{Im}(g_1)$, we can write also

$$\mathrm{coker}(g_1) = F_0/\mathrm{Im}(g_1) = F_0/\ker(g_0) \cong M.$$

[1] to explain notation: the letter "F" because it is a free module; the index "0" will be clear later in the course

Definition 2.2. A presentation of an R-module M is the datum of an R-linear map $g_1 \colon F_1 \to F_0$ between *free* R-modules, together with an isomorphism

$$f_0 \colon \mathrm{coker}(g_1) \to M.$$

We consider F_0 as giving the *generators* of M: indeed the composition $g_0 \colon F_0 \to M$ given by

$$F_0 \xrightarrow{\pi} \mathrm{coker}(g_1) \xrightarrow{f_0} M$$

is surjective, where π is the projection onto the cokernel; and we consider F_1 as giving the *relations* between the generators: indeed the kernel of the above composition $g_0 \colon F_0 \to M$ is $\mathrm{Im}(g_1)$.

One usually writes sloppily $g_1 \colon F_1 \to F_0$ for a presentation of M, neglecting to also specify an isomorphism $\mathrm{coker}(g_1) \cong M$. It is better to write $F_1 \to F_0 \to M$ for a presentation of M.

Example 2.3. Let $M = \bigoplus_{i \in I} R$ be a free module. Then there is a presentation of M with $F_0 = M$, $F_1 = 0$ (the zero module), so that $\mathrm{coker}(0 \colon F_1 \to F_0)$ is canonically identified with F_0 and hence with M. We see that M is "free from relations", and that's why we call it (a bit tautologically) a *free module*.

Example 2.4. Let $R = \mathbb{Q}[x, y]$ and consider $M = (x, y) \subset \mathbb{Q}[x, y]$, the ideal of polynomials in two variables with vanishing constant term. Define a map $g_0 \colon R \oplus R \cong R^2 \to M$ by sending $(1, 0) \mapsto x$ and $(0, 1) \mapsto y$.
Every polynomial in M is a sum of non-constant monomials, i.e. monomials which are multiples of x or of y (or both): this implies that g_0 is surjective. Moreover, let $(a(x, y), b(x, y)) \in R^2$ be a couple of polynomials in x, y which is in the kernel of g_0: then $a(x, y) \cdot x + b(x, y) \cdot y = 0 \in M = (x, y) \subset \mathbb{Q}[x, y]$. It follows that all monomials in a are multiples of y and all monomials in b are multiples of x, and moreover $a(x, y)/y$ is the same polynomial in x, y as $-b(x, y)/x$.
We can thus define an R-linear map $g_1 \colon R \to R^2$ by sending $1 \mapsto (y, -x)$, and thus $g_1(c(x, y)) = (c(x, y) \cdot y, -c(x, y) \cdot x)$. The image of g_1 is the kernel of g_0, and thus M admits a presentation with 2 generators and 1 relation. We can write this as

$$R \xrightarrow{g_1} R^2 \xrightarrow{g_0} M$$

2.4. **Tensor products over commutative rings.** The following is the "bad definition" of the tensor product. It is an explicit construction, but it produces an R-module that, in principle, is difficult to handle with: it has a lot of generators and a lot of relations. Only after proving Proposition 2.6 we will be able to understand what makes the tensor product so special.

Definition 2.5. Let M and M' be R-modules. We define an R module $M \otimes_R M'$ as follows. We start with the free module $F = \bigoplus_{(m,m') \in M \times M'} R$, with canonical basis elements denoted $(m, m') = \iota_{(m,m')}(1)$. We then consider the submodule N of F generated by all elements of the following forms, for all $m_1, m_2 \in M$, $m'_1, m'_2 \in M'$ and $r \in R$:

- $(m_1 + m_2, m'_1) - (m_1, m'_1) - (m_2, m'_1)$;
- $(m_1, m'_1 + m'_2) - (m_1, m'_1) - (m_1, m'_2)$;
- $(r \cdot m_1, m'_1) - r \cdot (m_1, m'_1)$;
- $(m_1, r \cdot m'_1) - r \cdot (m_1, m'_1)$.

Finally, we define $M \otimes_R M'$ as the quotient R-module F/N. The class of the generator (m, m') in F/N is also denoted $m \otimes m' \in M \otimes_R M'$.
The map of sets $\mu_\otimes \colon M \times M' \to M \otimes_R M'$ is defined by $\mu_\otimes(m, m') = m \otimes m'$, and it is by construction an R-bilinear map.

In a sense, Definition 2.5 produces an R-module P which is designed in order to receive a bilinear map from $M \times M'$. The following proposition makes this idea more precise.

Proposition 2.6. *Let $\mu \colon M \times M' \to P$ be any R-bilinear map, with target any R-module P. Then there exists a* unique *R-linear map $\theta \colon M \otimes_R M' \to P$ such that the following diagram of maps (of sets) commutes:*

$$\begin{array}{ccc} M \times M' & \xrightarrow{\mu_\otimes} & M \otimes_R M' \\ & \searrow^{\mu} & \downarrow{\scriptstyle\theta} \\ & & P. \end{array}$$

Proof. Since $M \otimes M'$ is a quotient F/N, giving an R-linear map $\theta \colon F/N \to P$ is equivalent to giving an R-linear map $\tilde{\theta} \colon F \to P$ that vanishes on N: the map $\tilde{\theta}$ is obtained from θ as the composite $F \twoheadrightarrow F/N \xrightarrow{\theta} P$.
If we want the diagram to commute, we must have the equality

$$\tilde{\theta}(m, m') = \theta(m \otimes m') = \mu(m, m')$$

for all $(m, m') \in M \times M'$. Thus the map $\tilde{\theta}$ is forced on the R-basis of F given by the elements (m, m'), and we can conclude that there are two possibilities:

- either $\tilde{\theta} \colon F \to P$ descends to an R-linear map $\theta \colon F/N \to P$, i.e. it vanishes on N;
- or $\tilde{\theta}$ does not descend to an R-linear map $\theta \colon F/N \to P$.

In the first case, we would have that θ exists and is unique; in the second case instead we would have that θ does not exist. Let us rule out the second case.
To prove that $\tilde{\theta}$ vanishes on N, it suffices to prove that it vanishes on generators (1)-(4) of N. Let us compute as example the image of a generator of N of type (3) along $\tilde{\theta}$:

$$\begin{aligned} \tilde{\theta}\big((r \cdot m_1, m_1') - r \cdot (m_1, m_1')\big) &= \tilde{\theta}(r \cdot m_1, m_1') - r \cdot \tilde{\theta}(m_1, m_1') \\ &= \mu(r \cdot m_1, m_1') - r \cdot \mu(m_1, m_1') = 0. \end{aligned}$$

In the first equality we use R-linearity of $\tilde{\theta} \colon F \to P$; in the second we use the definition of $\tilde{\theta}$, i.e. its evaluation on the basis of F; in the third we use that $\mu \colon M \times M' \to P$ is R-bilinear.
In a similar way one can check that all generators of N are sent to 0 along $\tilde{\theta}$. □

Motivated by Proposition 2.6, we give the following, which is the "good definition" of the tensor product, by *universal property*.

Definition 2.7. Let M and M' be R-modules. A universal bilinear map for $M \times M'$ is the datum of a couple $(\bar{P}, \bar{\mu})$, where $\bar{P}$ is an R-module and where $\bar{\mu} \colon M \times M' \to P$ is an R-bilinear map, satisfying the following property (called *universal property*): whenever (P, μ) is a (possibly different) couple with P being an R-module and

$\mu\colon M \times M' \to P$ an R-bilinear map, then there exists a *unique* R-linear map $\theta\colon \bar{P} \to P$ such that the following diagram of maps of sets commutes:

$$\begin{array}{ccc} M \times M' & \xrightarrow{\bar{\mu}} & \bar{P} \\ & \searrow^{\mu} & \downarrow{\scriptstyle \theta} \\ & & P. \end{array}$$

At first glance, it is not clear why the previous is a definition at all. A priori, such a wonderful $(\bar{P}, \bar{\mu})$ could not exist at all! But here Proposition 2.6 helps us: it tells us that in fact $(M \otimes M', \mu_{\otimes})$ satisfies the property required for $(\bar{P}, \bar{\mu})$. On the other hand, giving a property of an object often does not suffice to *define* the object. We have to check that, in fact, the property of $(\bar{P}, \bar{\mu})$ in Definition 2.7 suffices to determine this couple, at least up to isomorphism.

The argument is as follows. Let $(\bar{P}, \bar{\mu})$ and $(\check{P}, \check{\mu})$ be two couples satisfying the property required by Definition 2.7. If you wish, think that $(\bar{P}, \bar{\mu})$ is the tensor product from Definition 2.5, and $(\check{P}, \check{\mu})$ is instead obtained in another way.

Since $\bar{\mu}\colon M \times M' \to \bar{P}$ is an example of an R-bilinear map with source $M \times M'$, the universal property of $(\check{P}, \check{\mu})$ implies that there is a *unique* R-linear map $\theta_1\colon \check{P} \to \bar{P}$ such that the following commutes

$$\begin{array}{ccc} M \times M' & \xrightarrow{\check{\mu}} & \check{P} \\ & \searrow^{\bar{\mu}} & \downarrow{\scriptstyle \theta_1} \\ & & \bar{P}. \end{array}$$

Viceversa, since $\check{\mu}\colon M \times M' \to \check{P}$ is R-bilinear, the universal property of the couple $(\bar{P}, \bar{\mu})$ implies that there is a *unique* R-linear map $\theta_2\colon \bar{P} \to \check{P}$ such that the following commutes

$$\begin{array}{ccc} M \times M' & \xrightarrow{\bar{\mu}} & \bar{P} \\ & \searrow^{\check{\mu}} & \downarrow{\scriptstyle \theta_2} \\ & & \check{P}. \end{array}$$

Moreover, since $\bar{\mu}\colon M \times M' \to \bar{P}$ is R-bilinear, the universal property of $(\bar{P}, \bar{\mu})$ *itself* implies that there is a *unique* R-linear map $\bar{\theta}\colon \bar{P} \to \bar{P}$ such that the following commutes

$$\begin{array}{ccc} M \times M' & \xrightarrow{\bar{\mu}} & \bar{P} \\ & \searrow^{\bar{\mu}} & \downarrow{\scriptstyle \bar{\theta}} \\ & & \bar{P}. \end{array}$$

In the last diagram we have two natural candidates for $\bar{\theta}$: one is $\mathrm{Id}_{\bar{P}}$, and the other is $\theta_1 \circ \theta_2$: this second map makes the last diagram commute because we can glue the two previous diagrams, in which θ_1 and θ_2 appear.

By *uniqueness* of $\bar{\theta}$, we get that $\mathrm{Id}_{\bar{P}} = \theta_1 \circ \theta_2$. Similarly, using the universal property of $(\check{P}, \check{\mu})$ against $\check{\mu}$, one obtains that $\mathrm{Id}_{\check{P}} = \theta_2 \circ \theta_1$. This means that θ_1 and θ_2 are inverse R-linear isomorphisms between $\bar{P}$ and $\check{P}$, and that along these isomorphisms the bilinear maps $\bar{\mu}$ and $\check{\mu}$ are identified. In this sense, Definition 2.7 characterises a universal bilinear map out of $M \times M'$ up to canonical isomorphism.

Example 2.8. Let $f\colon M \to N$ and $f'\colon M' \to N'$ be R-linear maps. You can check that for any R-bilinear map $\mu_1\colon N \times N' \to P$ the composite map of sets

$$M \times M' \xrightarrow{f\times f'} N \times N' \xrightarrow{\mu_1} P$$

is an R-bilinear map $\mu_2\colon M\times M' \to P$. This holds in particular when $P = N\otimes_R N'$ and $\mu_1 = \mu_\otimes\colon N \times N' \to N \otimes_R N'$ is the universal bilinear map of $N \times N'$. The universal property of $M \otimes_R M'$ implies that there is a *unique* R-linear map $\theta\colon M \otimes_R M' \to N \otimes_R N'$ such that the following diagram commutes

$$\begin{array}{ccc} M \times M' & \xrightarrow{\mu_\otimes} & M \otimes_R M' \\ \big\downarrow{\scriptstyle f\times f'} & & \big\downarrow{\scriptstyle \theta} \\ N \times N' & \xrightarrow{\mu_\otimes} & N \otimes_R N. \end{array}$$

The map θ is often denoted as $f \otimes f'\colon M \otimes_R M' \to N \otimes_R N'$.

Exercise 2.9. Let $F = \bigoplus_{i\in I} R$ and $F' = \bigoplus_{i'\in I'} R$ be free R-modules, and let $F'' = \bigoplus_{(i,i')\in I\times I} R$ be also a free R-module. We have a bilinear map $\bar{\mu}\colon F \times F' \to F''$ given by the following formula, where elements of free modules are represented as (finite) linear combinations of basis elements:

$$\bar{\mu}\colon \left(\sum_{i\in I} r_i \cdot \iota_i(1), \sum_{i'\in I'} r'_{i'} \cdot \iota_{i'}(1)\right) \mapsto \sum_{(i,i')\in I\times I'} (r_i \cdot r'_{i'}) \cdot \iota_{(i,i')}(1).$$

Check that the previous assignment is indeed bilinear. Check moreover that $\bar{\mu}$ satisfies the universal property to be a universal R-bilinear map out of $F \times F'$. This implies that $(F'', \bar{\mu})$ is a model for the tensor product $F\otimes_R F'$. This model is much better that the one from Definition 2.5, as it exhibits immediately the tensor product as a free R-module. Thanks to this construction and thanks to the fact that the universal property characterises the tensor product up to canonical isomorphism, we of course get that also the tensor product as constructed in Definition 2.5 is a free R-module.

2.5. **Left and right modules.** At this point we abandon the hypothesis that R is a commutative ring, but we still assume that it is associative and it has a neutral element $1 \in R$.

Example 2.10. Here are two prominent examples of non-commutative rings that one really wants to study:

- Let $\mathbb{F}$ be a field and $k \geq 2$. Then the ring $M_k(\mathbb{F})$ of matrices of size $k \times k$ with coefficients in $\mathbb{F}$ is one of the fundamental objects to consider in linear algebra.
- Let G be a non-commutative group. Then the *group ring* $\mathbb{Z}[G]$ is the following ring: as an abelian group it is $\bigoplus_{g\in G} \mathbb{Z}$, where we denote simply by g the element $\iota_g(1)$; the product is defined by the rule

$$\left(\sum_{g\in G} a_g \cdot g\right) \cdot \left(\sum_{g'\in G} b_{g'} g'\right) = \sum_{(g,g')\in G\times G} (a_g \cdot b_{g'}) \cdot gg'.$$

  where all sums are assumed essentially finite (finitely many non-zero summands). Module theory over $\mathbb{Z}[G]$ is known as *representation theory of* G, and is one of the most central theories in mathematics.

The theory of modules becomes immediately more complicated, in that we can define two distinct types of modules.

Definition 2.11. A left R-module is an abelian group M with a multiplication by scalars

$$R \times M \to M, \qquad (r, m) \mapsto r \cdot m$$

satisfying the following properties, for all $r, r' \in R$ and all $m, m' \in M$:

(1) $r \cdot (m + m') = r \cdot m + r \cdot m'$;
(2) $(r + r') \cdot m = r \cdot m + r' \cdot m$;
(3) $r \cdot (r' \cdot m) = (r \cdot r') \cdot m$;
(4) $1 \cdot m = m$.

A right R-module is an abelian group M' with a multiplication by scalars

$$M \times R \to M, \qquad (m, r) \mapsto r \cdot m$$

satisfying the following properties, for all $r, r' \in R$ and all $m, m' \in M$:

(1) $(m + m') \cdot r = m \cdot r + m' \cdot r$;
(2) $m \cdot (r + r') = m \cdot r + m \cdot r'$;
(3) $(m \cdot r) \cdot r' = m \cdot (r \cdot r')$;
(4) $m \cdot 1 = m$.

As one can see, property (3) can be expressed both in the left and in the right module case as follows: no matter how we put parentheses, the product $r \cdot r' \cdot m$ (respectively $m \cdot r \cdot r'$) is always the same element in M (respectively in M'). We will use the following heuristic principle very often: axioms and definitions relative to left and right modules should be expressed as much as possible as equalities between expressions in which only the parentheses (and possibly a "$\otimes$" symbol) are moved; these equalities should therefore represent some sort of *associativity* between various types of multiplication.
Roughly speaking, there is a part of the theory that only deals with left R-modules, an analogue part that only deals with right R-modules, and a part of the theory that needs both notions at the same time. To simplify the exposition, whenever the theory only deals with one type of module, I will discuss it considering left modules, and leave to you to transform the statements into the setting of right modules.

2.6. **What can be recycled from the last lecture.** The following definitions, observations and constructions from last time can be repeated word by word for left modules:

- a submodule of a left R-module M is a sub-abelian group $N \subset M$ which is closed under scalar multiplication; the quotient M/N is also a left R-module;
- if $\mathcal{S} \subset M$ we have a submodule $\mathrm{Span}_R(\mathcal{S}) \subset M$;
- if $R \to S$ is a homomorphism of rings, then any left S-module becomes a left S-module;
- if $(M_i)_{i \in I}$ is a collection of left R-modules, then $\prod_{i \in I} M_i$ is a left R-module, containing a submodule $\bigoplus_{i \in I} M_i$.
- R is a left R-module; a *free* left R-module is one of the form $\bigoplus_{i \in I} R$.

2.7. **Homomorphisms of modules.** The first instance of a separation between left and right R-modules is that, in the case of a non-commutative ring R, it is only meaningful to define R-linear maps from a left to a left R-module (or from a right to a right). We first give the definition, and then see what goes wrong in trying naively to define R-linear maps from a left to a right R-module.

Definition 2.12. Let M and N be left R-modules. A function $f\colon M \to N$ is R-linear if it is a homomorphism of abelian groups and for all $r \in R$ and $m \in M$ the following equality holds:

$$f(r \cdot m) = r \cdot (f(m)).$$

The set of all R-linear maps from M to N is denoted $\mathrm{Hom}_R(M, N)$.

The formula in the previous definition is against our heuristic principle: if we remove parentheses, we get "frm" on left and "rfm" on right. This has a purely linguistic reason: when we talk of a function $f\colon M \to N$, we use to say "f of m" and write $f(m)$, instead of saying "of m, f" and writing $(m)f$. It might be surprising, but in the case of left R-modules it would be better to use the second terminology, and think of R-linear maps as acting *on right*. The axiom for an R-linear map of left R-modules becomes

$$(r \cdot m)f = r \cdot ((m)f),$$

which now satisfies our heuristic principle. For rights R-modules our old convention is perfectly fine: an R-linear map $f\colon M \to N$ between right R-modules should be thought of as acting *on left* and the axiom becomes

$$f(m \cdot r) = (f(m)) \cdot r.$$

If you are not convinced about the heuristic principle, see the following example.

Example 2.13. Let M be a left R-module and N be a right R-module. We would like to say that a homomorphism of abelian groups $f\colon M \to N$ is R-linear if it satisfies $f(r \cdot m) = f(m) \cdot r$. Of course, the previous formula doesn't respect our heuristic principle. What happens concretely is the following: let f be a naively R-linear map in the previous sense; then for all $r, s \in R$ and $m \in M$ we have

$$\begin{aligned} f(m) \cdot (r \cdot s) = (f(m) \cdot r) \cdot s = f(r \cdot m) \cdot s = f(s \cdot (r \cdot m)) \\ = f((s \cdot r) \cdot m) = f(m) \cdot (s \cdot r). \end{aligned}$$

This means that the image of f is contained in the subset (in fact a sub-right-R-module) N^{comm} of N of elements $n \in N$ that are sent to 0 by the right scalar multiplication by all elements of the form $rs - sr \in R$ (such elements are often called *commutators*). In other words, we are really considering an R-linear map $f\colon M \to N^{comm}$. Now N^{comm} is usually very small, and anyway it feels strange that a generic R-linear map with target N only uses a certain submodule for the image, right?

Let M, N be left R-modules. The set $\mathrm{Hom}_R(M, N)$ is naturally an abelian group by pointwise sum of functions: if $f, g\colon M \to N$ are R-linear, then $f + g$ is automatically R-linear. Unfortunately, however, there is no good way to upgrade this to an left or right R-module structure on $\mathrm{Hom}_R(M, N)$. The naive idea would be to define, for an R-linear map $f\colon M \to N$ and for $r \in R$, a new map $g\colon M \to N$ by the rule

$$(m)g = (r \cdot m)f = r \cdot ((m)f).$$

Note that the two expressions are automatically equal because f is R-linear. However, the map g is in general not R-linear: let $s \in R$, then we have

$$(s \cdot m)g = r \cdot s \cdot (m)f \neq s \cdot r \cdot (m)f = s \cdot (m)g,$$

where the inequality holds in general (but of course could be an equality sometimes).

2.8. **Bilinear maps and tensor products.** The heuristic principle really helps us when trying to understand what an R-bilinear map should be in the case of R non-commutative. It turns out that the only sensible definition is the following.

Definition 2.14. Let M be a *right* R-module and M' be a *left* R-module, and let P be an *abelian group*. An R-bilinear map $\mu\colon M \times M' \to P$ is a map of sets satisfying the following requirements, for all $m_1, m_2 \in M$, $m'_1, m'_2 \in M'$ and $r \in R$:

(1) $\mu(m_1 + m_2, m'_1) = \mu(m_1, m'_1) + \mu(m_2, m'_1) \in P$;
(2) $\mu(m_1, m'_1 + m'_2) = \mu(m_1, m'_1) + \mu(m_1, m'_2) \in P$.
(3) $\mu(m_1 \cdot r, m'_1) = \mu(m_1, r \cdot m'_1) \in P$;

Note that condition (3) is in line with our heuristic principle, as we see the letters $m_1 r m'_1$ as argument of μ on both sides. The tensor product $M \otimes_R M'$ will be only an abelian group, defined as follows.

Definition 2.15. Let M be a *right* R-module and M' be a *left* R-module. A universal bilinear map for $M \times M'$ is the datum of a couple $(\bar{P}, \bar{\mu})$, where $\bar{P}$ is an abelian group and where $\bar{\mu}\colon M \times M' \to P$ is an R-bilinear map, satisfying the following *universal property*: whenever (P, μ) is a (possibly different) couple with P an abelian group and $\mu\colon M \times M'$ an R-bilinear map, then there exists a *unique* homomorphism of abelian groups $\theta\colon \bar{P} \to P$ such that the following diagram of maps of sets commutes:

$$\begin{array}{ccc} M \times M' & \xrightarrow{\bar{\mu}} & \bar{P} \\ & \searrow^{\mu} & \downarrow \theta \\ & & P. \end{array}$$

The proof that such a couple $(\bar{P}, \bar{\mu})$, if it exists, is unique up to canonical isomorphism, is word by word the same as in the case of a commutative ring. The existence of a universal bilinear map is given by the following definition, which is analogue to Definition 2.5.

Definition 2.16. Let M be a *right* R-module and M' be a *left* R-module. Consider the *free abelian group* $A = \bigoplus_{(m,m')} \in M \times M'\mathbb{Z}$ with basis the elements of the set $M \times M'$. Denote $(m, m') = \iota_{(m,m')}(1) \in A$. Let $B \subset A$ be the subgroup generated by the following types of elements, for varying $m_1, m_2 \in M$, $m'_1, m'_2 \in M'$ and $r \in R$:

(1) $(m_1 + m_2, m'_1) - (m_1, m'_1) - (m_2, m'_1)$;
(2) $(m_1, m'_1 + m'_2) - (m_1, m'_1) - (m_1, m'_2)$.
(3) $(m_1 \cdot r, m'_1) - (m_1, r \cdot m'_1)$.

We set $M \otimes_R M'$ to be the abelian group A/B. We denote by $m \otimes m' = [(m, m')]_B$. We have a R-bilinear map $\mu_\otimes\colon M \times M' \to M \otimes_R M'$ given by $(m, m') \mapsto m \otimes m'$.

As an exercise, copy and adapt the proof of Proposition 2.6 to show that Definition 2.16 gives a universal R-bilinear map out of $M \times M'$.

2.9. **Some examples over the integers.** Let $R = \mathbb{Z}$ throughout the subsection, and consider first the modules $M = \mathbb{Z}/2$ and $N = \mathbb{Z}/2$. What is $M \otimes_{\mathbb{Z}} N$? We definitely have a $\mathbb{Z}$-bilinear map

$$\bar{\mu}\colon \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2, \qquad ([m]_2, [n]_2) \mapsto [mn]_2.$$

(Check that indeed this map is $\mathbb{Z}$-bilinear; it should be intuitive because it is the product of the ring $\mathbb{Z}/2$, but intuition is not always leading in the right place...). We now claim that $(\mathbb{Z}/2, \bar{\mu})$ is a universal bilinear map out of $\mathbb{Z}/2 \times \mathbb{Z}/2$, in the sense of Definition 2.7. To prove this, let (P, μ) be any couple where P is a $\mathbb{Z}$-module and $\mu\colon \mathbb{Z}/2 \times \mathbb{Z}/2 \to P$ is any $\mathbb{Z}$-bilinear map. Let $[m]_2, [n]_2 \in \mathbb{Z}/2$, and without loss of generality use $m, n \in \{0, 1\}$ as representative of classes modulo 2. We have

$$\mu([m]_2, [n]_2) = \mu(m \cdot [1]_2, n \cdot [1]_2) = mn \cdot \mu([1]_2, [1]_2).$$

Let's give the name $p = \mu([1]_2, [1]_2) \in P$. By the previous reasoning, we have $\mu([m]_2, [n]_2) = 0 \in P$ if either m or n is 0. Moreover we can compute

$$0 = \mu([2]_2, [1]_2) = \mu(2 \cdot [1]_2, [1]_2) = 2 \cdot \mu([1]_2, [1]_2) = 2 \cdot p.$$

Overall, μ sends 3 out of 4 elements of $\mathbb{Z}/2 \times \mathbb{Z}/2$ to 0 (the ones "containing an even coordinate"), and sends the last element to an element p satisfying $2p = 0 \in P$. But then we can define a $\mathbb{Z}$-linear map $\theta\colon \mathbb{Z}/2 \to P$ by setting $[1]_2 \mapsto p$ (and of course $[0]_2 \mapsto 0 \in P$). You can check that this map θ is the unique $\mathbb{Z}$-linear map $\mathbb{Z}/2 \to P$ (in fact it is also the unique map of sets!) such that $\mu = \theta \circ \bar{\mu}$. Conclusion: $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2$.

Let's now see what happens with $M = \mathbb{Z}/2$ and $N = \mathbb{Z}/3$. We obviously can take $P = 0$, the zero $\mathbb{Z}$-module, and consider the zero map $M \times N \to P$, which is $\mathbb{Z}$-bilinear. This is boring: can we do better? No! Let us check that the zero map $\bar{\mu}\colon \mathbb{Z}/2 \times \mathbb{Z}/3 \to 0$ is the univeral $\mathbb{Z}$-bilinear map out of $\mathbb{Z}/2 \times \mathbb{Z}/3$. Let therefore (P, μ) be another $\mathbb{Z}$-bilinear map towards a $\mathbb{Z}$-module P. Then for all m, n we have

$$\begin{aligned}\mu([m]_2, [n]_3) = \mu(3 \cdot [m]_2, [n]_3) = \mu([m]_2, 3 \cdot [n]_3) = \mu([m]_2, [0]_3)\\ = \mu([m]_2, 0 \cdot [1]_3) = 0 \cdot \mu([m]_2, [1]_3) = 0\end{aligned}$$

This means that μ is the constant 0 map. The constant 0 map factors uniquely through the zero $\mathbb{Z}$-module, so $(0, \bar{\mu})$ has the univeral property.

3. Properties of Hom and tensor, exact sequences

3.1. **A small ambiguity.** Let R be a commutative ring, and let M and M' be two R-modules. We have defined so far $M \otimes_R M'$ in two different ways:

- as an R-module, as a certain quotient of $F := \bigoplus_{(m,m') \in M \times M'} R$; let me call it $(M \otimes_R M')_1$;
- we can also forget that R is commutative and consider M as a right and M' as a left R-module; then we can use the recipe for non-commutative rings and define $M \otimes_R M'$ as an abelian groups, namely as a certain quotient of $A := \bigoplus_{(m,m') \in M \times M'} \mathbb{Z}$; let me call this $(M \otimes_R M')_2$.

It turns out that forgetting the R-scalar multiplication of $(M \otimes_R M')_1$, we do actually get an abelian group isomorphic to $(M \otimes_R M')_2$. A map of abelian groups

$$(M \otimes_R M')_2 \to (M \otimes_R M')_1$$

can be constructed as follows: the map $\mu_\otimes\colon M \times M' \to (M \otimes_R M')_1$ is "R-bilinear" in the first given definition, treating M and M as R-modules and treating $(M \otimes_R$

$M')_1$ as an R-module; in particular, it is also "R-bilinear" in the second given definition, treating M as a right R-module and M' as a left R-module, and treating $(M \otimes_R M')_1$ only as an abelian group. so the universal property of $(M \otimes_R M')_2$ gives a map of abelian groups $(M \otimes_R M')_2 \to (M \otimes_R M')_1$.

Exercise 3.1. Prove that the previous map of abelian groups is an isomorphism.

3.2. **Universal property of products and direct sums of modules.** Let R be a (possibly non-commutative) ring. We have introduced left and right R-modules, and R-linear maps. We thus obtain the following two categories.

Definition 3.2. The category ${}_R\mathrm{Mod}$ has left R-modules as objects and R-linear maps of left R-modules as morphisms. Composition of maps is defined by composing R-linear maps as maps of sets and recognising that the resulting composite map is again R-linear.
Similarly, Mod_R is the category with objects right R-modules and morphisms the R-linear maps of right R-modules.

In this subsection we focus on left R-modules, but we could make an analogue discussion about right R-modules. Let $(M_i)_{i\in I}$ be a collection of left R-modules. We have constructed the left R-modules $\prod_{i\in I} M_i$ and inside it we have identified $\bigoplus_{i\in I} M_i$. Both these modules can be characterised by a universal property.

Definition 3.3. Let $\mathcal{C}$ be a category and let $(x_i)_{i\in I}$ be a collection of objects in $\mathcal{C}$. A *product* of the x_i's is the datum $(\bar{y}, (\bar{\pi}_i)_{i\in I})$ of an object $\bar{y}$ and a collection of $\mathcal{C}$-morphisms $\bar{\pi}_i \colon y \to x_i$ with the following universal property: whenever $(y, (\pi_i)_{i\in I})$ is the datum of an object $y \in \mathcal{C}$ and a collection of $\mathcal{C}$-morphisms $\pi_i \colon y \to x_i$, there is a unique map $\theta \colon y \to \bar{y}$ such that for all $i \in I$ we have $\pi_i = \bar{\pi}_i \circ \theta$.

Similarly, a *coproduct* of the x_i's is the datum $(\bar{z}, (\bar{\iota}_i)_{i\in I})$ of an object $\bar{z}$ and a collection of $\mathcal{C}$-morphisms $\bar{\iota}_i \colon x_i \to z$ with the following universal property: whenever $(z, (\iota_i)_{i\in I})$ is the datum of an object $z \in \mathcal{C}$ and a collection of $\mathcal{C}$-morphisms $\iota_i \colon x_i \to z$, there is a unique map $\theta \colon \bar{y} \to y$ such that for all $i \in I$ we have $\iota_i = \theta \circ \iota_i$.

For a generic category $\mathcal{C}$ and a generic collection $(x_i)_{i\in I}$ of objects, there are two possibilities: either there exists a product (respectively, a coproduct) of the collection, and in this case it is unique up to a canonical isomorphism identifying also the structure maps to (from) the x_i's; or there exists no product (no coproduct).

Notation 3.4. If a product $(\bar{y}, (\bar{\pi}_i)_{i\in I})$ of the x_i's exists, the object $\bar{y}$ is usually denoted $\prod_{i\in I} x_i \in \mathcal{C}$, but remember that this is only half of the information of a product: the other half are the morphisms $\bar{\pi}_i$ towards the x_i's.
Similarly, if a coproduct of the x_i's exists, its underlying object is usually denoted $\coprod_{i\in I} x_i \in \mathcal{C}$, and unfortunately there seems to be no standard notation to denote the structure morphisms from the x_i's to the coproduct (I'm using the letter "ι" for them).

In the category ${}_R\mathrm{Mod}$, the product module $\prod_{i\in I} M_i$ with coordinate projections $\pi_j \colon \prod_{i\in I} \to M_j$ is a model for the categorical product, i.e. it satisfies the universal property.
Similarly, in ${}_R\mathrm{Mod}$, the direct sum $\bigoplus_{i\in I} M_i$ with the maps $\iota_j \colon M_i \to \bigoplus_{i\in I} M_j$ is a model for the categorical coproduct, i.e. it has the universal property.

For a generic left R-module N, we then have isomorphisms of sets

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, N);$$

$$\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \cong \prod_{i\in I} \mathrm{Hom}_R(N, M_i).$$

Check that the previous are a isomorphisms of abelian groups, and if R is commutative they are even isomorphisms of R-modules.
If I is finite, we saw that the inclusion $\bigoplus_{i\in I} M_i \subset \prod_{i\in I} M_i$ is in fact an equality. This is a special property of the category ${}_R\mathrm{Mod}$: finite coproducts "coincide" with finite products. In particular, there is an object, the zero module, which is both the empty categorical coproduct (initial object) and the empty categorical product (final object).
This is for example not the case for the category Set of sets: a disjoint union of two sets is usually not the same as the cartesian product of sets. And the empty set (initial object) is not isomorphic to the one-point set (terminal object).

3.3. **Distributivity of Hom and tensor.** In this subsection we do not assume R commutative; if however R is commutative, we write in parentheses what happens more. Let M, M', N, N' be left R-modules; then we have isomorphisms of abelian groups (of R-modules)

- $\mathrm{Hom}_R(M \oplus M', N) \cong \mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(M', N)$;
- $\mathrm{Hom}_R(M, N \oplus N') \cong \mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(M, N')$;

This follows from the previous subsection, using I of cardinality 2. What happens with the tensor product? Let $(M_i)_{i\in I}$ be a collection of right R-modules and let N be a left R-module. For each $j \in I$ the composition

$$M_j \times N \xrightarrow{\iota_j \times \mathrm{Id}_N} \left(\bigoplus_{i\in I} M_i\right) \times N \xrightarrow{\mu_\otimes} \left(\bigoplus_{i\in I} M_i\right) \otimes_R N$$

is the composition of a cartesian product of R-linear maps with an R-bilinear map, hence is R-bilinear, hence it induces a map of abelian groups (of R-modules)

$$\iota_j \otimes \mathrm{Id}_N \colon M_j \otimes_R N \to \left(\bigoplus_{i\in I} M_i\right) \otimes_R N.$$

By the universal property of direct sum, these maps assemble into a map

$$\bigoplus_{j\in I} \iota_j \otimes \mathrm{Id}_N \colon \bigoplus_{j\in I} (M_j \otimes_R N) \to \left(\bigoplus_{i\in I} M_i\right) \otimes_R N.$$

Proposition 3.5. *The above map is an isomorphism.*

One usually says that the tensor product is "distributive with respect to direct sum", or it is "biadditive". I leave the proof of the previous proposition as exercise. A good way to attack the exercise is to try to prove that the abelian group $\bigoplus_{j\in I} (M_j \otimes_R N)$ receives a universal R-bilinear map from the product of R-modules (one right, one left) $\left(\bigoplus_{i\in I} M_i\right) \times N$.

3.4. **More properties of Hom and tensor over commutative rings.** Let R be a commutative ring throughout the subsection, and let M and M' be R-modules.

Example 3.6. We give a hands-on proof that if $\mathcal{S} \subset M$ and $\mathcal{S}' \subset M'$ are generating sets for M, M' as R-modules, then the set $\mathcal{S}'' = \{s \otimes s' : (s, s') \in \mathcal{S} \times \mathcal{S}'\} \subset M \otimes_R M'$ generates $M \otimes_R M'$ *as an R-module.*
Surely, the set $\{m \otimes m' : (m, m') \in M \times M'\} = \mathrm{Im}(\mu_\otimes) \subset M \otimes_R M'$ suffices to generate the entire $M \otimes_R M'$ as an R-module[2]. So we want to prove that for all $(m, m') \in M \times M'$, the element $m \otimes m'$ can be generated using the set $\mathcal{S}''$. Write $m = \sum_{i=1}^{k} r_i \cdot s_i$ and $m' = \sum_{j=1}^{k'} r'_j \cdot s'_j$, with $r_i, r'_j \in R$, $s_i \in \mathcal{S}$ and $s'_j \in \mathcal{S}'$. Then we have

$$m \otimes m' = \left(\sum_{i=1}^{k} r_i \cdot s_i \right) \otimes \left(\sum_{j=1}^{k'} r'_j \cdot s'_j \right) = \sum_{i=1}^{k} \sum_{j=1}^{k'} (r_i r'_j) \cdot (s_i \otimes s'_j).$$

In particular, if both M and M' are finitely generated, then also $M \otimes M'$ is finitely generated *as an R-module* (but possibly not as an abelian group!).

Ask yourself how much one can adapt the previous example to the case of non-commutative rings R[3].

Example 3.7. Let I be an ideal of R. We set $M' = R/I$[4] and want to compute $M \otimes_R R/I$. Let $IM \subset M$ be the submodule generated by all elements of the form $r \cdot M$, for $r \in I$ and $m \in M$, and consider the quotient M/IM. There is an R-bilinear map

$$\bar{\mu} \colon M \times R/I \to M/IM, \qquad (m, [r]_I) \mapsto [r \cdot m]_{IM}.$$

Check that the previous map is well-defined and R-bilinear. We want now to check the universal property for the couple $(M/IM, \bar{\mu})$. Let $\mu \colon M \times R/I \to P$ be an R-bilinear map towards some R-module P. Then the map[5]

$$\mu(-, [1]_I) \colon M \to P, \qquad m \mapsto \mu(m, [1]_I)$$

is R-linear from M to P, and vanishes on the generating set $\{r \cdot m : r \in I, m \in M\}$ of IM: we have

$$\begin{aligned} \mu(r \cdot m, [1]_I) &= \mu(m, r \cdot [1]_I) = \mu(m, [r]_I) = \mu(m, [0]_I) \\ &= \mu(m, 0 \cdot [0]_I) = 0 \cdot \mu(m, [0]_I) = 0. \end{aligned}$$

Thus we get an R-linear map $\theta \colon M/IM \to P$. Check that $\theta \circ \bar{\mu} = \mu$ as maps of sets, and that θ is the only R-linear map $M/IM \to P$ with this property.

As an application, let $I, I' \subset M$ be two ideals. Then $R/I \otimes_R R/I' \cong R/(I, I')$ as R-modules, where (I, I') is the ideal generated by I and I'. Another application: $M \otimes_R R \cong M$ as R-module. Adapt the previous example to study $R/I \otimes M$, or just prove the following general lemma.

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[2]in fact, it suffices to generate $M \otimes_R M'$ even as an abelian group!

[3]Probably not much, for generic rings. In particular, one now needs to find generators of the tensor product *as an abelian group*, so one in general needs a lot of generator to compensate the lack of scalar multiplication times R in the tensor product.

[4]An R-module of the form R/I is usually called a *cyclic* R-module, and it is characterised by admitting a generating set with a single element

[5]This is a special case of a general phenomenon: if R is commutative and $\mu \colon M \otimes M' \to P$ is R-bilinear, then for all $m' \in M'$ the map $\mu(-, m') \colon M \to P$ is R-linear. And similarly fixing the first variable and letting the second vary.

Lemma 3.8. *Let R be a commutative ring and let M, M', P be R-modules. Then a map of sets $\mu\colon M \times M' \to P$ is R-bilinear if and only if the composite $\mu \circ \mathrm{swap}\colon M' \times M \to P$ is R-bilinear, where* $\mathrm{swap}\colon M' \times M \to M \times M'$ *is the map of sets swapping coordinates. Hence, the two R-modules $M \otimes_R M'$ and $M' \otimes_R M$ have equivalent universal properties and are thus isomorphic.*

Ask yourself how much one can adapt the previous example to the case of non-commutative rings R. Be careful with left and right ideals!

Example 3.9. Let again I be an ideal of R and set $M' = R/I$ again. We want to understand $\mathrm{Hom}_R(R/I, M)$ as an R-module. An R-linear map $f\colon R/I \to M$ is equivalent to an R-linear map $g\colon R \to M$ that vanishes on the submodule $I \subset R$. On the other hand, an R-linear map $g\colon R \to M$ is uniquely determined by $g(1) \in M$, and if we want $g|_I \equiv 0$, we need to ask $r \cdot g(1) = 0$ for all $r \in I$. This is a condition on the element $g(1)$ that we pick as image of M.
If we define $M[I] \subset M$ the sub-R-module of elements m such that $r \cdot m = 0$ for all $r \in I$, we get a bijection of sets $\mathrm{Hom}_R(R/I, M) \cong M[I]$. Check that this is in fact an isomorphism of R-modules.

Ask yourself how much one can adapt the previous example to the case of non-commutative rings R. Again, be careful with left and right ideals!
As applications of all previous examples to the case of $\mathbb{Z}$-modules with a single generator, we get the following classical table of isomorphisms, where $a, b \geq 1$ are integers, and $\gcd(a, b)$ is the greatest common divisor of a and b. Every mathematician should know this table by memory and repeat it before every meal.

- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/b) \cong \mathbb{Z}/b$;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}) \cong 0$;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) \cong \mathbb{Z}/\gcd(a, b)$;
- $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b \cong \mathbb{Z}/b$;
- $\mathbb{Z}/a \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/a$;
- $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$;
- $\mathbb{Z}/a \otimes_{\mathbb{Z}} \mathbb{Z}/b \cong \mathbb{Z}/\gcd(a, b)$.

Using the distributivity of Hom and tensor over direct sums in both entries, one can compute Hom and tensors for every finitely generated $\mathbb{Z}$-module. Indeed all finitely generated $\mathbb{Z}$-modules are finite direct sums of 1-generated modules (i.e. modules which are isomorphic to a quotient of $\mathbb{Z}$).
The same holds more generally for PIDs: if R is a domain with principal ideals, then every finitely generated module over R is isomorphic to a module of the form $\bigoplus_{i=1}^k R/I_i$, where I_i is either the zero ideal (in which case $R/I_i \cong R$) or it is the ideal (r_i) of multiples of some element $r_i \neq 0$. Moreover one can adapt the table above, by replacing all instances of "$\mathbb{Z}$" with "R", and by reinterpreting R/a as $R/(a)$ (and similarly for b).

Example 3.10. Let $R = \mathbb{Q}[x]$ and consider the modules $\mathbb{Q}^\lambda$, which now we can call with a more proper name, that is, $\mathbb{Q}[x]/(x - \lambda)$. For $\lambda \neq \lambda'$ we already saw that

$$\mathrm{Hom}_{\mathbb{Q}[x]}\left(\mathbb{Q}[x]/(x - \lambda), \mathbb{Q}[x]/(x - \lambda')\right) \cong 0.$$

Now, using that $\mathbb{Q}[x]$ is a PID, we recover this fact from the above argument, as a consequence of the fact that $\gcd(x - \lambda, x - \lambda') = 1$, i.e. the two polynomials have no common factor of positive degree, or i.e. the ideal $(x-\lambda, x-\lambda') \subset \mathbb{Q}[x]$ generated by the two polynomials coincides with the entire $\mathbb{Q}[x]$. In a similar way we have

$$\mathbb{Q}[x]/(x-\lambda) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-\lambda') \cong 0.$$

In the previous example $\mathbb{Q}$ can be replaced by any field. But it is important to remember that only if R is a field the polynomial ring $R[x]$ is a PID.

3.5. **Exact sequences.** In this subsection we let R be a possibly non-commutative ring, and we focus on left R-modules. Everything can be adapted to right R-modules. In the case of an inclusion $N \hookrightarrow M$ of a submodule into a module, the cokernel coincides with the quotient module M/N from Definition 1.6. We then have three modules intertwined by two R-linear maps, as follows:

$$N \xrightarrow{\;i\;} M \xrightarrow{\;p\;} M/N,$$

where $i\colon N \to M$ is the inclusion and $p\colon M \to M/N$ is the projection.

Definition 3.11. A short exact sequence (SES) of R-modules is the datum of three modules M', M, M'' and two R-linear maps $i\colon M' \to M$ and $p\colon M \to M''$ with the following properties:

(1) i is injective;
(2) p is surjective;
(3) $\ker(p) = \mathrm{Im}(i)$; in particular this implies that $i \circ p$ is the zero map[6].

The previous definition extrapolates precisely what happens in the situation of N, M and M/N. The idea of a short exact sequence is to "decompose" the middle module M into two "smaller" modules M' and M''. Note however that the adjective *smaller* has a quite different meaning in the two cases: M' is smaller in that it is a submodule, M'' is smaller in that it is a quotient module.

Example 3.12. Let $f\colon S \to R$ be a homomorphism of rings, and let

$$M' \xrightarrow{\;i\;} M \xrightarrow{\;p\;} M''$$

be a sequence of three R-modules and two R-linear maps. Then we have a functor $f^*\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ by restrictions of scalars: every left R-module becomes a left S-module, and R-linear maps are automatically S-linear. Since the condition for being exact does not really involve the ring acting on our modules, we have that the previous sequence is exact of left R-modules if and only if it is exact of left S-modules. In fact, one can take $S = \mathbb{Z}$, which is the initial ring with unit, and thus consider the previous as a sequence of $\mathbb{Z}$-modules/abelian groups. Upshot: a short sequence of left R-modules is exact if and only if it is exact when considered as a short sequence of abelian groups.

The equality "$\ker(p) = \mathrm{Im}(i)$" occurring in Definition 3.11 is formally the same as we encountered when defining a presentation $F_1 \to F_0 \to M$ of an R-module M as a cokernel of an R-linear map between free modules. Since we encountered the same situation twice, we should make this into a definition.

[6]Since we consider left modules, homomorphisms act on right, so are also composed in the corresponding way. Here $i \circ p$ is the map such that $(m')i \circ p = ((m')i)p$ for all $m' \in M'$

Definition 3.13. An *exact sequence* of R-modules is the datum of an interval[7] $I \subset \mathbb{Z}$ and R-modules $(M_i)_{i\in I}$ and R-linear maps $(g_i\colon M_i \to M_{i-1})_{i,i-1\in I}$ such that whenever $i+1, i, i-1 \in I$ the following equality of submodules of M_i holds[8]: $\mathrm{Im}(g_{i+1}) = \ker(g_i)$.

An exact sequence has thus the form

$$\dots \xrightarrow{g_{i+2}} M_{i+1} \xrightarrow{g_{i+1}} M_i \xrightarrow{g_i} M_{i-1} \xrightarrow{g_{i-1}} \dots.$$

It has the property that the composition of every two consecutive maps $g_{i+1} \circ g_i$ is the zero map: this is equivalent to the containment $\mathrm{Im}(g_{i+1}) \subseteq \ker(g_i)$, whenever both terms are defined. But more holds: the last containment must be an equality. We can anticipate the definition of chain complex.

Definition 3.14. A *chain complex* of R-modules is the datum of R-modules $(M_k)_{k\in\mathbb{Z}}$ and R-linear maps $(g_k\colon M_k \to M_{k-1})_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$ the following containment of submodules of M_k holds: $\mathrm{Im}(g_{k+1}) \subseteq \ker(g_k)$. This condition is equivalent to the equality $g_{k+1} \circ g_k = 0$ as maps $M_{k+1} \to M_{k-1}$.

In fact Example 3.12 generalises to the fact that the functor $f^*\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ sends exact sequences to exact sequences, and it sends chain complexes to chain complexes.
We gave Definition 3.13 with generic intervals $I \subseteq \mathbb{Z}$ for the sake of generality, but one usually can reduce to the case $I = \mathbb{Z}$. We remark the following, where all capital letters are R-modules and 0 is the zero module, and where we omit the indices:

- $0 \to M \xrightarrow{g} M'$ is exact if and only if $g\colon M \to M'$ is injective, as the image of $0 \to M$ is 0;
- $M \xrightarrow{g} M' \to 0$ is exact if and only if $g\colon M \to M'$ is surjective, as the kernel of $M' \to 0$ is the entire M';
- $0 \to M \to 0$ is exact if and only if M is the trivial module, as M must inject into the trivial module (or equivalently, as the trivial module must surject onto M);
- $0 \to M \xrightarrow{g} M' \to 0$ is exact if and only if $g\colon M \to M'$ is an isomorphism, putting together the first two observations;
- $0 \to M' \xrightarrow{g'} M \xrightarrow{g} M'' \to 0$ is exact if and only if $M' \xrightarrow{g'} M \xrightarrow{g} M''$ is a short exact sequence in the sense of Definition 3.11.

Hopefully the last example explains why the word “short” has been assigned to exact sequences of length 5 with 0 at the beginning and at the end: all shorter situations, starting and ending with a 0, can be described completely with the words “zero module” and “isomorphism of modules”.

3.6. **Split short exact sequences.** Again, R is possibly non-commutative and we focus on R-modules, but the following can be repeated almost word by word for right R-modules.

[7] An interval is a subset $I \subset \mathbb{Z}$ with the property that for all $a \le b \le c \in \mathbb{Z}$, if $a, c \in I$, then also $b \in I$

[8] referring to the single equality at $i \in I$, we say that the sequence is exact “at M_i”

Example 3.15. Let M and N be two left R-modules. Then there is a short exact sequence

$$N \xrightarrow{\iota_N} N \oplus M \xrightarrow{\pi_M} M,$$

using the maps from Examples 1.21 and 1.24.

The previous SES has a special property: the surjective map $\pi_M \colon N \oplus M \to M$ admits a *section*, namely the R-linear map ι_M. By "section" we mean an R-linear map $s \colon M \to N \oplus M$ such that the composite $M \xrightarrow{s} N \oplus M \xrightarrow{\pi_M} M$ is the identity of M. The fact that π_M is surjective only implies that there is a section $s \colon M \to N \oplus M$ which is a map of sets. Finding a section which is also R-linear is much more difficult, and *not always possible* for a generic SES.

Definition 3.16. Let

$$M' \xrightarrow{i} M \xrightarrow{p} M''$$

be a short exact sequence of left R-modules. We say that it is a *split* short exact sequence if the surjective, R-linear map $p \colon M \to M''$ admits an R-linear section.

Being split is a property of a SES. The following proposition gives equivalent characterisations of split SES.

Proposition 3.17. *Let*

$$M' \xrightarrow{i} M \xrightarrow{p} M''$$

be a SES. Then the following are equivalent (each condition implies each of the other):

- *there exists an R-linear map $s \colon M'' \to M$, called* section*, such that $s \circ p = \mathrm{Id}_{M''}$;*
- *there exists an R-linear map $r \colon M \to M'$, called* retraction*, such that $i \circ r = \mathrm{Id}_{M'}$;*
- *there is an isomorphism $\Phi \colon M \to M' \oplus M''$ making the following diagram commute*

$$\begin{array}{ccccc} M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' \\ \downarrow{\scriptstyle \mathrm{Id}_{M'}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{Id}_{M''}} \\ M' & \xrightarrow{\iota_{M'}} & M' \oplus M'' & \xrightarrow{\pi_{M''}} & M'' \end{array}$$

I leave the proof of Proposition 3.17 as exercise. Constructing r from s or viceversa as maps of abelian groups is not difficult; checking that the obtained maps are indeed R-linear is an essential part of the proof.

Example 3.18. Let R be a field. Then every short exact sequence is split: given a basis $\mathcal{S}''$ of the vector space M'', we can define $s \colon M'' \to M$ by mapping each basis element $s'' \in \mathcal{S}''$ to some element in the preimage $p^{-1}(s'') \subset M$, and extending linearly over R. Check that the R-linear extension is indeed a section in the sense of Proposition 3.17.
The same holds when R is any ring and M'' is a free module: also in this case we have a basis with the same, needed formal properties to make the previous argument work.

In the previous example, note that in general each of the fibres $p^{-1}(s'')$ contains more than one element; thus there are in general more possibilities to choose a section s of $p\colon M \to M''$. In general, if $M' \to M \to M''$ is a split SES, it can be split "in many different ways", i.e. there can be multiple choices of a section witnessing the splitness. Similarly, there can be many retractions $r\colon M \to M'$, as soon as at least one exists.

Example 3.19. Let ℓ be a prime number and consider the SES of $\mathbb{Z}$-modules

$$\mathbb{Z}/\ell \xrightarrow{\iota} \mathbb{Z}/\ell^2 \xrightarrow{\pi} \mathbb{Z}/\ell,$$

where the first map sends $[m]_\ell \mapsto [m\ell]_{\ell^2}$ and the second map sends $[n]_{\ell^2}$ to $[n]_\ell$. Then the preimage $\pi^{-1}([1]_\ell)$ contains the elements $[\ell n + 1]_{\ell^2}$ for varying n, and all these elements don't vanish when multiplied by ℓ. As a result, there is only a set-theoretic section of the surjective map π, but not a $\mathbb{Z}$-linear section.
Compare now with the following SES

$$\mathbb{Z}/\ell \xrightarrow{\iota} \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell \xrightarrow{\pi} \mathbb{Z}/\ell,$$

which is a particular case of Example 3.15. The latter SES is split, and it has the same left and right term. Upshot: there can be SES with equal first and third term, and yet with very different behaviour (e.g. one is split, the other is not). This does not happen over a field.

Example 3.20. Let $\mathbb{F}$ be a field and consider $R = \mathbb{F}[x]$. Then we have a SES of $\mathbb{F}[x]$-modules

$$\mathbb{F}[x]/(x) \xrightarrow{\iota} \mathbb{F}[x]/(x^2) \xrightarrow{\pi} \mathbb{F}[x]/(x),$$

where the first map sends $[f(x)]_x \mapsto [xf(x)]_{x^2}$ and the second map sends $[g(x)]_{x^2} \mapsto [g(x)]_x$. We want to argue that there is no $\mathbb{F}[x]$-linear section of π; suppose by absurd that such $s\colon \mathbb{F}[x]/(x) \to \mathbb{F}[x]/(x^2)$ exists, and let $g(x)$ be a polynomial such that $s([1]_x) = [g(x)]_{x^2}$. Then since s is a section, we must have that $g(x)$ has constant term equal to 1, i.e. $g(x)$ is of the form $1 + xf(x)$ for some polynomial $f(x)$. We then have

$$[0]_{x^2} = s([0]_x) = s(x \cdot [1]_x) = x \cdot s([1]_x) = x \cdot [1 + xf(x)]_{x^2} = [x]_{x^2} \neq [0]_{x^2},$$

which is a contradiction. Thus the previous is a non-split SES of $\mathbb{F}[x]$-modules. Note however that if we consider it as a sequence of $\mathbb{F}$-modules, i.e. $\mathbb{F}$-vector spaces, then it is split. We can find a $\mathbb{F}$-linear section, but not a $\mathbb{F}[x]$-linear section.
Upshot: being a SES is a property of the underlying abelian groups; being split strongly depends on the ring we are working on.

4. Bimodules and additive functors

4.1. **Bimodules.** We can combine the notion of left and right module into a single notion of bimodule.

Definition 4.1. Let R and R' be two associative, unital rings. An $R-R'$-bimodule M is an abelian groups M endowed with a structure of left R-module and also with a structure of right R'-module, such that for all $r \in R$, $m \in M$ and $r' \in R'$ we have

$$r \cdot (m \cdot r') = (r \cdot m) \cdot r'.$$

The last condition is a form of compatibility between left and right multiplication, or a form of associativity of all types of multiplication.

Example 4.2. Each ring R is an $R-R$-bimodule, by using the ring multiplication in both cases. The compatibility of left and right scalar multiplication is precisely the associative property of the product of R.

Example 4.3. Let R be commutative. Then each R-module M is naturally an $R-R$-bimodule. The scalar multiplication $R \times M \to M$, which is traditionally expressed as a left scalar multiplication $(r, m) \mapsto r \cdot m$, gives rise to a map of sets $M \times R \to R$ just by setting $(m, r) \mapsto r \cdot m$. Commutativity of R ensures that the second map makes M into a right R-module; moreover the two maps together make M into an $R-R$-bimodule.

Example 4.4. Let R be non-commutative and M be a left R-module. Then $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ is an abelian group (pointwise sum of R-linear maps), but is also endowed with a binary, associative operation, given by composing functions: $(f, g) \mapsto f \circ g$. Check that the composition is distributive with respect to the sum, i.e. the following holds, for all $f, f', g \in \mathrm{End}_R(M)$:

- $(f + f') \circ g = f \circ g + f' \circ g$;
- $g \circ (f + f') = g \circ f + g \circ f'$.

Moreover Id_M is a left and right neutral element for $\circ$. It follows that $\mathrm{End}_R(M)$ is an associative ring with unit. Finally, we can consider M as a *right* module over this ring, by setting simply, for $m \in M$ and $f \in \mathrm{End}_R)(M)$

$$m \cdot f := (m)f.$$

The two structures on M combine to an $R-\mathrm{End}_R(M)$-bimodule structure on M.

Note that in the previous example it is very important to use the *same* ring R when introducing $\mathrm{End}_R(M)$. For instance, we can also consider M as an abelian group and thus as a right $\mathrm{End}_{\mathbb{Z}}(M)$-module, but then it is in general not true that M becomes an $R-\mathrm{End}_{\mathbb{Z}}(M)$-bimodule.

Example 4.5. Let M be a left R-module. Then the fact that M is an abelian group makes M into a $\mathbb{Z}$-module, and we can consider M as a $R-\mathbb{Z}$-bimodule.

Recall that in the non-commutative setting, in general, Hom_R and $\otimes_R$ only produce abelian groups. However, if we use bimodules as inputs, we usually get modules as outputs. See the following examples, where R and S are associative unital rings.

Example 4.6. Let M be an $R-S$-bimodule, and let M' be a left R-module. Let $\mathrm{Hom}_R(M, M')$ be the set of R-linear maps of left R-modules. Then $\mathrm{Hom}_R(M, M')$ is naturally a left S-module, by setting, for $s \in S$, $f \in \mathrm{Hom}_R(M, M')$ and $m \in M$,

$$(m)(s \cdot f) = (m \cdot s)f.$$

The map $s \cdot f$ is indeed R-linear, as for all $r \in R$ we have

$$r \cdot ((m)(s \cdot f)) = r \cdot ((m \cdot s)f) = (r \cdot (m \cdot s))f = ((r \cdot m) \cdot s)f = (r \cdot m)(s \cdot f).$$

Example 4.7. Let M' be an $R-S$-bimodule, and let M be a left R-module. Then $\mathrm{Hom}_R(M, M')$ is naturally a right S-module, by setting, for $s \in S$, $f \in \mathrm{Hom}_R(M, M')$ and $m \in M$,

$$(m)(f \cdot s) = (m \cdot f) \cdot s.$$

Check that the map $f \cdot s$ is indeed R-linear.

Example 4.8. Let S' be a third associative ring[9], and let M be a $S-R$-bimodule and M' be a $R-S'$-bimodule. Then the abelian group $M \otimes_R M'$ has a natural structure of $S-S'$-bimodule, as we argue in the following.
In order to give a left S-module structure on $M \otimes_R M'$, we need to define, for all $s \in S$, a map of abelian groups $s \cdot -\colon M \otimes_R M' \to M \otimes_R M'$. We wish to define this map on simple tensors $m \otimes m'$, which generate the abelian group $M \otimes_R M'$, by the formula

$$s \cdot (m \otimes m') = (s \cdot m) \otimes m' \in M \otimes_R M',$$

and then "extend $\mathbb{Z}$-linearly/additively". One has to check that the previous assignment really induces a map of abelian groups. If simple tensors were a basis of $M \otimes_R M'$ as free abelian group, this would be automatic; however, it is rather true that the elements (m, m') are a basis of a free abelian group $A = \bigoplus_{(m,m') \in M \times M'} \mathbb{Z}$, and $M \otimes_R M'$ is a quotient of A by a sub-abelian group B (see Definition 2.16. To be precise, we should therefore first define a homomorphism of abelian groups $s_*\colon A \to A$ by setting $s_*(m, m') = (s \cdot m, m')$ on the standard basis of A, and then check that this homomorphism descends to a homomorphism $s \cdot -\colon M \otimes_R M' \to M \otimes_R M'$ on the quotient. Among other things one has to check the equality, for all $r \in R$,

$$[s_*(m \cdot r, m')]_B = [s_*(m, r \cdot m')]_B.$$

By definition of s_* on basis elements, we have indeed

$$\begin{aligned} s_*(m \cdot r, m') = (s \cdot (m \cdot r), m') = (s \cdot m) \cdot r, m'); \\ s_*(m, r \cdot m') = (s \cdot m, r \cdot m'); \\ [(s \cdot m) \cdot r, m')]_B = [(s \cdot m, r \cdot m')]_B. \end{aligned}$$

The first equality follows from the axioms of $S-R$-bimodule on M; the third is a consequence of property (3) in Definition 2.16 applied to the elements $(s \cdot m) \in M$, $(m' \cdot s') \in M'$ and $r \in R$. This shows that for all $s \in S$ we get an additive map $s \cdot -\colon M \otimes_R M' \to M \otimes_R M'$. One then has to check that these maps fit together in a left S-module structure on $M \otimes_R M'$. For instance, one has to check that $s_1 s_2 \cdot -$ is the composition $(s_1 \cdot -) \circ (s_2 \cdot -)$. But to check that two maps of abelian groups $M \otimes_R M \to M \otimes_R M$ coincide it really suffices to check that they coincide on a generating set of $M \otimes_R M$ as abelian groups. For $m \in M$ and $m' \in M'$ we have indeed

$$\begin{aligned} (s_1 s_2) \cdot (m \otimes m') &= ((s_1 s_2) \cdot m) \otimes m' = (s_1 \cdot (s_2 \cdot m)) \otimes m' \\ &= s_1 \cdot ((s_2 \cdot m) \otimes m') = s_1 \cdot (s_2 \cdot (m \otimes m')). \end{aligned}$$

After that, one has to repeat the argument on the other side, and check that there is a right S'-module structure on $M \otimes_R M'$ satisfying, for all $m \in M$, $m' \in M'$ and $s' \in S'$,

$$(m \otimes m') \cdot s' = m \otimes (m' \cdot s') \in M \otimes_R M'.$$

And eventually one has to check that the two structures are compatible, i.e. one really get an $S-S'$-bimodule structure on $M \otimes_R M'$: again, the maps of abelian groups $(s \cdot -) \circ (- \cdot s')$ and $(- \cdot s') \circ (s \cdot -)$ agree on the generating set of $M \otimes_R M'$ as abelian group given by simple tensors (check this!), hence they are the same map of abelian groups $M \otimes_R M' \to M \otimes_R M'$.

[9] We sake the greatest generality; setting some of the rings to be $\mathbb{Z}$ yields more specific examples of the construction

The heuristic principle governing the previous examples is the following: let M and N be abelian groups with an action of R by scalar multiplication, such that $\mathrm{Hom}_R(M, N)$ is defined (action on the same side), or such that $M \otimes_R N$ is defined (action on opposite sides). Then every action of a ring S on either M or N, which is compatible with the R-action, should induce an action of S on $\mathrm{Hom}_R(M, N)$, or respectively on $M \otimes_R N$.

4.2. **Hom-tensor adjunction.** Let R be a ring, M be a right R-module, M' be a left R-module and P be an abelian group. On the one hand we can consider the abelian group

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R M', P);$$

on the other hand we have a left R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(M, P)$, using that M is a $\mathbb{Z} - R$-bimodule; this leads to a well-defined abelian group

$$\mathrm{Hom}_R\left(M', \mathrm{Hom}_{\mathbb{Z}}(M, P)\right).$$

Proposition 4.9. *In the above setting, there is a canonical bijection of abelian groups*

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R M', P) \cong \mathrm{Hom}_R\left(M', \mathrm{Hom}_{\mathbb{Z}}(M, P)\right).$$

In the following it is convenient to consider M and P as left $\mathbb{Z}$-modules, and thus let $\mathbb{Z}$-linear maps act on right. Given an additive map $f\colon M \otimes M' \to P$, one can define for each $m' \in M'$ an additive map

$${}_{m'}f\colon M \to P, \qquad m \mapsto (m \otimes m')f.$$

The assignment $m' \mapsto {}_{m'}f$ is R-linear: besides being additive (check it!) we have, for all $r \in R$

$$(m)(r \cdot {}_{m'}f) = (m \cdot r)_{m'}f = ((m \cdot r) \otimes m')f = (m \otimes (r \cdot m'))f = (m)_{r \cdot m'}f.$$

Thus we get an R-linear map $g\colon M' \to \mathrm{Hom}_{\mathbb{Z}}(M, P)$.
In the other direction, let $g\colon M' \to \mathrm{Hom}_{\mathbb{Z}}(M, P)$ be an R-linear map. Then the assignment

$$M \times M' \to P, \qquad (m, m') \mapsto (m)((m')g)$$

is an R-bilinear map out of $M \times M'$ (check it carefully!), hence it induces an additive map $f\colon M \otimes_R M' \to P$.

Exercise 4.10. Prove Proposition 4.9 by showing that the above constructions give inverse bijections of the sets $\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R M', P)$ and $\mathrm{Hom}_R\left(M', \mathrm{Hom}_{\mathbb{Z}}(M, P)\right)$, and checking that these bijections are compatible with the structure of abelian groups that these sets have.

A variation of the previous is the following: if M is an $S - R$-bimodule, M' is a left R-module and P is a left S-module, then $M \otimes_R M'$ is a left S-module, $\mathrm{Hom}_S(M, P)$ is a left R-module, and there is a canonical isomorphism of abelian groups

$$\mathrm{Hom}_S(M \otimes_R M', P) \cong \mathrm{Hom}_R(M', \mathrm{Hom}_S(M, P)).$$

Why is this subsection called Hom-tensor *adjunction*? Focusing on this second example, we note that an $S - R$-bimodule M gives rise to a functor

$$M \otimes_R -\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod},$$

sending an object M' to $M \otimes M'$, and an R-linear map $f\colon M'_1 \to M'_2$ to the S-linear map $\mathrm{Id}_M \otimes_R f\colon M \otimes_R M'_1 \to M \otimes_R M'_2$. We also note that an $S-R$-bimodule M gives rise to a functor

$$\mathrm{Hom}_R(M,-)\colon {}_S\mathrm{Mod} \to {}_R\mathrm{Mod},$$

sending an object P to $\mathrm{Hom}_S(M,P)$, and an S-linear map $g\colon P_1 \to P_2$ to the map $-\circ g\colon \mathrm{Hom}_S(M,P_1) \to \mathrm{Hom}_S(M,P_2)$. The previous discussion shows that the two functors are adjoint, in particular the functor $M \otimes_R -$ is left adjoint to the functor $\mathrm{Hom}_R(M,-)$.[10] But you have noticed that there is more to be said: the two naturally equivalent functors

$$\mathrm{Hom}_S(M \otimes_R -,-)\ ,\ \mathrm{Hom}_R(-,\mathrm{Hom}_S(M,-))\colon {}_R\mathrm{Mod}^{op} \boxtimes {}_S\mathrm{Mod} \to \mathrm{Set}$$

actually don't only land in the category of sets, but in the "richer" category of abelian groups.

4.3. **$\mathbb{Z}$-linear categories and additive categories.** In this subsection we focus on left modules; similar considerations hold for right modules.

A priori, for a (locally small) category $\mathcal{C}$ and two objects x, y in $\mathcal{C}$, the morphisms $\mathrm{Hom}_{\mathcal{C}}(x,y)$ only form a *set*. However, we saw that for two left R-modules M, N, the set $\mathrm{Hom}_R(M,N)$ has a natural structure of abelian group, by pointwise sum of functions.

Moreover, for three objects x, y, z in $\mathcal{C}$, the composition law is defined as a map of sets

$$\circ_{x,y,z}\colon \mathrm{Hom}_{\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z);$$

however, if M, N, P are left R-modules, the map

$$\circ_{M,N,P}\colon \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(N,P) \to \mathrm{Hom}_R(M,P)$$

is a $\mathbb{Z}$-bilinear map from a product of $\mathbb{Z}$-modules (abelian groups) to a $\mathbb{Z}$-module (abelian group): for example, for $f, f'\colon M \to N$ and $g\colon N \to P$ we have, for $m \in M$,

$$\begin{aligned}(m)((f+f')\circ g) = ((m)(f+f'))g = ((m)f + (m)f')g = ((m)f)g + ((m)f')g \\ = (m)f\circ g + (m)f'\circ g,\end{aligned}$$

i.e. $(f+f')\circ g = f\circ g + f'\circ g$. We abstract this in the following definition.

Definition 4.11. A category $\mathcal{C}$ is *enriched in abelian groups/ enriched in $\mathbb{Z}$-modules/ a $\mathbb{Z}$-linear category* if for all objects $x, y \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is endowed with a structure of abelian group, in such a way that all composition laws $\mathrm{Hom}_{\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z)$ are $\mathbb{Z}$-bilinear.

Example 4.12. The category ${}_R\mathrm{Mod}$ is enriched in $\mathbb{Z}$-modules. Any subcategory of ${}_R\mathrm{Mod}$ obtained by selecting some left R-modules, and taking all possible R-linear maps between the selected modules[11] is again a $\mathbb{Z}$-linear category. Example: the category ${}_R\mathrm{Mod}^{free}$ of free left R-modules and R-linear maps between free left R-modules, is a $\mathbb{Z}$-linear category.

[10] To be precise, checking that for all M' and P we have a bijection of sets $\mathrm{Hom}_S(M \otimes_R M', P) \cong \mathrm{Hom}_R(M', \mathrm{Hom}_S(M,P))$ does not suffice: one needs to check that these bijections, taken together, form a natural transformation of functors ${}_R\mathrm{Mod}^{op} \boxtimes {}_S\mathrm{Mod} \to \mathrm{Set}$.

[11] This type of subcategory is called a *full* subcategory

Example 4.13. If $\mathcal{C}$ is a $\mathbb{Z}$-linear category, then the opposite category $\mathcal{C}^{op}$ is also $\mathbb{Z}$-linear, by declaring the abelian group structure on the set $\mathrm{Hom}_{\mathcal{C}^{op}}(x, y)$ to be that of the set $\mathrm{Hom}_{\mathcal{C}}(y, x)$, leveraging on the fact that these sets are in bijection. The fact that compositions are $\mathbb{Z}$-bilinear is a consequence of Lemma 3.8.

Example 4.14. Let R be an associative ring. We define a category $\mathcal{C}_R$ with a single object $*$, and we set $\mathrm{Hom}_{\mathcal{C}_R}(*, *) = R$. The identity of $*$ is $1 \in R$; composition of morphisms is given by multiplication in R. Moreover, the additive structure of R is used to make $\mathrm{Hom}_{\mathcal{C}_R}(*, *)$ into an abelian group, and the axioms of ring ensure that the composition map $\mathrm{Hom}_{\mathcal{C}_R}(*, *) \times \mathrm{Hom}_{\mathcal{C}_R}(*, *) \to \mathrm{Hom}_{\mathcal{C}_R}(*, *)$ is $\mathbb{Z}$-bilinear. Hence $\mathcal{C}_R$ is a category enriched in abelian groups.

For a ring R, we already saw that the category ${}_R\mathrm{Mod}$ admits all products and coproducts, even infinite ones; we also saw that finite products are isomorphic to the corresponding finite coproducts (direct sums); in particular the terminal object of ${}_R\mathrm{Mod}$ is isomorphic to the initial object of ${}_R\mathrm{Mod}$.
Example 4.14 shows that these additional properties don't follows from being enriched in $\mathbb{Z}$-modules: the category $\mathcal{C}_R$, for example, admits no initial nor terminal object, and for instance there is no object in $\mathcal{C}_R$ with the properties of a categorical coproduct $* \sqcup *$ or of a categorical product $* \times *$. We will be interested in the existence of *finite* products and coproducts, so we give the following definition.

Definition 4.15. A category $\mathcal{C}$ is *additive* if it is enriched in $\mathbb{Z}$-modules and if, moreover, for all finite collection $(x_i)_{i\in I}$ of objects in $\mathcal{C}$, there exists a product $\prod_{i\in I} x_i$ in $\mathcal{C}$.

Exercise 4.16. Prove that the above definition is complete: show that if $\mathcal{C}$ is $\mathbb{Z}$-linear and admits all finite products, then it also admits all finite coproducts and these coincide with the products.
More precisely, if $(x_i)_{i\in I}$ is a finite collection of objects, and if $\left(\prod_{i\in I} x_i, (\pi_i)_{i\in I}\right)$ is a product, then we can define maps $\iota_j \colon x_j \to \prod_{i\in I} x_i$ for all $j \in I$, by using the universal property of the product and declaring $\pi_j \circ \iota_j = \mathrm{Id}_{x_j}$ and $\pi_i \circ \iota_j = 0$ for all $i \neq j$, where "0" is the neutral element in the abelian group $\mathrm{Hom}_{\mathcal{C}}(x_j, x_i)$.
Prove that $\left(\prod_{i\in I} x_i, (\iota_i)_{i\in I}\right)$ satisfies the universal property for being a coproduct of $(x_i)_{i\in I}$ in $\mathcal{C}$.
Hint: start by proving that a terminal object $t \in \mathcal{C}$ (an empty product) is also initial: if $x \in \mathcal{C}$ and $f \colon t \to x$, then $f = f \circ \mathrm{Id}_t = f \circ 0 = 0$; justify all steps of the previous chain of equalities.

In particular, in an additive category all finite products are isomorphic to the corresponding finite coproducts. Saying that ${}_R\mathrm{Mod}$ is additive acknowledges the existence of finite direct sums of left R-modules. Note that being an *additive* category is a *property* that a $\mathbb{Z}$-linear category may or may not have; instead, being $\mathbb{Z}$-linear is an additional structure put on a *category*.[12]

Example 4.17. If $\mathcal{C}$ is additive, then so is $\mathcal{C}^{op}$. Example 4.13 shows

[12] In fact, if a category is such that all finite products and coproducts exist and coincide, then it automatically becomes enriched in abelian monoids; as a consequence, one can define additive categories as those categories with the property that all finite products and coproducts exist and coincide, and such that the automatic enrichment in abelian monoids is in fact an enrichment in abelian groups.

4.4. **Additive functors.**

Definition 4.18. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ between $\mathbb{Z}$-linear categories is a *$\mathbb{Z}$-linear functor/ functor enriched in $\mathbb{Z}$-modules* if for all $x_1, x_2 \in \mathcal{C}$ the map of sets

$$F_{x_1,x_2}\colon \mathrm{Hom}_{\mathcal{C}}(x_1,x_2) \to \mathrm{Hom}_{\mathcal{C}'}(F(x_1),F(x_2))$$

is in fact a homomorphism of abelian groups. Especially when both $\mathcal{C}$ and $\mathcal{C}'$ are additive, one also says that F is an *additive functor*.

In the case in which $\mathcal{C}$ and $\mathcal{C}'$ are both additive, a $\mathbb{Z}$-linear functor has an additional property: it *preserves finite products and coproducts*.

Exercise 4.19. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a $\mathbb{Z}$-linear functor between *additive* categories. Let $(x_i)_{i\in I}$ be a finite collection of objects in $\mathcal{C}$, let $(\bar{y}, (\bar{\pi}_i)_{i\in I})$ be a categorical product of $(x_i)_{i\in I}$, and let $(\bar{z}, (\bar{\iota}_i)_{i\in I})$ be a categorical coproduct of $(x_i)_{i\in I}$.[13]
Then $(F(\bar{y}), (F(\bar{\pi}_i))_{i\in I})$ is a categorical product of the collection $(F(x_i))_{i\in I}$, and $(F(\bar{z}), (F(\bar{\iota}_i))_{i\in I})$ is a categorical coproduct of the collection $(F(x_i))_{i\in I}$.
For instance, let x be a zero object in $\mathcal{C}$, i.e. an object which is both initial and terminal. Let $x' = F(x) \in \mathcal{C}$. Since $\mathrm{Hom}_{\mathcal{C}}(x,x)$ is the zero abelian group, we have $\mathrm{Id}_x = 0 \in \mathrm{Hom}_{\mathcal{C}}(x,x)$. It follows that $\mathrm{Id}_{x'} = F(\mathrm{Id}_x) = F(0) = 0 \in \mathrm{Hom}_{\mathcal{C}'}(x',x')$: here we use that a functor sends identity morphisms to identity morphisms, and an additive functor sends 0 morphisms to 0 morphisms. Conclude that x' is a zero object in $\mathcal{C}'$: for all $y \in \mathcal{C}'$, any morphism $f\colon x' \to y$ can be written as $f \circ \mathrm{Id}_{x'} = f \circ 0 = 0$; and any morphism $f\colon y \to x'$ similarly vanishes.

Recall Example 4.13. If $\mathcal{C}$ and $\mathcal{C}'$ are $\mathbb{Z}$-linear categories, then $\mathcal{C}^{op}$ is also $\mathbb{Z}$-linear, so we can make sense of a *$\mathbb{Z}$-linear contravariant functor* from $\mathcal{C}$ to $\mathcal{C}'$, which is just a $\mathbb{Z}$-linear functor $F\colon \mathcal{C}^{op} \to \mathcal{C}'$.

Example 4.20. Let R be a commutative ring and consider the category ${}_R\mathrm{Mod}$ of R-modules. Consider the following three functors $F_1, F_2, F_3\colon {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$:

- $F_1(M) = R$ and $F_1(f\colon M \to N) = (\mathrm{Id}_R\colon R \to R)$;
- $F_2(M) = M \oplus M$ and $F_2(f\colon M \to N) = (f \oplus f\colon M \oplus M \to N \oplus N)$, where $f \oplus f$ maps (m_1, m_2) to $(f(m_1), f(m_2))$;
- $F_3(M) = M \otimes_R M$ and $F_3(f\colon M \to N) = (f \otimes_R f\colon M \otimes_R M \to N \otimes_R N)$.

Then F_2 is additive, as the map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M \oplus M, N \oplus N)$ given by $f \mapsto f \oplus f$ is a homomorphism of abelian groups, for all M, N. Instead neither F_1 nor F_3 are additive, at least for $R = \mathbb{Z}$: for example, setting $M = N = \mathbb{Z}$, the map

$$\mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}(M,N) \to \mathrm{Hom}_{\mathbb{Z}}(F_1(M), F_1(N)) \cong \mathbb{Z}$$

induced by F_1 is the map $\mathbb{Z} \to \mathbb{Z}$ sending all integers to 1, which is not a homomorphism of abelian groups; similarly the map

$$\mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}(M,N) \to \mathrm{Hom}_{\mathbb{Z}}(F_3(M), F_3(N)) \cong \mathbb{Z}$$

induced by F_2 is the map $\mathbb{Z} \to \mathbb{Z}$ sending $n \mapsto n^2$, at least after identifying $M \otimes_{\mathbb{Z}} M = N \otimes_{\mathbb{Z}} N = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ with $\mathbb{Z}$ in a canonical way. This map is not a homomorphism of abelian groups.

Example 4.21. Let R be any ring and let N be a left R-module. Then we have the following additive functors:

[13] Note that since $\mathcal{C}$ is additive, you can assume that $\bar{y}$ and $\bar{z}$ are *the same* object of $\mathcal{C}$; after all, they are isomorphic objects.

- $\mathrm{Hom}_R(N,-)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$;
- $\mathrm{Hom}_R(-,N)\colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$;
- $-\otimes_R N\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$.

Note that the first is a covariant functor out of ${}_R\mathrm{Mod}$, whereas the second is a contravariant functor out of ${}_R\mathrm{Mod}$, i.e. a functor out of ${}_R\mathrm{Mod}^{op}$. The third is a covariant functor out of the category Mod_R of right R-modules. Similarly, if N' is a right R-module, we obtain an additive functor

- $\mathrm{Hom}_R(N',-)\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$;
- $\mathrm{Hom}_R(-,N')\colon {\mathrm{Mod}_R}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$;
- $N'\otimes_R -\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$.

Using bimodules one can change the target categories with more interesting categories of modules over some other ring S.

Check carefully that the functors from Example 4.21 are additive. For example, let us check that the first functor $\mathrm{Hom}_R(N,-)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is additive. Let M, M' be left R-modules. We then have a map of sets

$$\alpha\colon \mathrm{Hom}_R(M,M') \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(N,M),\mathrm{Hom}_R(N,M')), \qquad f \mapsto (g \mapsto g\circ f).$$

If we evaluate $\alpha(f_1+f_2)$, we obtain the map $g \mapsto g\circ(f_1+f_2)$, which coincides with the sum of the maps $g\mapsto g\circ f_1$ and $g \mapsto g\circ f_2$. So α is in fact a homomorphism of abelian groups, i.e. the first functor is additive.

4.5. **Exact functors.** In this section we focus for simplicity on left modules, and consider functors $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ for two rings R and S; but similar considerations could be done for right modules, or one can consider functors from a category of left modules to a category of right modules. As we will see, what we really need are categories where there is a well-behaved notion of "kernel" of morphisms. We will expand in this abstract direction in a future lecture.
Consider an exact sequence of left R-modules

$$\dots \xrightarrow{g_{i+2}} M_{i+1} \xrightarrow{g_{i+1}} M_i \xrightarrow{g_i} M_{i-1} \xrightarrow{g_{i-1}} \dots.$$

and let $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be an additive functor. Then applying F to the sequence above we obtain a sequence

$$\dots \xrightarrow{F(g_{i+2})} F(M_{i+1}) \xrightarrow{F(g_{i+1})} F(M_i) \xrightarrow{F(g_i)} F(M_{i-1}) \xrightarrow{F(g_{i-1})} \dots.$$

Recall that the condition $\mathrm{Im}(g_{i+1}) \subseteq \ker(g_i)$ can be expressed by the equality $g_{i+1}\circ g_i = 0$. Applying F, which is additive, we obtain $F(g_{i+1})\circ F(g_i) = F(0) = 0$, which can be reformulated as $\mathrm{Im}(F(g_{i+1})) \subseteq \ker(F(g_i))$, i.e. the second sequence is a chain complex. Can we also use that F is additive to prove that the second sequence is in fact an exact sequence? In other words, can we use that F is additive to prove the other containment $\mathrm{Im}(F(g_{i+1})) \supseteq \ker(F(g_i))$? *Unfortunately, no.* The fact that an additive functor in general only transforms exact sequences into chain complexes is one of the reason for the existence of homological algebra.

Example 4.22. Consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{[-]_2} \mathbb{Z}/2 \longrightarrow 0$$

If we apply the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}/2$, we obtain a short exact sequence of the form

$$0 \longrightarrow \mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow \mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2\otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow 0.$$

Using the table from Subsection 3.4 we can replace the above with

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

It doesn't take long to prove that the left map $\mathbb{Z}/2 \to \mathbb{Z}/2$ is the zero map, whereas the right map $\mathbb{Z}/2 \to \mathbb{Z}/2$ is an isomorphism. However, it should be clear that the last short sequence cannot be exact: otherwise there would be a surjection of $\mathbb{Z}/2$ onto $\mathbb{Z}/2$ with kernel isomorphic to $\mathbb{Z}/2$, and this is impossible by simply counting cardinalities.
This shows that an additive functor can indeed break the exactness, returning only a chain complex.

Example 4.23. The SES in the previous example is not split, and this is not a case. Indeed, if $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ is an additive functor, and if

$$0 \longrightarrow M' \xrightarrow{\ i\ } M \xrightarrow{\ p\ } M'' \longrightarrow 0$$

is a *split* short exact sequence of left R-modules, then one can use any R-linear section $s\colon M'' \to M$ to exhibit M as the direct sum $M' \oplus M''$, using that the map $i \oplus s\colon M' \oplus M'' \to M$ given by $(m', m'') \mapsto i(m') + s(m'')$ is an isomorphism. Since F is additive, we have a natural isomorphism $F(M' \oplus M'') \cong F(M') \oplus F(M'')$; moreover $F(i \oplus s)$ is an isomorphism between $F(M' \oplus M'')$ and $F(M)$. It follows that

$$0 \longrightarrow F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'') \longrightarrow 0$$

is a *split* exact sequence, with an example of section of $F(p)$ given by $F(s)$.

Motivated by the previous, we give a definition.

Definition 4.24. An additive functor $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ is *exact* if it sends exact sequences to exact sequences.

For instance, the functor F_2 from Example 4.20 is exact: check that if

$$\dots \xrightarrow{g_{i+2}} M_{i+1} \xrightarrow{g_{i+1}} M_i \xrightarrow{g_i} M_{i-1} \xrightarrow{g_{i-1}} \dots$$

is exact, then also the following sequence is exact

$$\dots \xrightarrow{g_{i+2}\oplus g_{i+2}} M_{i+1} \oplus M_{i+1} \xrightarrow{g_{i+1}\oplus g_{i+1}} M_i \oplus M_i \xrightarrow{g_i\oplus g_i} M_{i-1} \oplus M_{i-1} \xrightarrow{g_{i-1}\oplus g_{i-1}} \dots.$$

Another example of exact functor is, for a map of rings $f\colon R \to S$, the restriction of scalars functor $f^*\colon {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$, as in fact we already checked last time.
We will prove next week the following proposition (but you can try to prove it as an exercise).

Proposition 4.25. *Let $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be an additive functor that sends short exact sequences to short exact sequences. Then F is an exact functor.*

In fact, all examples of additive functors from Example 4.21 are not exact functors for a generic ring R (if R is a field, all these functors are exact and life is wonderful). However, these functors fail very little from being exact, especially when evaluated at exact sequences of the form $0 \to M' \to M \to M''$ or $M' \to M \to M'' \to 0$:

(1) if F is one of the following covariant functors
   - $\mathrm{Hom}_R(N,-)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$;
   - $\mathrm{Hom}_R(N',-)\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$;

   then an exact sequence in the source category of the form $0 \to M' \to M \to M''$ is sent to the sequence $0 = F(0) \to F(M') \to F(M) \to F(M'')$ which turns out to be exact at $F(M')$ and at $F(M)$;

(2) if F is one of the following contravariant functors
   - $\mathrm{Hom}_R(-,N)\colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$;
   - $\mathrm{Hom}_R(-,N')\colon {\mathrm{Mod}_R}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$;

   then an exact sequence in the source category of the form $M' \to M \to M'' \to 0$ is sent to the sequence $0 = F(0) \to F(M'') \to F(M) \to F(M')$ which turns out to be exact at $F(M'')$ and at $F(M)$;

(3) if F is one of the following covariant functors
   - $N' \otimes_R -\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$.
   - $- \otimes_R N\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$.

   then an exact sequence in the source category of the form $M' \to M \to M'' \to 0$ is sent to the sequence $F(M') \to F(M) \to F(M'') \to F(0) = 0$ which turns out to be exact at $F(M)$ and at $F(M'')$.

Exercise 4.26. Prove the previous statements. For example, let us prove that $- \otimes_R N\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$ sends an exact sequence of right R-modules of the form

$$M' \xrightarrow{g'} M \xrightarrow{g} M'' \longrightarrow 0$$

to an exact sequence of abelian groups

$$M' \otimes_R N \xrightarrow{g' \otimes_R \mathrm{Id}_N} M \otimes_R N \xrightarrow{g \otimes_R \mathrm{Id}_N} M'' \otimes_R N \longrightarrow 0.$$

First, we have to check that $g \otimes_R \mathrm{Id}_N$ is surjective. This follows from recalling that $M'' \otimes_R N$ is generated as abelian group by the simple tensors $m'' \otimes n$; using that g is surjective (and using also $\mathrm{Id}_N\colon N \to N$ is surjective!), each simple tensor $m'' \otimes n$ can be written as the image along $g \otimes_R \mathrm{Id}_N$ of a simple tensor in $M \otimes_R N$. Thus the image of $g \otimes_R \mathrm{Id}_N$ contains generators of $M'' \otimes_R N$ as abelian groups, hence $g \otimes_R \mathrm{Id}_N$ is surjective.
Next, we can leverage on the fact that $- \otimes_R N\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$ is an additive functor and get the containment $\mathrm{Im}(g' \otimes_R \mathrm{Id}_N) \subseteq \ker(g \otimes_R \mathrm{Id}_N)$.
Finally, we have to prove the inclusion $\mathrm{Im}(g' \otimes_R \mathrm{Id}_N) \supseteq \ker(g \otimes_R \mathrm{Id}_N)$. At this point it helps to remember Definition 2.16. The tensor product $M \otimes_R N$ can be constructed as a quotient of the free abelian group $A_{M,N} = \bigoplus_{(m,n)\in M\times N} \mathbb{Z}$, with basis the set $M \times N$, by a certain subgroup $B_{M,N}$. Similarly, $M'' \otimes_R N$ can be constructed as a quotient of the free abelian group $A_{M'',N}$ by a subgroup $B_{M'',N}$. The following composition of maps of abelian groups is surjective, as both of them are surjective (for the first, use that g is surjective)

$$A_{M,N} \xrightarrow{g\times \mathrm{Id}_N} A_{M'',N} \xrightarrow{p_{M'',N}} M'' \otimes_R N,$$

where $g \times \mathrm{Id}_N\colon A_{M,N} \to A_{M'',N}$ is defined on basis elements by $(m,n) \mapsto (g(m),n)$, and $p_{M'',N}\colon A_{M'',N} \to A_{M'',N}/B_{M'',N}$. It follows that the kernel of the composite $\psi := p_{M'',n} \circ (g \times \mathrm{Id}_N)$ can be generated as a sub-abelian group of $A_{M,N}$ by the union $\mathcal{S} \cup \mathcal{S}'$, where $\mathcal{S}$ and $\mathcal{S}'$ are well-chosen subsets of $A_{M,N}$:

- we choose a set $\mathcal{S} \subset A_{M,N}$ generating the kernel of $g \times \mathrm{Id}_N \colon A_{M,N} \to A_{M'',N}$;
- we choose a set $\mathcal{S}' \subset A_{M,N}$ such that the image $(g \times \mathrm{Id}_N)(\mathcal{S}') \subset A_{M'',N}$ generates the kernel of $p_{M'',N}$, that is, $B_{M'',N}$.

We set $\mathcal{S}$ to be the set of all differences $(m+g'(m'),n)-(m,n)$ for varying $m \in M$, $m' \in M'$ and $n \in N$: here we use that two elements of M sent along g to the same element of M'' have the form m and $m+g'(m')$ for some $m \in M$ and $m' \in M'$, because $\mathrm{Im}(g') = \ker(g)$.
We set $\mathcal{S}'$ to be a set of generators of $B_{M,N}$; check that indeed $B_{M,N}$ surjects onto $B_{M'',N}$, using that $g \colon M \to M''$ is surjective, and the description of generators of both abelian groups from Definition 2.16.
Now take an element $x \in M \otimes_R N$ with $x \in \ker(g \otimes \mathrm{Id}_N)$, and lift x to an element $y \in A_{M,N}$ along $p_{M,N} \colon A_{M,N} \to M \otimes_R N$, i.e. choose y such that $p_{M,N}(y) = x$. The hypothesis on x ensures that $\psi(y) = 0$. It follows that y can be generated $\mathbb{Z}$-linearly using the elements of $\mathcal{S}$ and of $\mathcal{S}'$. It follows that x can be generated $\mathbb{Z}$-linearly using the elements of $p_{M,N}(\mathcal{S})$ and of $p_{M,N}(\mathcal{S}')$. Now note that $p_{M,N}(\mathcal{S}') = \{0\}$, whereas $p_{M,N}(\mathcal{S})$ is the set of elements of $M \otimes_R N$ of the form $(m+g'(m')) \otimes n - m \otimes n$, for varying $m \in M$, $m' \in M'$ and $n \in N$; the previous element can be written as $g'(m) \otimes n$ in $M \otimes_R N$, i.e. it is a simple tensor in the image of $g' \otimes_R \mathrm{Id}_N$. As a result, x is generated by simple tensors in the image of $g' \otimes_R \mathrm{Id}_N$, which suffices to ensure $x \in \mathrm{Im}(g' \otimes \mathrm{Id}_N)$. The conclusion is that $\ker(g \otimes_R \mathrm{Id}_N) \subseteq \mathrm{Im}(g' \otimes \mathrm{Id}_N)$.

Motivated by the previous discussion, we give a definition.

Definition 4.27. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories of left or of right modules over a ring, and let $F \colon \mathcal{C} \to \mathcal{C}'$ be a covariant or a contravariant functor. We say that F is *right exact* if either of the following holds:

- F is covariant and sends exact sequences of the form $M' \to M \to M'' \to 0$ to exact sequences of the form $F(M') \to F(M) \to F(M'') \to 0$;
- F is contravariant and sends exact sequences of the form $0 \to M' \to M \to M''$ to exact sequences of the form $F(M'') \to F(M) \to F(M') \to 0$.

We say that F is *left exact* if either of the following holds:

- F is covariant and sends exact sequences of the form $0 \to M' \to M \to M''$ to exact sequences of the form $0 \to F(M') \to F(M) \to F(M'')$;
- F is contravariant and sends exact sequences of the form $M' \to M \to M'' \to 0$ to exact sequences of the form $0 \to F(M'') \to F(M) \to F(M')$.

The convention in the terminology is that a right exact functor produces exact sequences in which "0" is on right; and similarly for a left exact functor.

Example 4.28. An additive functor may be neither left nor right exact. For example, let R be commutative, let N be an R-module, and take the covariant functor $F \colon {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$ given by $F(M) = M \otimes_R N \oplus \mathrm{Hom}_R(N, M)$. Then F is additive (in general, a direct sum of additive functors is additive), but it is in general neither left exact nor right exact.

Thinking again of the functors from Example 4.21, it would be nice to identify those left R-modules N for which any of the functors $\mathrm{Hom}_R(N, -)$, $\mathrm{Hom}_R(-, N)$ and $- \otimes_R N$ is an exact functor.

Definition 4.29. A left R-module N is *projective* if the following functor is exact

$$\mathrm{Hom}_R(N,-)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}.$$

A left R-module N is *injective* if the following functor is exact

$$\mathrm{Hom}_R(-,N)\colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}.$$

A left R-module N is *flat* if the following functor is exact

$$-\otimes_R N\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}.$$

Similarly, one defines projective, injective and flat right R-modules.

Example 4.30. Consider R as an R–R-bimodule. We can consider $\mathrm{Hom}_R(R,-)$ as a functor ${}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$, and this functor is naturally isomorphic to the identity functor of the category ${}_R\mathrm{Mod}$, which is an exact functor. The forgetful functor ${}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, sending a module to its underlying abelian group, can be seen as a particular case of restriction of scalars, corresponding to the unique map of rings $\mathbb{Z} \to R$; in particular, this forgetful functor is also exact. The composition is the functor $\mathrm{Hom}_R(R,-)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, which then is also exact: this is the functor that one immediately obtains by considering R only as a left R-module. It follows that R is a projective left R-module.
Consider now again R as an $R-R$-bimodule. Use that $-\otimes R$ can be considered as a functor $\mathrm{Mod}_R \to \mathrm{Mod}_R$, and as such it is naturally isomorphic to the identity functor of Mod_R. composing with a forgetful functor as above, we conclude that R is a flat left R-module.

We will see more examples next time.

5. Projective, injective, flat modules

Also in this lecture, we focus on categories of left modules, but all results can be adapted to right modules as well.

5.1. **Proof of Proposition 4.25.** Let $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be a (covariant) additive functor sending short exact sequences to short exact sequences: that is, for every short exact sequence of left R-modules

$$0 \longrightarrow M' \xrightarrow{\ i\ } M \xrightarrow{\ p\ } M'' \longrightarrow 0$$

also the following is a short exact sequence, of left S-modules

$$0 \longrightarrow F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'') \longrightarrow 0.$$

Consider now a generic exact sequence of left R-modules

$$\dots \xrightarrow{g_{j+2}} M_{j+1} \xrightarrow{g_{j+1}} M_j \xrightarrow{g_j} M_{j-1} \xrightarrow{g_{j-1}} \dots$$

We can write kernel and image of every map, and expand the diagram as follows

$$\begin{array}{ccccccccc}
\dots & = & \ker(g_{j+1}) & & \mathrm{Im}(g_j) & = & \ker(g_{j-1}) & & \dots \\
 & & \downarrow{\scriptstyle i_{j+1}} & & \uparrow{\scriptstyle p_j} & & \downarrow{\scriptstyle i_{j-1}} & & \\
\dots & \xrightarrow{g_{j+2}} & M_{j+1} & \xrightarrow{g_{j+1}} & M_j & \xrightarrow{g_j} & M_{j-1}, & \xrightarrow{g_{j-1}} & \dots \\
 & & \downarrow{\scriptstyle p_{j+1}} & & \uparrow{\scriptstyle i_j} & & \downarrow{\scriptstyle p_{j-1}} & & \\
\dots & & \mathrm{Im}(g_{j+1}) & = & \ker(g_j) & & \mathrm{Im}(g_{j-1}) & = & \dots
\end{array}$$

Note that every column is a short exact sequence of left R-modules: the map i_j is the inclusion of $\ker(g_j)$ into M_j, and the map p_j is the same as the map g_j, but considering $\mathrm{Im}(g_j) \subseteq M_{j-1}$ as target. Equalities are meant as equalities of submodules, and are guaranteed by the exactness. Every square is commutative. Now we apply F and obtain a diagram of left S-modules

$$\begin{array}{ccccccccc}
\dots & = & F(\ker(g_{j+1})) & & F(\mathrm{Im}(g_j)) & = & F(\ker(g_{j-1})) & & \dots \\
 & & \downarrow{\scriptstyle F(i_{j+1})} & & \uparrow{\scriptstyle F(p_j)} & & \downarrow{\scriptstyle F(i_{j-1})} & & \\
\dots & \xrightarrow{F(g_{j+2})} & F(M_{j+1}) & \xrightarrow{F(g_{j+1})} & F(M_j) & \xrightarrow{F(g_j)} & F(M_{j-1}), & \xrightarrow{F(g_{j-1})} & \dots \\
 & & \downarrow{\scriptstyle F(p_{j+1})} & & \uparrow{\scriptstyle F(i_j)} & & \downarrow{\scriptstyle F(p_{j-1})} & & \\
\dots & & F(\mathrm{Im}(g_{j+1})) & = & F(\ker(g_j)) & & F(\mathrm{Im}(g_{j-1})) & = & \dots
\end{array}$$

The hypothesis on F ensures that every column of the last diagram is a short exact sequence. Since F is a functor, we also obtain that equalities of objects keep being equalities of objects, and commutative squares are sent to commutative squares. The map $F(g_j)\colon F(M_j) \to F(M_{j-1})$ is written as a composition $F(p_j) \circ F(i_{j-1})$ of a (surjective) map $F(p_j)$ and an injective map $F(i_j)$. It follows that $\ker(F(g_j)) = \ker(F(p_j)) \subseteq M_j$. By exactness of the middle column, we also have $\ker(F(p_j)) = \mathrm{Im}(F(i_j) \subseteq M_j$. Finally, we also have a factorisation of $F(g_{j+1})\colon F(M_{j+1}) \to F(M_j)$ as a composition of a surjective map $F(p_{j+1})$ and an (injective) map $F(i_j)$. It follows that $\mathrm{Im}(F(i_j)) = \mathrm{Im}(F(g_{j+1})) \subseteq M_j$. Putting together all equalities of submodules of M_j, we obtain that the sequence is exact at M_j.

5.2. **Projective modules.** A left R-module N is projective if the additive and left exact functor $\mathrm{Hom}_R(N, -)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is in fact an exact functor (i.e. it is also right exact).

Example 5.1. Let $N = \bigoplus_{i \in I} R$ be a free left R-module. We want to check that N is projective, i.e. the functor $\mathrm{Hom}_R(N, -)$ is exact. Let $M' \xrightarrow{i} M \xrightarrow{p} M''$ be a SES of left R-modules. Then the sequence

$$0 \to \mathrm{Hom}_R(N, M') \xrightarrow{\mathrm{Hom}_R(N,i)} \mathrm{Hom}_R(N, M) \xrightarrow{\mathrm{Hom}_R(N,p)} \mathrm{Hom}_R(N, M'') \to 0$$

is exact at $\mathrm{Hom}_R(N, M')$ and at $\mathrm{Hom}_R(N, M)$, and we want to prove that it is also exact at $\mathrm{Hom}_R(N, M'')$, i.e. the map $\mathrm{Hom}_R(N, p)$ is surjective. This means that for any R-linear map $(f\colon N \to M'') \in \mathrm{Hom}_R(N, M'')$ we want to prove the existence of an R-linear map $(g\colon N \to M) \in \mathrm{Hom}_R(N, M)$ such that $f = g \circ p$. Recall that p is surjective: we can therefore choose elements $m_i \in M$, for all $i \in I$, such that $p(m_i) = f(\iota_i(1)) \in M''$. The universal property of direct sum allows us to define an R-linear map $g\colon N \to M$ by declaring $g(\iota_i(1)) = m_i$. And now: is it true that this g does the job? Is it true that $f = g \circ p$? Let $N' \subseteq N$ be the subset of all $n \in N$ for which $(n)f = (n)g \circ p$; then N' is a sub-R-module of N and contains the basis of elements $\iota_i(1)$, so we must have $N' = N$.

For example, if R is a field $\mathbb{F}$, then every $\mathbb{F}$-vector space N is free (admits a basis), and hence it is projective. This also follows from the fact that every SES is split over a field, hence the additive functor $\mathrm{Hom}_{\mathbb{F}}(N, -)$ sends SES (i.e. split SES) to SES (i.e split SES).

Example 5.2. Let R be a ring, $F = \bigoplus_{i\in I} R$ be a free left R-module, and suppose that N_1 and N_2 are submodules of F such that the inclusions $\iota_1 \colon N_1 \to F$ and $\iota_2 \colon N_2 \to F$ exhibit F as isomorphic to the direct sum $N_1 \oplus N_2$. This can happen, for instance, if we split $I = I_1 \sqcup I_2$ and set $N_1 = \bigoplus_{i\in I_1} R$ and $N_2 = \bigoplus_{i\in I_2} N_2$; but we will see soon that more weird examples exist. We want to show that N_1 (and similarly N_2) is projective.

Let $M' \xrightarrow{i} M \xrightarrow{p} M''$ be a SES of left R-modules. As in Example 5.1, it suffices to check that the sequence

$$0 \to \mathrm{Hom}_R(N_1, M') \xrightarrow{\mathrm{Hom}_R(N_1,i)} \mathrm{Hom}_R(N_1, M) \xrightarrow{\mathrm{Hom}_R(N_1,p)} \mathrm{Hom}_R(N_1, M'') \to 0$$

is exact at $\mathrm{Hom}_R(N_1, M'')$, i.e. that the map $\mathrm{Hom}_R(N_1, p)$ is surjective. Let then $(f\colon N_1 \to M'') \in \mathrm{Hom}_R(N_1, M'')$ be an R-linear map. We can use the universal property of F as direct sum $N_1 \oplus N_2$ to extend f to an R-linear map $f'\colon F \to M''$: we define f' by declaring its restriction on N_1 to be f, and its restriction on N_2 to be the zero map (or any other R-linear map $N_2 \to M''$ of your choice). Now we use the argument from Example 5.1 to define a map $g\colon F \to M$ such that $f' = g \circ p\colon F \to M''$. Finally, we restrict to N_1 and obtain

$$f = \iota_1 \circ f' = \iota_1 \circ (g \circ p) = (\iota_1 \circ g) \circ p = \mathrm{Hom}_R(N_1, p)(\iota_1 \circ g) \in \mathrm{Hom}_R(N_1, M'').$$

This shows that $\mathrm{Hom}_R(N_1, p)$ is surjective.

Both in Example 5.1 and 5.2 we have used left exactness of $\mathrm{Hom}_R(N, -)$ to quickly reduce to a question about surjectivity of a map. Generalising this idea, we get the following characterisation of a projective left R-module.

Lemma 5.3. *Let N be a left R-module. Then N is projective if and only if for every surjective map of left R-modules $p\colon M \to M''$ and for every R-linear map $f\colon N \to M''$ there exists an R-linear map $g\colon N \to M$ such that $f = g \circ p$.*

Proof. If N is projective, then given $p\colon M \to M''$ we can construct a SES $\ker(p) \to M \xrightarrow{p} M''$. Applying the exact functor $\mathrm{Hom}_R(N, -)$, we in particular obtain that the map $\mathrm{Hom}_R(N, p) = - \circ p\colon \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'')$ is surjective, and this is exactly the condition that for every f there is g with the required property. Viceversa, if N satisfies the property involving f and g, then given a SES $M' \xrightarrow{i} M \xrightarrow{p} M''$ of left R-modules, we have in particular that p is surjective; therefore the induced map $\mathrm{Hom}_R(N, p)\colon \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'')$ is surjective. We then have that the sequence

$$0 \to \mathrm{Hom}_R(N, M') \xrightarrow{\mathrm{Hom}_R(N,i)} \mathrm{Hom}_R(N, M) \xrightarrow{\mathrm{Hom}_R(N,p)} \mathrm{Hom}_R(N, M'') \to 0$$

is exact: it is exact at $\mathrm{Hom}_R(N, M'')$ as we just saw, and it is exact at $\mathrm{Hom}_R(N, M')$ and $\mathrm{Hom}_R(N, M)$ because $\mathrm{Hom}_R(N, -)$ is a left exact functor. □

Example 5.4. Let $R = \mathbb{Z}$ and let $N = \mathbb{Z}/2$. Then the map $\mathbb{Z} \xrightarrow{[-]_2} \mathbb{Z}/2$ is surjective; however the induced map $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is not surjective, since it has the 0 abelian group as source and an abelian group isomorphic to $\mathbb{Z}/2$ as target. Therefore $\mathbb{Z}/2$ is not projective, as it does not satisfy the characterisation from Lemma 5.3.

The previous example is of course in contrast with what happens over a field: every module is free, and hence projective.

Example 5.5. Let $R = \mathbb{Z}/6$, and consider $\mathbb{Z}/2$ as an R-module: you can think of $\mathbb{Z}/2$ as being $R/([2]_6)$, i.e. the quotient of the ring by an ideal. Similarly, consider $\mathbb{Z}/3$ as an R-module. The chinese reminder theorem implies that the map

$$\mathbb{Z}/6 \to \mathbb{Z}/2 \times \mathbb{Z}/3, \qquad [n]_6 \mapsto ([n]_2, [n]_3)$$

is bijective. Note that this map is also R-linear, and note that $\mathbb{Z}/2 \times \mathbb{Z}/3$ is equal to the direct sum $\mathbb{Z}/2 \oplus \mathbb{Z}/3$. It follows that both $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective R-modules.

The previous example shows that it is possible to have projective modules that are not free: indeed the cardinality of a free $\mathbb{Z}/6$-module is either a power of 6 or infinite.

Example 5.6. Let $\mathbb{F}$ be a field and consider $\mathbb{F}[x]/(x)$ as a module over the following rings:

- $\mathbb{F}[x]/(x)$ is free over $\mathbb{F}$, hence it is projective;
- $\mathbb{F}[x]/(x)$ is not free over $\mathbb{F}[x]$: for instance, because scalar multiplication by x is injective on any free $\mathbb{F}[x]$-module; try to prove that $\mathbb{F}[x]/(x)$ is not free over $\mathbb{F}[x]$, or see in the following either Proposition 5.8 or Theorem 5.10;
- $\mathbb{F}[x]/(x)$ is projective but not free over $\mathbb{F}[x](x^2 - x)$: use that a non-zero, free $\mathbb{F}[x](x^2 - x)$-module admits elements that are not sent to 0 by the scalar multiplication by $[x]_{x^2-x}$, and use the chinese reminder theorem to show that $\mathbb{F}[x](x^2 - x) \cong \mathbb{F}[x]/(x) \oplus \mathbb{F}[x]/(x - 1)$ as $\mathbb{F}[x](x^2 - x)$-modules.

Here is a non-commutative example.

Example 5.7. Let $R = \mathrm{Mat}_{k\times k}(\mathbb{F})$ be the ring of $k \times k$ matrices with coefficients in a field $\mathbb{F}$, and let $M = \mathbb{F}^k$, considered as a left R-module. For each $1 \le i \le k$ we can define $R_i \subset R$ to be the sub-$\mathbb{F}$-vector space of those matrices whose entries outside the i^{th} column vanish (whereas we allow any entries in $\mathbb{F}$ on the i^{th} column). Then the inclusions $\iota_i\colon R_i \to R$ exhibit R as the direct sum of $\mathbb{F}$-vector spaces $R_1 \oplus \cdots \oplus R_k$. In fact every R_i is a left R-submodule of R, so there is an isomorphism of left R-modules

$$R \cong R_1 \oplus \cdots \oplus R_k;$$

moreover each R_i is isomorphic to M as a left R-module. It follows that M is a projective left R-module.

All examples of projective R-modules M seen so far follow the pattern of Example 5.2, is this a chance? No!

Proposition 5.8. *Let N be a left R-module. Then N is projective if and only if N is isomorphic to a direct summand of a free left R-module.*

Proof. Example 5.2 shows that a direct summand of a free module is projective. Viceversa, let N be projective, and choose any R-linear surjective map $p\colon F \to N$ from a (sufficiently big) free left R-module. Lemma 5.3 can be applied to $f = \mathrm{Id}_N\colon N \to N$: since p is surjective, there exists $g\colon N \to F$ such that $f = g \circ p$. The SES $\ker(p) \to F \xrightarrow{p} N$ is thus split, because g is an example of a section of p. It follows from Proposition 3.17 that F is isomorphic to the direct sum $\ker(p) \oplus N$, and thus both $\ker(p)$ and N are direct summands of a free module. □

It is already the second time that we use an argument involving, for a given R-module M, the existence of a surjective R-linear map $F \to M$ from a free module F. In fact, this property of the category of left R-modules will be used very often: more precisely, in many situations we will have an R-module M and we will have to invoke the existence of a *projective* module N with a surjective R-linear map $N \to M$. The fact that N can be taken to be even free will be often irrelevant.

Notation 5.9. We will say that each of the categories ${}_R\mathrm{Mod}$ and Mod_R "has enough projectives" to mean that every module M in either of the categories receives a surjective R-linear map from some projective R-module.

Finally, we mention the following theorem about modules over a PID, leaving the proof as an exercise (or see [Rot, Corollary 4.15]).

Theorem 5.10. *If R is a PID, then every submodule of a free R-module is again a free R-module.*

In particular, putting together Proposition 5.8 and theorem 5.10, we obtain that all projective R-modules over a PID R are in fact free modules.

Example 5.11. $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module, as it is not a free $\mathbb{Z}$-module. To see the latter, you can prove either of the following properties of $\mathbb{Q}$, whose analogue does not hold for a non-zero free $\mathbb{Z}$-module:

- $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/k = 0$ for $k \geq 1$ (use identification with $\mathbb{Q}/k\mathbb{Q}$);
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.

The last, general fact about projective modules is the following exercise.

Exercise 5.12. Let R be a ring and let $(M_i)_{i \in I}$ be a collection of projective left R-modules. Then $\bigoplus_{i \in I} M_i$ is also a projective R-module.

5.3. **Injective modules.** A left R-module N is injective if the additive and left exact functor $\mathrm{Hom}_R(-, N) \colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is in fact an exact functor (i.e. it is also right exact). By Proposition 4.25, and since we know that $\mathrm{Hom}_R(-, N)$ is left exact, we have the following characterisation of injective modules, which is completely analogue to the one for projective modules from Lemma 5.3, and whose proof is left as exercise.

Lemma 5.13. *Let N be a left R-module. Then N is injective if and only if for every injective map of left R-modules $i \colon M' \to M$ and for every R-linear map $f \colon M' \to N$ there exists an R-linear map $g \colon M \to N$ such that $f = i \circ g$.*

Thinking of i as an inclusion, the characterisation says the following: any R-linear map f defined on a sub-module $M' \subseteq M$ and with values in an injective module N can be extended to an R-linear map g defined on the entire M.
The following proposition, known as Baer's criterion to recognise injective modules.[14]

Proposition 5.14 (Baer's criterion)**.** *Let R be a ring and let N be a left R-module. Then N is injective if and only if the following holds: for every left ideal $I \subset R$ and for every R-linear map of left R-modules $f \colon I \to N$, there exists an R-linear map of left R-modules $g \colon R \to N$ extending f.*

[14]We state and prove the proposition for left R-modules, but the same statement and the same proof work also for right R-modules: just consider right ideals.

Proof. If N is injective, then the mentioned property must hold, as it is a very special case of the characterising property for injective R-modules from Lemma 5.13.

Viceversa, let N have the mentioned extension property with respect to all left ideals of R, and consider a generic injective R-linear map $i\colon M' \to M$ and a generic R-linear map $f\colon M' \to N$. Without loss of generality, assume that $M' \subseteq M$ is a submodule and that i is the inclusion.

The first part of the proof is set-theoretical, and relies on the Zorn's lemma: its effect is to enlarge M' as much as possible among submodules of M on which f can be extended. This part of the proof does not use any of the hypotheses on N. Let $\mathscr{F}$ be the set of all couples (P, h) such that $P \subseteq M$ is a left R-submodule containing M', and $h\colon P \to N$ is an R-linear map extending f on M'. For example, $(M', f) \in \mathscr{F}$, which is thus non-empty. We define a partial order on $\mathscr{F}$ as follows: we set $(P_1, h_1) \preceq (P_2, h_2)$ if $P_1 \subseteq P_2$ and $h_2\colon P_2 \to N$ is an extension of $h_1\colon P_1 \to N$. For example, (M', f) is the minimum of $\preceq$. We are however interested in "large" extensions of f, so we look for $\preceq$-big elements in $\mathscr{F}$, possibly maximal ones. Now, if $\mathscr{G} \subset \mathscr{F}$ is a $\preceq$-chain, then the union $\hat{P} = \bigcup_{(P,h)\in\mathscr{G}} P$, which a priori is only a subset of M, is a sub-R-module of M, and there is a unique function of sets $\hat{h}\colon \hat{P} \to N$ that, for each $(P, h) \in \mathscr{G}$, restricts to $h\colon P \to N$ on P: check that $\hat{h}\colon \hat{P} \to N$ is in fact an R-linear map. It follows that $(\hat{P}, \hat{h})$ is a $\preceq$-upper bound for the chain $\mathscr{G}$; in particular, every $\preceq$-chain in $\mathscr{F}$ admits an upper bound. Zorn's lemma guarantees then that there exist at least one element $(\bar{P}, \bar{h})$ which is maximal with respect to $\preceq$. Concretely, this means that for every submodule $\bar{P} \subset P \subseteq M$ with $\bar{P} \neq P$, there exists no R-linear extension $h\colon P \to N$ of $\bar{h}$. If this happens because $\bar{P} = M$, then we are done, as $\bar{h}\colon M \to N$ is an extension of f as required.

We suppose now by contradition that $\bar{P} \neq M$, fix $m \in M \setminus P$ and declare $P = \mathrm{Span}_R(\bar{P} \cup \{m\})$. Using the hypotheses on N, we will construct an R-linear extension $h\colon P \to N$ of $\bar{h}$, contradicting maximality of $(\bar{P}, \bar{h})$ in $\mathscr{F}$.

Define an R-linear map $\pi\colon \bar{P} \oplus R \to P$ by setting $\pi|_{\bar{P}} \equiv i$ and $\pi(r) = r \cdot m$ for $r \in R$. The map π is surjective, since it hits generators of P; its kernel consists of those couples $(\bar{p}, r)$ such that $\bar{p} + r \cdot m = 0 \in M$. We thus have $P \cong (\bar{P} \oplus R)/\ker(\pi)$, and thus in order to define an R-linear map $h\colon P \to N$ it suffices to define an R-linear map $\tilde{h}\colon \bar{P} \oplus R \to N$ with the property that $\tilde{h}|_{\ker(\pi)} \equiv 0$; moreover we want $\tilde{h}|_{\bar{P}} = \bar{h}\colon \bar{P} \to N$, so that the induced map $h\colon P \to N$ is an extension of $\bar{h}$.

Thus we are only left with choosing $\tilde{h}|_R$. Let $I \subset R$ be the left ideal of elements $r \in R$ such that $r \cdot m \in \bar{P}$; then every element $(-r \cdot m, r)$, for $r \in I$, should belong to $\ker(\tilde{h})$; it follows that $\tilde{h}|_R$ must send $r \mapsto \bar{h}(-r \cdot m)$ whenever $r \in I$. Otherwise, there are no other constraints in choosing $\tilde{h}|_R\colon R \to N$.

Now we finally use the hypotheses on N: the map $f\colon I \to N$ given by $f(r) = \bar{h}(-r \cdot m)$ can be extended R-linearly to a map $g\colon R \to N$, and we set $\tilde{h}|_R = g$. □

Baer's criterion is unfortunately difficult to apply in general, but at least there are two easy applications.

Example 5.15. Let $\mathbb{F}$ be a field. Then there are two ideals in $\mathbb{F}$, namely 0 and $\mathbb{F}$ itself. Every $\mathbb{F}$-linear map defined on 0 is the zero map, which can be extended (e.g. by the zero map!) on $\mathbb{F}$; every map defined on $\mathbb{F}$ is already defined on the entire $\mathbb{F}$. Conclusion: every $\mathbb{F}$-vector space is injective.

Example 5.16. Let R be a PID. Then Baer criterion for injectivity of an R-module becomes the following: and R-module N is injective if and only if for all $a \neq 0$ in R and for all $n \in N$ there exists an element $n' \in N$ such that $a \cdot n' = n$. Such an R-module is also called a *divisible* R-module.
For example, $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module, and every quotient of $\mathbb{Q}$ is also divisible and hence $\mathbb{Z}$-injective, as $\mathbb{Z}$ is a PID. The most famous quotient of $\mathbb{Q}$ as a $\mathbb{Z}$-module is probably $\mathbb{Q}/\mathbb{Z}$, which has the remarkable property of being a divisible $\mathbb{Z}$-module in which every element is a *torsion* element (see later Definition 5.22).

The last, general fact about injective modules is the following exercise, which parallels Exercise 5.12.

Exercise 5.17. Let R be a ring and let $(M_i)_{i\in I}$ be a collection of injective left R-modules. Then $\prod_{i\in I} M_i$ is also an injective R-module.

Finally, we introduce a terminology which is parallel with that of Notation 5.9.

Notation 5.18. We will say that each of the categories ${}_R\mathrm{Mod}$ and Mod_R "has enough injectives" to mean that every module M in either of the categories is the source of an injective R-linear map to some injective R-module.

We will prove in the next lecture that the previous terminology is not void!

5.4. **Flat modules.** A left R-module N is flat if the additive and right exact functor $-\otimes_R N \colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$ is in fact an exact functor (i.e. it is also left exact). Note that to define "flat" we must use simultaneously both notions of left and right R-modules. By Proposition 4.25, and since we know that $-\otimes_R N$ is right exact, we have the following characterisation of flat modules, which is completely analogue to the one for projective modules from Lemma 5.3, and whose proof is left as exercise.

Lemma 5.19. *A left R-module N is flat if and only if the following holds: whenever $i \colon M' \to M$ is an injective R-linear map between right R-modules, then the map $i \otimes_R \mathrm{Id}_n \colon M' \otimes_R N \to M \otimes_R N$ is also injective.*

Example 5.20. Let $F = \bigoplus_{i\in I} R$ be a free left R-module; then F is flat. To see this, let $f \colon M' \to M$ be an injective map of right R-modules. Consider F as an $R-R$-bimodule, leveraging on the fact that each copy of R is an $R-R$-bimodule. Then there is a diagram of right R-modules

$$\begin{array}{ccc} \bigoplus_{i\in I} M' & \xrightarrow{\bigoplus_{i\in I} f} & \bigoplus_{i\in I} M \\ \downarrow\cong & & \downarrow\cong \\ M' \otimes_R F & \xrightarrow{f\otimes_R \mathrm{Id}_F} & M \otimes_R F. \end{array}$$

The map $\bigoplus_{i\in I} f$ is defined by sending $\iota_i(m) \mapsto \iota_i(f(m))$ for all $i \in I$ and $m \in M$; check that it is an injective map. The vertical isomorphisms follow from Proposition 3.5. Hence also $f \otimes_R \mathrm{Id}_F$ is injective.
Now let N be a projective left R-module, and let N' be a projective left R-module such that $F = N \oplus N'$ is a free module. We claim that N is flat (and similarly N'

is flat). Let $f\colon M'\to M$ as above; then we have a commutative diagram

$$\begin{array}{ccc} M'\otimes_R N & \xrightarrow{f\otimes_R \mathrm{Id}_N} & M\otimes_R N \\ \big\downarrow{\scriptstyle \mathrm{Id}_{M'}\otimes_R\ \iota_N} & & \big\downarrow{\scriptstyle \mathrm{Id}_M\otimes_R\ \iota_N} \\ M'\otimes_R F & \xrightarrow{f\otimes_R \mathrm{Id}_F} & M\otimes_R F. \end{array}$$

The vertical maps are injective by Proposition 3.5, and the bottom horizontal map is injective as argued above; it follows that also the top horizontal map is injective.

Using the previous example, we can give the following terminology, which is however non-standard, and in fact is not going to be a very useful one.

Notation 5.21. We will say that each of the categories ${}_R\mathrm{Mod}$ and Mod_R "has enough flat modules" to mean that every module M in either of the categories receives a surjective R-linear map from some flat R-module.

Definition 5.22. Let R be a (commutative) domain and let M be an R-module. An element $m\in M$ is a *torsion* element if there is $a\in\mathbb{Z}$ such that $a\cdot m=0\in M$. We denote by $\mathrm{tors}(M)\subset M$ the subset of torsion elements

For example, 0 is always a torsion element. Check that $\mathrm{tors}(M)$ is a sub-R-module of M (here the hypothesis that R is a domain is essential).

Example 5.23. Let R be a domain and let M be an R-module. Assume that $\mathrm{tors}(M)\neq 0$. Then M is not flat. Indeed, let $m\neq 0\in\mathrm{tors}(M)$, and let $a\neq 0\in R$ such that $a\cdot m=0$. Let $(a)\subset R$ be the principal ideal generated by a. Then the map $R\overset{a\cdot -}{\to} R$ is an injective map of R-modules, yet the map $(a\cdot -)\otimes_R \mathrm{Id}_M\colon R\otimes_R M\to R\otimes_R M$ can be identified with the map $a\cdot -\colon M\to M$, which is not injective.

Example 5.24. Let $R=\mathbb{F}[x,y]$ for some field $\mathbb{F}$, and consider $M=(x,y)\subset R$. Note that M is torsion-free, as R is a domain. Yet, we claim that M is not flat. To see this, consider the inclusion $i\colon (x,y)\to R$; tensoring over R with M we obtain the map $i\otimes_R \mathrm{Id}_M\colon M\otimes_R M\to M\otimes_R R$. The element $x\otimes y-y\otimes x\in M\otimes_R M$ is sent to 0 in $M\otimes_R R$, as in this second tensor product we can compute

$$x\otimes y-y\otimes x=x\otimes y\cdot 1-y\otimes x\cdot 1=xy\otimes 1-yx\otimes 1=0.$$

However this computation does not make sense in $M\otimes_R M$, since 1 is not an element of M. In fact we can prove that $x\otimes y-y\otimes x\neq 0\in M\otimes_R M$.
To see this, consider the quotient R-modules $M_1=(x,y)/(y,x^2)$ and $M_2=(x,y)/(x,y^2)$. The surjective R-linear maps $p_1\colon M\to M_1$ and $p_2\colon M\to M_2$ give rise to a surjective map of R-modules

$$p_1\otimes_R p_2\colon M\otimes_R M\to M_1\otimes_R M_2.$$

The element $x\otimes y-y\otimes x\in M\otimes_R M$ is sent to the element $[x]\otimes[y]$, as the other summand $-[y]\otimes[x]$ vanishes in $M_1\otimes_R M_2$. We claim that $[x]\otimes[y]\neq 0\in M_1\otimes_R M_2$. Note that both M_1 and M_2 are isomorphic to $R/(x,y)$; more precisely, there is an R-linear isomorphism $R/(x,y)\to M_1$ sending $[1]\mapsto[x]$ and an R-linear isomorphism $R/(x,y)\to M_2$ sending $[1]\mapsto[y]$. There is therefore an R-linear isomorphism $M_1\otimes_R M_2\to R/(x,y)\otimes_R R/(x,y)$ under which the element $[x]\otimes[y]$ corresponds to $[1]\otimes[1]$; and now remember that $R/(x,y)\otimes_R R/(x,y)\cong R/(x,y)$, with an isomorphism sending $[1]\otimes[1]$ to $[1]\neq[0]\in R/(x,y)$.

The fact that in the previous example R was not a PID was crucial, as we see in the following.

Proposition 5.25. *Let R be a ring and let N be a left R-module with the following property: every finitely generated sub-R-module $N' \subset N$ is flat. Then N is flat.*

Proof. Let $i\colon M' \to M$ be an injective homomorphism of right R-modules. Suppose by absurd that $i \otimes_R \mathrm{Id}_N\colon M' \otimes_R N \to M \otimes_R N$ is not an injective $\mathbb{Z}$-linear map. Then there is an element $x = \sum_{j=1}^k m'_j \otimes n_j \neq 0 \in M' \otimes_R N$ such that $(i \otimes_R \mathrm{Id}_N)(x) = \sum_{j=1}^k i(m'_j) \otimes n_j = 0 \in M \otimes_R N$.
Recall that $M \otimes_R N = A_{M,N}/B_{M,N}$, where $A_{M,N} = \bigoplus_{(m,n)\in M\times N} \mathbb{Z}$ and $B_{M,N}$ is the sub-abelian group generated by the elements from Definition 2.16. We get that $y = \sum_{i=1}^k (i(m_j), n_j) \in B_{M,N} \subseteq A_{M,N}$, and thus we can express y as a $\mathbb{Z}$-linear combination of generators of B:

$$\begin{aligned} y =& \sum_{j=1}^a \alpha_j \cdot \left[(m^1_{j,1} + m^1_{j,2}, n^1_j) - (m^1_{j,1}, n^1_j) - (m^1_{j,2}, n^1_j)\right] \\ &+ \sum_{j=1}^b \beta_j \cdot \left[(m^2_j, n^2_{j,1} + n^2_{j,2}) - (m^2_j, n^2_{j,1}) - (m^2_j, n^2_{j,2})\right] \\ &+ \sum_{j=1}^c \gamma_j \cdot \left[(m^3_j \cdot r_j, n^3_j) - (m^3_j, r \cdot n^3_j)\right]. \end{aligned}$$

Here each letter m, n and n represents an element in M, N or R; the exponents $^1,^2,^3$ are just to distinguish three different families of elements, of sizes $a, b, c \geq 0$; and α_j, β_j and γ_j are coefficients in $\mathbb{Z}$. Let now

$$N' = \mathrm{Span}_R\left(\{n_j\}_{j=1}^k \cup \left\{n^1_j\right\}_{j=1}^a \cup \left\{n^2_{j,1}, n^2_{j,2}\right\}_{j=1}^b \cup \left\{n^3_j\right\}_{j=1}^c\right) \subseteq N.$$

Then N' is a finitely generated R-submodule of N, and denote by $\iota\colon N' \to N$ the inclusion. We can consider the element $x' = \sum_{i=1}^k m'_j \otimes n_j \in M' \otimes_R N'$, given by the same formula used for x; since the map of abelian groups $\mathrm{Id}_{M'} \otimes_R \iota$ sends x' to $x \neq 0$, we have $x' \neq 0 \in M' \otimes_R N'$. Consider now the map $i \otimes_R \mathrm{Id}_{N'}\colon M' \otimes_R N' \to M \otimes_R N'$; then $(i \otimes_R)(x') = \sum_{i=1}^k i(m'_j) \otimes n_j$ vanishes in $M \otimes_R N'$, as witnessed by the fact that $y' = \sum_{i=1}^k (i(m_j), n_j) \in A_{M,N'}$ can be written as

$$\begin{aligned} y' =& \sum_{j=1}^a \alpha_j \cdot \left[(m^1_{j,1} + m^1_{j,2}, n^1_j) - (m^1_{j,1}, n^1_j) - (m^1_{j,2}, n^1_j)\right] \\ &+ \sum_{j=1}^b \beta_j \cdot \left[(m^2_j, n^2_{j,1} + n^2_{j,2}) - (m^2_j, n^2_{j,1}) - (m^2_j, n^2_{j,2})\right] \\ &+ \sum_{j=1}^c \gamma_j \cdot \left[(m^3_j \cdot r_j, n^3_j) - (m^3_j, r \cdot n^3_j)\right]. \end{aligned}$$

and is thus an element in $B_{M,N'} \subset A_{M,N'}$. This shows that the finitely generated sub-R-module N' of N is not flat, which contraditcts the hypothesis on N. □

The idea in the previous proof is more or less the following: to show that a map is not injective, one only needs to describe an element that vanishes, and this requires

finitely many generators in the source. To witness the vanishing of the image of the chosen element, one needs finitely many of the relations holding in the target. In few words, one needs a finite amount of information to witness that an R-linear map is *not* injective.
We apply the previous proposition to the case of a PID.

Corollary 5.26. *Let R be a PID and let N be an R-module with* $\mathrm{tors}(N) = 0$ *(we say, N is* torsion-free*). Then N is flat.*

Proof. Let's check that every finitely generated sub-R-module N' of N is flat. Surely, N' is torsion-free; a finitely generated module over a PID is a direct sum of cyclic modules, and since N' torsion-free, then all cyclic summands of N' must be of the form R. This means that N' is a finitely generated *free* R-module, and in particular it is flat. □

Example 5.27. $\mathbb{Q}$ is flat over $\mathbb{Z}$.

6. Enough injectives

6.1. **The category $_R\mathrm{Mod}$ has enough injectives.** We already saw that $_R\mathrm{Mod}$ has enough projectives: every left R-module receives a surjective R-linear map from a projective (in fact, from a free) R-module. We want now to prove a "dual" statement. We begin with the simplest ring R, namely $\mathbb{Z}$.

Lemma 6.1. *Every $\mathbb{Z}$-module M injects into some injective $\mathbb{Z}$-module N.*

Proof. Recall that a $\mathbb{Z}$-module is injective if and only if it is divisible, as $\mathbb{Z}$ is a PID. In particular $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. For every element $m \neq 0 \in M$ we can construct a $\mathbb{Z}$-linear map $f_m \colon M \to \mathbb{Q}/\mathbb{Z}$ as follows:

- first, we choose a non-zero $\mathbb{Z}$-linear map $f'_m \colon \mathrm{Span}_{\mathbb{Z}}(m) \to \mathbb{Q}/\mathbb{Z}$: this is possible, since $\mathrm{Span}_{\mathbb{Z}}(m)$ is either isomorphic to $\mathbb{Z}$ (in which case we map m to any non-zero element of $\mathbb{Q}/\mathbb{Z}$, and extend $\mathbb{Z}$-linearly), or it is isomorphic to $\mathbb{Z}/k$ for some $k \geq 1$ (in which case we map m to any non-zero element of order k in $\mathbb{Q}/\mathbb{Z}$, and extend $\mathbb{Z}$-linearly);[15]
- second, we use that $\mathbb{Q}/\mathbb{Z}$ is injective to extend f'_m to a $\mathbb{Z}$-linear map $f_m \colon M \to \mathbb{Q}/\mathbb{Z}$.

The map f_m is guaranteed to map $m \in M$ to something non-zero in $\mathbb{Q}/\mathbb{Z}$, but it could be non-injective. However, if we consider *all* maps f_m at the same time, we obtain an injective $\mathbb{Z}$-linear map into the product (using the universal property of the product, btw)

$$f = \prod_{m \in M \setminus \{0\}} f_m \colon M \to \prod_{m \in M \setminus \{0\}} \mathbb{Q}/\mathbb{Z}.$$

And now we remember from Exercise 5.17 that a product of injective modules is injective. □

That was great, but after all $\mathbb{Z}$ is a very special ring, namely a PID. Over a generic *non-commutative* ring what can we do? Actually, are we even sure that over a generic, possibly non-commutative ring R there exist injective left R-modules at all, apart from the zero module?

[15]We use $\mathbb{Q}/\mathbb{Z}$ rather than a conceptually simpler injective $\mathbb{Z}$-module as, for example, $\mathbb{Q}$, precisely because $\mathbb{Q}/\mathbb{Z}$ has torsion elements of all orders.

Lemma 6.2. *Let R be a ring, and consider R as a $\mathbb{Z} - R$-bimodule. If A is an injective (left) $\mathbb{Z}$-module, then $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is an injective left R-module.*

Proof. Let $i\colon M' \to M$ be an injective map of left R-modules. We want to prove that the induced map of abelian groups

$$\mathrm{Hom}_R(i, \mathrm{Hom}_{\mathbb{Z}}(R, A)) = i\circ - \colon \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \to \mathrm{Hom}_R(M', \mathrm{Hom}_{\mathbb{Z}}(R, A))$$

is surjective. We can now use the Hom-tensor adjunction, and replace the above map by another, equivalent map of abelian groups

$$\mathrm{Hom}_{\mathbb{Z}}(R \otimes_R i, A) = (\mathrm{Id}_R \otimes_R i) \circ - \colon \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R M, A) \to \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R M', A)$$

And now we can compute easily tensor products over R with R, and simplify the above further as follows, where i is now treated only as a $\mathbb{Z}$-linear map

$$\mathrm{Hom}_{\mathbb{Z}}(i, A) = i \circ - \colon \mathrm{Hom}_{\mathbb{Z}}(M, A) \to \mathrm{Hom}_{\mathbb{Z}}(M', A).$$

A magic has occurred: the ring R has disappeared from the formula! Now we just use that A is an injective $\mathbb{Z}$-module, and that $i\colon M' \to M$ is an injective $\mathbb{Z}$-linear map. □

Using Lemma 6.2 we can now repeat the strategy of Lemma 6.1. Given a left R-module M, for each $m \in M$ we first define a $\mathbb{Z}$-linear map $f_m\colon M \to \mathbb{Q}/\mathbb{Z}$ with the property that $f_m(m) \neq 0$; we then first interpret f_m as a $\mathbb{Z}$-linear map

$$f_m\colon R \otimes_R M \to \mathbb{Q}/\mathbb{Z}$$

and then use the Hom-tensor adjunction to transform f_m into an R-linear map

$$g_m\colon M \to \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/bZ).$$

The property that $f_m(m) \neq 0$ is transformed into the property that $g_m(m)$ sends $1 \in R$ to a non-zero element in $\mathbb{Q}/\mathbb{Z}$; in particular $g_m(m) \neq 0 \in \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. And now we take the big product:

$$\prod_{m \in M \setminus \{0\}} g_m\colon M \to \prod_{m \in M \setminus \{0\}} \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$$

is an injective, R-linear map with target an injective left R-module.

7. Localisations and abelian categories

7.1. **Localisation of (commutative) rings.** In this subsection we let R be a commutative ring. The zero ring $R = \{0\}$ is allowed: it is the unique ring in which $0 = 1$, and it is commutative; all arguments will not need the hypothesis $0 \neq 1$, and it will actually happen to encounter the zero ring often, when dealing with localisations.
The basic idea behind the notion of *localisation* is that given a ring R and a set $\mathcal{S} \subset R$ of admissible "denominators", we can construct a new ring $\mathcal{S}^{-1}R$ containing fractions of elements of R with an element of $\mathcal{S}$ as denominator. This should generalise the construction of $\mathbb{Q}$ as ring of fractions of elements of $\mathbb{Z}$, with an element of $\mathbb{Z} \setminus \{0\}$ as denominator.

Definition 7.1. An element $r \in R$ is *invertible* if there exists an element $r' \in R$ such that $rr' = 1$.[16] The subset of invertible elements of R is denoted $R^\times \subseteq R$.

[16] We are in the commutative setting, so also the other equality $r'r = 1$ is immediately satisfied. In the non-commutative setting one has to impose both equalities!

For example, if $\mathbb{F}$ is a field, then $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$; instead $\mathbb{Z}^\times = \{\pm 1\}$; in the zero ring the unique element is $0 = 1$, which is invertible (with itself as inverse). We note the following:

- $R^\times$ is an abelian group[17], with $1 \in R$ as neutral element and with multiplication in R as composition;
- Every map $f\colon R \to S$ of commutative rings sends invertible elements to invertible elements, and thus restricts to a map of abelian group $f^\times\colon R^\times \to S^\times$. Moreover, the subset $f^{-1}(S^\times) \subseteq R$ is closed under multiplication.

Definition 7.2. Let R be a commutative ring and $\mathcal{S} \subseteq R$ be a subset. A homomorphism of commutative rings $f\colon R \to S$ is *$\mathcal{S}$-local* if $f(\mathcal{S})$ is contained in $S^\times$.
A *localisation* of R at $\mathcal{S}$ is a couple $(\bar{S}, \bar{f})$ of a commutative ring $\bar{S}$ and an $\mathcal{S}$-local homomorphism of rings $\bar{f}\colon R \to \bar{S}$, satisfying the following universal property: whenever (S, f) is a couple of a commutative ring S and a $\mathcal{S}$-local homomorphism of rings $f\colon R \to S$, there exists a unique homomorphism of rings $\theta\colon \bar{S} \to S$ such that the following diagram of commutative rings commutes

$$\begin{array}{ccc} R & \xrightarrow{\bar{f}} & \bar{S} \\ & \searrow^{f} & \downarrow{\scriptstyle\theta} \\ & & S. \end{array}$$

As usual, when defining something by universal property, we can easily check that if R admits a localisation at $\mathcal{S}$, then this couple $(\bar{S}, \bar{f})$ is unique up to canonical isomorphism. But does a localisation exist? Before attempting to construct a localisation, we note the following.
If $\mathcal{S} \subseteq R$ is a subset, we can define $\bar{\mathcal{S}}$ to be the "multiplicative and invertible closure" of $\mathcal{S}$, i.e. the smallest subset of R containing $\mathcal{S} \cup R^\times$ and closed under multiplication. Then it is easy to check that a homomorphism of rings $f\colon R \to S$ is $\mathcal{S}$-local if and only if it is also $\bar{\mathcal{S}}$-local. As a consequence, if a localisation of R at $\mathcal{S}$ exists, then it will have also the universal property for being a localisation of R at $\bar{\mathcal{S}}$; and viceversa, any localisation of R at $\bar{\mathcal{S}}$ is also a localisation of R at $\mathcal{S}$.

Definition 7.3. Let R be a commutative ring and let $\mathcal{S} \subset R$ be a subset containing $R^\times$ and closed under multiplication. We define $\mathcal{S}^{-1}R$ as the set of equivalence classes of couples $(r, s) \in R \times \mathcal{S}$: two couples (r, s) and (r', s') are equivalent if there exists $t \in \mathcal{S}$ such that $trs' = tr's$. Check that this is indeed an equivalence relation! The equivalence class of (r, s) is usually denoted as a fraction $\frac{r}{s}$.
We define a sum on the set $\mathcal{S}^{-1}R$ by setting $\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$; the neutral element of the sum is $\frac{0}{1}$, and the additive inverse of $\frac{r}{s}$ is $\frac{-r}{s}$.
We define a product on the set $\mathcal{S}^{-1}R$ by setting $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$; the neutral element of the product is $\frac{1}{1}$. The set $\mathcal{S}^{-1}R$ becomes thus a commutative ring (check associativity, distributivity, well-definition of the operations...).
Moreover, we have a map of rings $\eta_{\mathcal{S},R}\colon R \to \mathcal{S}^{-1}R$ given by $r \mapsto \frac{r}{1}$.

We can now check that $(\mathcal{S}^{-1}R, \eta_{\mathcal{S},R})$ has the universal property for being a localisation of R at $\mathcal{S}$. Let $f\colon R \to S$ be a homomorphism of commutative rings sending

[17] If R is non-commutative, we get nevertheless that $R^\times$ is a (possibly non-abelian) group.

$\mathcal{S}$ to $S^\times$. Then the map $\theta\colon \mathcal{S}^{-1}R \to S$, if it exists, must send $\frac{r}{1} = \eta_{\mathcal{S},R}(r)$ to the element $f(r) \in S$, for all $r \in R$. Moreover, for all $s \in \mathcal{S}$, we must have

$$f(s)\cdot\theta\left(\frac{1}{s}\right) = \theta\left(\frac{s}{1}\right)\cdot\theta\left(\frac{1}{s}\right) = \theta\left(\frac{s}{1}\cdot\frac{1}{s}\right) = \theta\left(\frac{s}{s}\right) = \theta\left(\frac{1}{1}\right) = 1,$$

using the easy-to-check equality $\frac{s}{s} = \frac{1}{1}$ in $\mathcal{S}^{-1}R$. This means that θ must send $\frac{1}{s}$ to the multiplicative inverse of $f(s)$ in S, which fortunately, by the hypothesis on f, exists. Last, for every $\frac{r}{s} \in \mathcal{S}^{-1}R$ we can now write the constraint $\theta(\frac{r}{s}) = f(r)\cdot\theta(\frac{1}{s})$, and therefore there are two possibilities:

- either the above constraints are compatible with each other, and give rise to a map of sets $\theta\colon \mathcal{S}^{-1}R \to S$ which happens to be a homomorphism of rings: in this case θ exists and is unique;
- or something goes wrong: in this case the required homomorphism of rings $\theta\colon \mathcal{S}^{-1}R \to S$ does not exist.

Check that indeed everything goes fine.

Notation 7.4. If $\mathcal{S} \subset R$ is any subset, one defines $\mathcal{S}^{-1}R$ as $\bar{\mathcal{S}}^{-1}R$, where $\bar{\mathcal{S}}$ is the "multiplicative and invertible closure" of $\mathcal{S}$ in the sense above.

In fact, one can take even a larger "closure" of $\mathcal{S}$ without changing the universal property of localisation.

Definition 7.5. Let $r, r' \in R$ be two elements. We say that r' is a *divisor* of r if there exists $r'' \in R$ with $r'r'' = r$. Note that a divisor of an element in $R^\times$ is again in $R^\times$.
Given a subset $\mathcal{S} \subset R$, we define the "multiplicative and divisible closure" $\hat{\mathcal{S}} \subset R$ as the smallest subset of R which contains $\mathcal{S} \cup R^\times$, is closed under multiplication, and contains any divisor of any of its elements.

We remark that a localisation of R at $\mathcal{S}$ and a localisation of R at $\hat{\mathcal{S}}$ have equivalent universal properties, and thus are isomorphic rings. In few words: if we change R by allowing denominators in $\mathcal{S}$, then there is no harm in allowing denominators equal to products of elements of $\mathcal{S}$, or divisors of elements of $\mathcal{S}$, or a combination of the two.

Example 7.6. Let $\mathcal{S} = \mathbb{Z} \setminus \{0\} \subset \mathbb{Z}$; then $\mathcal{S}^{-1}\mathbb{Z}$ is precisely the ring $\mathbb{Q}$, as defined in elementary school.
Let $\mathcal{S} = \{2\} \subset \mathbb{Z}$; then $\mathcal{S}^{-1}\mathbb{Z}$ is the subring of $\mathbb{Q}$ containing all fractions whose denominator has the form $\pm 2^k$, for some $k \geq 0$. The same description holds for the ring $\{4\}^{-1}\mathbb{Z}$.
In general, if $\mathcal{S} \subseteq \mathbb{Z} \setminus \{0\}$, then $\mathcal{S}^{-1}\mathbb{Z}$ is the subring of $\mathbb{Q}$ containing all reduced fractions $\frac{a}{b}$ such that each prime factor of b is a divisor of some element of $\mathcal{S}$.

Example 7.7. More generally, if R is a domain, one can define the *fraction field* $\mathrm{Frac}(R)$ as the localisation $(R \setminus \{0\})^{-1}R$.

Example 7.8. If $\mathcal{S} \subseteq R^\times \subset R$, then the couple (R, Id_R) has the universal property for being a localisation of R at $\mathcal{S}$. It follows that $\mathcal{S}^{-1}R$ is canonically isomorphic to R; the canonical isomorphism is the map $\eta_{\mathcal{S},R}\colon R \to \mathcal{S}^{-1}R$.

The above examples suggests that $\mathcal{S}^{-1}R$ is a *larger* ring than R, in particular the map $\eta_{\mathcal{S},R}\colon R \to \mathcal{S}^{-1}R$ is injective. This is however false in general!

Example 7.9. Let $\mathcal{S} = R \subseteq R$. Then every fraction $\frac{r}{s} \in \mathcal{S}^{-1}R$ is equal to zero: to check that $\frac{r}{s} = \frac{0}{1}$, we need to find $t \in \mathcal{S}$ such that $t \cdot r \cdot 1 = t \cdot 0 \cdot s$, and we may take $t = 0$. This means that $R^{-1}R$ is the zero ring; the map $\eta_{R,R}\colon R \to R^{-1}R$ will be surjective, but not injective (unless R itself is the zero ring).

Example 7.10. Let now $R = \mathbb{Z}/6$ and let $\mathcal{S} = \{[2]_6\}$. The map of rings $\bar{f}\colon \mathbb{Z}/6 \to \mathbb{Z}/3$ sending $[n]_6 \mapsto [n]_3$ has the property of sending $[2]_6$ to the element $[2]_3$, which is invertible in $\mathbb{Z}/3$. Let us check that $(\mathbb{Z}/3, \bar{f})$ has the universal property to be a localisation of $\mathbb{Z}/6$ at $[2]_6$.
Let $f\colon \mathbb{Z}/6 \to S$ be a homomorphism of commutative rings, sending $[2]_6$ to an invertible element in S. Since $[1]_6 \mapsto 1_S \in S$, we must have that $2_S \colon = 1_S + 1_S$ is invertible. Moreover $6_S = f([6]_6) = f([0]_6) = 0_S$. Finally, note that $2_S \cdot 3_S = 6_S$, hence this product vanishes. Multiplying by $(2_S)^{-1}$, i.e. the multiplicative inverse of 2_S in S, we obtain that $3_S = 0_S$, which implies that $f([3]_6) = 0$. This means that f factors through the quotient $\mathbb{Z}/3 \cong (\mathbb{Z}/6)/([3]_6)$ of $\mathbb{Z}/6$ by the ideal generated by $[3]_6$, and this is precisely saying that f factors (uniquely) through $\bar{f}$.
Again we note that $\eta_{\{[2]_6\},\mathbb{Z}/6}\colon \mathbb{Z}/6 \to \mathbb{Z}/3$ is not injective.

Let R be a commutative ring and let $\mathcal{S}$ be any subset of R. Then $\mathcal{S}^{-1}R$ can be considered as an R-module, using the map of rings $\eta_{\mathcal{S},R}\colon R \to \mathcal{S}^{-1}R$. We now have the following remarkable fact.

Proposition 7.11. *The R-module $\mathcal{S}^{-1}R$ is flat.*

Assuming the proposition, we observe that $\mathcal{S}^{-1}R$ can be also considered as a $\mathcal{S}^{-1}R$–R-bimodule; this implies that $\mathcal{S}^{-1}R \otimes_R -$ can be considered as a functor ${}_R\mathrm{Mod} \to {}_{\mathcal{S}^{-1}R}\mathrm{Mod}$, and the proposition says that this functor is exact[18].

Example 7.12. $\mathbb{Q}$ is a flat and injective $\mathbb{Z}$-module, but it is not a projective $\mathbb{Z}$-module (see Example 5.11).

Example 7.13. Let $\mathbb{Q}(x)$ be the field of fractions of polynomials in $\mathbb{Z}[x]$; then $\mathbb{Q}(x)$ is a flat $\mathbb{Z}[x]$-module. Clearly, $\mathbb{Q}(x)$ is also isomorphic to the field of fractions of the domain $\mathbb{Q}[x]$, so it is also flat over $\mathbb{Q}[x]$.

Exercise 7.14. Let $a \geq 1$ be an integer. Prove that $\mathbb{Z}/2$ is

- not at all a $\mathbb{Z}/a$-module, if a is odd;
- a projective, hence flat $\mathbb{Z}/a$-module, if a is even but $a/2$ is odd;
- a non-flat, hence non-projective $\mathbb{Z}/a$-module, if a is divisible by 4.

Generalise the above by replacing 2 with p^k, for some p prime and $k \geq 1$.

7.2. **Localisation of modules.** To prove Proposition 7.11, it is convenient to do the following three steps:

- define a new functor $\mathcal{S}^{-1}-\colon {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$;
- prove that the functor $\mathcal{S}^{-1}-$ is exact;
- prove that the functor $\mathcal{S}^{-1}-$ is naturally isomorphic to the functor $\mathcal{S}^{-1}R \otimes_R -\colon {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$.

[18]Actually, the proposition says that the composition of the functor $\mathcal{S}^{-1}R \otimes_R -\colon {}_R\mathrm{Mod} \to {}_{\mathcal{S}^{-1}R}\mathrm{Mod}$ with the functor $\bar{f}^*\colon {}_{\mathcal{S}^{-1}R}\mathrm{Mod} \to {}_R\mathrm{Mod}$ is exact: but checking one statement is equivalent to checking the other, as anyway any functor towards a category of modules if exact if and only if its composition with the forgetful functor all the way to ${}_{\mathbb{Z}}\mathrm{Mod}$ is exact.

Definition 7.15. Let R be a commutative ring and $\mathcal{S} \subset R$. An R-module M is *$\mathcal{S}$-local* if for all $r \in \mathcal{S}$ the map $r \cdot -: M \to M$ is bijective.

Note that if M is $\mathcal{S}$-local, then it is automatically also $\hat{\mathcal{S}}$-local:

- any element of $R^\times$ acts bijectively on M, just because M is an R-module;
- if $r, r' \in R$ act bijectively on M, then so does rr';
- using that R is commutative, if r acts bijectively on M and $r = r'r'' = r''r'$, then both $r' \cdot -$ and $r'' \cdot -$ must be self-bijections of M.

Example 7.16. Let M be an $\mathcal{S}^{-1}R$-module, and consider M as an R-module by restriction of scalars along the map of rings $R \to \mathcal{S}^{-1}R$. Then M is a $\mathcal{S}$-local R-module: indeed for every element $r \in \mathcal{S}$ the map $r \cdot -: M \to M$ coincides with the map $\frac{r}{1} \cdot -: M \to M$, which has as inverse the (forgotten) map $\frac{1}{r} \cdot -: M \to M$. In fact the converse also holds: if M is a $\mathcal{S}$-local R-module, it is *because* it is obtained from a $\mathcal{S}^{-1}R$-module by restriction of scalars. Concretely, for a fraction $\frac{r}{s} \in \mathcal{S}^{-1}R$ we can define the action $\frac{r}{s} \cdot -: M \to M$ as the composition of $r \cdot -: M \to M$ and the inverse of the bijection $s \cdot -: M \to M$:

$$\tfrac{r}{s} \cdot -: M \xrightarrow{r \cdot -} M \xrightarrow{(s \cdot -)^{-1}} M.$$

Check that two equivalent fractions induce the same map $M \to M$; check that these maps assemble into an action of $\mathcal{S}^{-1}R$ on M, so that the abelian group M becomes an $\mathcal{S}^{-1}R$-module, and so that the old R-module structure can be retrieved using the restriction of scalars.

The previous example shows that the information of a $\mathcal{S}$-local R-module is equivalent to the information of a $\mathcal{S}^{-1}R$-module. To make this precise, you can solve the following exercise.

Exercise 7.17. The restriction of scalars functor is a functor ${}_{\mathcal{S}^{-1}R}\mathrm{Mod} \to {}_R\mathrm{Mod}$ with the following properties:

- it is fully faithful (this means that a map of sets between $\mathcal{S}^{-1}R$-modules is an R-linear map if and only if it is also an $\mathcal{S}^{-1}R$-linear map);
- its essential image is the full subcategory of ${}_R\mathrm{Mod}$ spanned by $\mathcal{S}$-local R-modules (this means that precisely the $\mathcal{S}$-local modules can be obtained, up to isomorphism, through the functor).

Definition 7.18. Let M be an R-module. A localisation of M at $\mathcal{S}$ is a couple $(\bar{N}, \bar{f})$ consisting of a $\mathcal{S}$-local R-module $\bar{N}$ and an R-linear map $\bar{f}: M \to \bar{N}$ satisfying the following universal property: whenever (N, f) is a couple of a $\mathcal{S}$-local R-module and an R-linear map $f: M \to N$, there exists a unique R-linear map $\theta: \bar{N} \to N$ such that the following diagram of R-modules and R-linear maps commutes:

$$\begin{array}{ccc} M & \xrightarrow{\bar{f}} & \bar{N} \\ & \searrow^{f} & \downarrow \theta \\ & & N. \end{array}$$

Uniqueness up to canonical isomorphism of a localisation of M at $\mathcal{S}$ can be proved in the usual way. The existence is given by the following definition, which is very similar to Definition 7.3

Definition 7.19. Let R be a commutative ring, $\mathcal{S} \subset R$ be a subset containing $R^\times$ and closed under multiplication, and let M be an R-module. We define $\mathcal{S}^{-1}M$ as the set of equivalence classes of couples $(m, s) \in M \times \mathcal{S}$: two couples (m, s) and (m', s') are equivalent if there exists $t \in \mathcal{S}$ such that $tms' = tm's$. Check that this is indeed an equivalence relation! The equivalence class of (m, s) is usually denoted as a fraction $\frac{m}{s}$.
We define a sum on the set $\mathcal{S}^{-1}M$ by setting $\frac{m}{s} + \frac{m'}{s'} = \frac{s' \cdot m + s \cdot m'}{ss'}$; the neutral element of the sum is $\frac{0}{1}$, and the additive inverse of $\frac{m}{s}$ is $\frac{-m}{s}$.
We define an action of $\mathcal{S}^{-1}R$ by scalar multiplication on $\mathcal{S}^{-1}M$ by setting $\frac{r}{s} \cdot \frac{m}{s'} = \frac{r \cdot m}{ss'}$. The set $\mathcal{S}^{-1}M$ becomes thus a $\mathcal{S}^{-1}R$-module, and hence by Example 7.16 it can be also considered as a $\mathcal{S}$-local R-module (check that everything goes fine with the above definition of $\mathcal{S}^{-1}R$-module structure on $\mathcal{S}^{-1}M$).
Moreover, we have a map of R-modules $\eta_{\mathcal{S},M} \colon M \to \mathcal{S}^{-1}M$ given by $m \mapsto \frac{m}{1}$.

You can now adapt the proof of the fact that $\mathcal{S}^{-1}R$ is a localisation of R at $\mathcal{S}$ as rings, and prove that $(\mathcal{S}^{-1}M, \eta_{\mathcal{S},M})$ is a localisation of M at $\mathcal{S}$ as R-modules.

Notation 7.20. If $\mathcal{S} \subset R$ is any subset, we define $\mathcal{S}^{-1}M$ as $\bar{\mathcal{S}}^{-1}M$, where $\bar{\mathcal{S}}$ is the "multiplicative and invertible closure" of $\mathcal{S}$.

We can now explore the functoriality of the construction transforming an R-module M into a $\mathcal{S}^{-1}R$-module $\mathcal{S}^{-1}M$. Let $g \colon M \to N$ be an R-linear map. Then the composition $M \overset{g}{\to} N \overset{\eta_{\mathcal{S},N}}{\to} \mathcal{S}^{-1}N$ is an R-linear map $M \to \mathcal{S}^{-1}N$ with source M and target a $\mathcal{S}$-local R-module: the universal property of M ensures that there exists a unique R-linear map $\theta \colon \mathcal{S}^{-1}M \to \mathcal{S}^{-1}N$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & N \\ \big\downarrow{\scriptstyle \eta_{\mathcal{S},M}} & & \big\downarrow{\scriptstyle \eta_{\mathcal{S},N}} \\ \mathcal{S}^{-1}M & \xrightarrow{\ \theta\ } & \mathcal{S}^{-1}N. \end{array}$$

The map θ is usually denoted $\mathcal{S}^{-1}g$; concretely, it sends $\frac{m}{s} \mapsto \frac{g(m)}{s}$. This construction gives rise to a functor $\mathcal{S}^{-1}- \colon {}_R\mathrm{Mod} \to {}_{\mathcal{S}^{-1}R}\mathrm{Mod}$, sending $M \mapsto \mathcal{S}^{-1}M$ and $(g \colon M \to N) \mapsto (\mathcal{S}^{-1}g \colon \mathcal{S}^{-1}M \to \mathcal{S}^{-1}N)$.

Lemma 7.21. *Let R be a commutative ring and $\mathcal{S} \subset R$. Consider $\mathcal{S}^{-1}R$ as a $\mathcal{S}^{-1}R - R$-bimodule, and regard $\mathcal{S}^{-1}R \otimes_R -$ as a functor ${}_R\mathrm{Mod} \to {}_{\mathcal{S}^{-1}R}\mathrm{Mod}$. Then there is a natural equivalence of functors $(\mathcal{S}^{-1}-) \cong (\mathcal{S}^{-1}R \otimes_R -)$.*

Proof. Let M be an R-module. We can define an R-bilinear map $\mathcal{S}^{-1}R \times M \to \mathcal{S}^{-1}M$ by sending $(\frac{r}{s}, m) \mapsto \frac{r \cdot m}{s}$; this gives rise to an R-linear map $\epsilon_M \colon \mathcal{S}^{-1}R \otimes_R M \to \mathcal{S}^{-1}M$. This map has the property of sending $\frac{r}{s} \otimes m \mapsto \frac{r \cdot m}{s}$.
Viceversa, we can define an R-linear map $M \to \mathcal{S}^{-1}R \otimes_R M$ as the composition

$$M \xrightarrow{\quad \cong \quad} R \otimes M \xrightarrow{\eta_{\mathcal{S},R} \otimes_R \mathrm{Id}_M} \mathcal{S}^{-1}R \otimes M;$$

Since the target R-module is $\mathcal{S}$-local, the universal property of $\mathcal{S}^{-1}M$ gives rise to an R-linear map $\epsilon'_M \colon \mathcal{S}^{-1}M \to \mathcal{S}^{-1}R \otimes_R M$. This map has the property of sending $\frac{m}{s} \mapsto \frac{1}{s} \otimes_R m$.
The maps ϵ_M and ϵ'_M are inverse of each other, as can be checked on generators. To conclude the proof, one needs to check that:

- the collection of all maps ϵ_M, for $M \in {}_R\mathrm{Mod}$, assemble into a natural transformation $\epsilon\colon \big(\mathcal{S}^{-1}R \otimes_R -\big) \Rightarrow \big(\mathcal{S}^{-1}-\big)$;
- the collection of all maps ϵ'_M, for $M \in {}_R\mathrm{Mod}$, assemble into a natural transformation $\epsilon'\colon \big(\mathcal{S}^{-1}-\big) \Rightarrow \big(\mathcal{S}^{-1}R \otimes_R -\big)$.

This last check is left as exercise. □

The last step to prove Proposition 7.11 is the following Lemma.

Lemma 7.22. *Let R be a commutative ring, and let $\mathcal{S} \subset R$ be a subset. Then the functor $\mathcal{S}^{-1}-\colon {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$ is exact.*

Proof. Without loss of generality assume that $\mathcal{S}$ contains $R^\times$ and is closed under multiplication. By Lemma 7.21 we know that $\mathcal{S}^{-1}-$ can be identified with a *right exact* functor, namely $\mathcal{S}^{-1}R\otimes_R-$. So we do not really need to check whether $\mathcal{S}^{-1}-$ is enriched over abelian groups, is additive, is right exact. All we have to check is the following: if $i\colon M' \to M$ is an injective R-linear map, then $\mathcal{S}^{-1}i\colon \mathcal{S}^{-1}M' \to \mathcal{S}^{-1}M$ is injective as well.
Let $\frac{m'}{s} \in \mathcal{S}^{-1}M'$ and assume that $\frac{i(m')}{s} = \frac{0}{1} \in \mathcal{S}^{-1}M$. Then there exists a $t \in \mathcal{S}$ such that $t\cdot i(m') = ts\cdot 0 = 0$. The injectivity of i implies the equality $t\cdot m' = 0 \in M'$, and this witnesses that $\frac{m'}{s} = 0 \in \mathcal{S}^{-1}M'$. □

7.3. **A glimpse into abelian categories.**

Definition 7.23. Let $\mathcal{C}$ be a category enriched in $\mathbb{Z}$-modules, and let $f\colon x \to x'$ be a morphism. A *kernel* for f is an *equaliser* of the diagram

$$x \overset{f}{\underset{0}{\rightrightarrows}} x'$$

i.e. it is a couple $(\bar{y}, \bar{i})$ with $\bar{y} \in \mathcal{C}$ and $\bar{i}\colon \bar{y} \to x$, such that $0 \circ \bar{i} = f \circ \bar{i}$ (of course, the first composition is also $0 \in \mathrm{Hom}_\mathcal{C}(\bar{y}, x')$, and such that whenever (y, i) is another couple with $i\colon y \to x$ satisfying $0 \circ i = f \circ i$, then there is a unique $\theta\colon y \to \bar{y}$ such that $i = \bar{i} \circ \theta$.

Check that Definition 7.23 is equivalent to requiring that

$$(\bar{y}\ ;\ \bar{i}\colon \bar{y} \to x,\ f \circ \bar{i}\colon \bar{y} \to x')$$

is a *limit* in $\mathcal{C}$ of the diagram from Definition 7.23. Check that in ${}_R\mathrm{Mod}$ the categorical kernel of an R-linear map $f\colon M \to N$ exists and a model for it is given by the classical notion of kernel $\ker(f)$, together with the inclusion $i\colon \ker(f) \subset M$.

Notation 7.24. In a generic category $\mathcal{C}$, a morphism $f\colon x \to x'$ is called a *monomorphism* if the following holds: whenever $g, g'\colon y \to x$ are distinct morphisms from some object y to x, then $g \circ f \neq g' \circ f \in \mathrm{Hom}_\mathcal{C}(y, x')$.
If $\mathcal{C}$ is a $\mathbb{Z}$-linear category admitting a zero object 0 (e.g., an additive category), a morphism $f\colon x \to x'$ is a monomorphism if and only if the zero object satisfies the universal property for being a categorical kernel of f.

In ${}_R\mathrm{Mod}$, a monomorphism is an injective R-linear map.

Definition 7.25. Let $\mathcal{C}$ be a category enriched in $\mathbb{Z}$-modules, and let $f\colon x \to x'$ be a morphism. A *cokernel* for f is a *coequaliser* of the diagram

$$x \overset{f}{\underset{0}{\rightrightarrows}} x'$$

i.e. it is a couple $(\bar{z}, \bar{p})$ with $\bar{z} \in \mathcal{C}$ and $\bar{p}\colon x' \to \bar{z}$, such that $\bar{p} \circ 0 = \bar{p} \circ f$ (of course, the first composition is also $0 \in \mathrm{Hom}_{\mathcal{C}}(x, \bar{z})$, and such that whenever (z, p) is another couple with $p\colon x' \to z$ satisfying $p \circ 0 = p \circ f$, then there is a unique $\theta\colon \bar{z} \to z$ such that $p = \theta \circ \bar{p}$.

Again, a categorical cokernel can be equivalently defined as a colimit of the diagram from Definition 7.25. In ${}_R\mathrm{Mod}$ a model for the categorical cokernel is given by the classical cokernel $\mathrm{coker}(f)$, together with the projection to the quotient $p\colon N \to \mathrm{coker}(f)$, using the notation above.

Notation 7.26. In a generic category $\mathcal{C}$, a morphism $f\colon x \to x'$ is called an *epimorphism* if the following holds: whenever $g, g'\colon x' \to y$ are distinct morphisms from x' to some object y, then $f \circ g \neq f \circ g' \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.
If $\mathcal{C}$ is a $\mathbb{Z}$-linear category admitting a zero object 0 (e.g., an additive category), a morphism $f\colon x \to x'$ is an epimorphism if and only if the zero object satisfies the universal property for being a categorical cokernel of f.

In ${}_R\mathrm{Mod}$, an epimorphism is a surjective R-linear map.

Example 7.27. Let ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$ be the subcategory of ${}_{\mathbb{Z}}\mathrm{Mod}$ containing all *free* abelian groups and all $\mathbb{Z}$-linear maps between them. It is a $\mathbb{Z}$-linear category and it is also additive, as finite products of free abelian groups are again free abelian groups.
For a homomorphism $f\colon A \to B$ of free abelian groups, the classical $\ker(f)$ is a subgroup of a free abelian group, which is again a free abelian group[19] Check that $\ker(f)$, with its inclusion in A, is a categorical kernel for f in the category ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$ (Hint: if $\ker(f)$ has the universal property to be a categorical kernel in the larger category ${}_{\mathbb{Z}}\mathrm{Mod}$, then it also has the universal property in the smaller category ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$, in which it also lies.
We sketch now an argument showing that not every map in ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$ admits a cokernel. We start from the following black-boxed fact about the $\mathbb{Z}$-module $\prod_{i \in \mathbb{N}} \mathbb{Z}$:

$$\mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i \in \mathbb{N}} \mathbb{Z}; \mathbb{Z}\right) \cong \bigoplus_{j \in \mathbb{N}} \mathbb{Z},$$

where the isomorphism is given explicitly as the map from $\bigoplus_{j \in \mathbb{N}} \mathbb{Z}$ to $\mathrm{Hom}_{\mathbb{Z}}(\prod_{i \in \mathbb{N}} \mathbb{Z}; \mathbb{Z})$ sending the basis element $\iota_j(1)$ to the projection on the j^{th} coordinate $\pi_j\colon (n_i)_{i \in \mathbb{N}} \mapsto n_j$. In particular, note that $\prod_{i \in \mathbb{N}} \mathbb{Z}$ is *not* a free abelian groups: if it were, it would have a more-than-countable basis (indeed, the infinite product is a more-than-countable set), but then $\mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i \in \mathbb{N}} \mathbb{Z}; \mathbb{Z}\right)$ would also be a more-than-countable set, whereas by the black-boxed fact $\mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i \in \mathbb{N}} \mathbb{Z}; \mathbb{Z}\right)$ is a countable group.
Now let $g_1\colon F_1 \to F_0$ be a presentation of the $\mathbb{Z}$-module $\prod_{i \in \mathbb{N}} \mathbb{Z}$, i.e. $\mathrm{coker}(g_1) \cong \prod_{i \in \mathbb{N}} \mathbb{Z}$, where F_1 and F_0 are free abelian groups (likely of more-than-countable size). We claim that g_1 admits no categorical cokernel in ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$. Suppose instead that there is a categorical cokernel $(\bar{A}, \bar{p})$, where $\bar{A} = \bigoplus_{i \in \bar{I}} \mathbb{Z}$ is a free abelian group, and $\bar{p}\colon F_0 \to \bar{A}$. Then there are isomorphisms of abelian groups

$$\bigoplus_{j \in \mathbb{N}} \mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i \in \mathbb{N}} \mathbb{Z}; \mathbb{Z}\right) \cong \{f\colon F_0 \to \mathbb{Z} \,|\, f \circ g_1 \equiv 0\colon F_1 \to \mathbb{Z}\} \cong \mathrm{Hom}_{\mathbb{Z}}(\bar{A}, \mathbb{Z}).$$

[19]It is in general true that if R is a PID (or a field!), then a sub-R-module of a free R-module is again free. This is false for generic rings.

The first bijection is given by the black-boxed fact; the second is given by the fact that $\prod_{i\in\mathbb{N}}\mathbb{Z}$ is a cokernel of g_1 in the category ${}_{\mathbb{Z}}\mathrm{Mod}$; the third is given by the assumption that $\bar{A}$ is a cokernel of g_1 in the category ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$. And now we see a contradiction: if $\bar{I}$ is finite, then the first group is an infinitely generated free abelian group, and the last is a finitely generated free abelian group; if instead $\bar{I}$ is infinite, then the first is a countable group and the last is a more-than-countable group.

The previous example shows that it is possible to find an *additive* category $\mathcal{C}$ that does not admit all categorical cokernels of its morphisms. Taking $\left({}_{\mathbb{Z}}\mathrm{Mod}^{free}\right)^{op}$, we also get an example of an additive category that does not admit all categorical kernels. The next definition will thus discriminate good additive categories like ${}_R\mathrm{Mod}$ and Mod_R from less well-behaved additive categories like ${}_{\mathbb{Z}}\mathrm{Mod}^{free}$.

Definition 7.28. An additive category $\mathcal{C}$ is an *abelian category* if the following hold[20]:

- every morphism $f\colon x \to y$ in $\mathcal{C}$ admits a categorical kernel and a categorical cokernel;
- if $i\colon x \to y$ is a monomorphism and $(\bar{y}, \bar{p})$ is a categorical cokernel of i, then (x, i) is a categorical kernel of $\bar{p}$;
- if $p\colon x \to y$ is an epimorphism and $(\bar{x}, \bar{i})$ is a categorical kernel of p, then (y, p) is a categorical cokernel of $\bar{i}$.

The categories ${}_R\mathrm{Mod}$ and Mod_R are abelian. An abelian category is the right place where to define exact sequences, and in general where to study homological algebra. For example, a priori one can define the image of a morphism $f\colon x \to y$ in an abelian category $\mathcal{C}$ in two different ways:

- $\mathrm{Im}_1(f)$ is a categorical kernel of the natural map $y \to \mathrm{coker}(f)$;
- $\mathrm{Im}_2(f)$ is a categorical cokernel of the natural map $\ker f \to x$.

The map $f\colon x \to y$ is such that the composition $\ker f \to x \to y$ is zero, hence it induces a map $\mathrm{Im}_2(f) \to y$; this map has the property that the composition $\mathrm{Im}_2(f) \to y \to \mathrm{coker}(f)$ is zero (check this carefully!), hence it induces a map $\mathrm{Im}_2(f) \to \mathrm{Im}_1(f)$. The axioms of abelian category imply that this last map is an isomorphism. It follows that in an abelian category there is a well-behaved notion of “image of a morphism”.

A sequence $\dots x_{i+1} \overset{g_{i+1}}{\to} x_i \overset{g_i}{\to} x_{i-1} \dots$ in an abelian category can be declared to be exact at x_i if $g_i \circ g_{i+1} = 0$ is the zero map, and the natural map $\mathrm{Im}_2(g_{i+1}) \to \ker(g_i)$ is an isomorphism (here it is convenient to use the second definition of image, as a coker, in order to define a map out of it).

Exercise 7.29. Define projective and injective objects in a generic abelian category, mimicking Lemmas 5.3 and 5.13.

8. Chain complexes and homology

We fix a ring R throughout the section and work in the category ${}_R\mathrm{Mod}$ of left R-modules. An analogue discussion could be carried out in Mod_R, or in any abelian category $\mathcal{A}$.

[20]here 0 denotes the zero object of $\mathcal{C}$ i.e. the initial and terminal object; we use that $\mathcal{C}$ is additive to invoke the existence of this object.

8.1. **Chain complexes.** We already saw the notion of chain complex in Definition 3.14, but let us repeat it, with a slightly different notation that is somehow more standard.

Definition 8.1. A chain complex in ${}_R\mathrm{Mod}$ is the datum $(C_\bullet, d)$ of a sequence $C_\bullet(C_i)_{i\in\mathbb{Z}}$ of left R-modules $C_i \in {}_R\mathrm{Mod}$, and a sequence $d = (d_i\colon C_i \to C_{i-1})_{i\in\mathbb{Z}}$ of R-linear maps between them, with the requirement that for all $i \in \mathbb{Z}$ the composition $d_i \circ d_{i-1}\colon C_i \to C_{i-2}$ is the zero map. A chain complex can be represented by a diagram as follows:

$$\dots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \dots.$$

The condition that $d_i \circ d_{i-1} = 0$ can be reformulated by saying that $\mathrm{Im}(d_i) \subseteq \ker(d_{i-1})$. However, we do not require equality in the containment (otherwise we would get an exact sequence, which is a special case of chain complex). The elements of C_i are called *i-chains / chains of degree i*, and the map d_i is called the i^{th} *differential / boundary map.*

Definition 8.2. Let $(C_\bullet, d)$ be a chain complex in ${}_R\mathrm{Mod}$. For all $n \in \mathbb{Z}$ we define the n^{th} *homology group* of $(C_\bullet, d)$, denoted $H_n(C_\bullet, d)$, as the quotient $\ker(d_n)/\mathrm{Im}(d_{n+1})$, which is again a left R-module.[21]

Note that $H_n(C_\bullet, d)$ is neither a quotient module nor a submodule of C_n: it is rather a *subquotient*, that is, it is the quotient of a submodule of C_n by a subsubmodule; or equivalently, it is a submodule of a quotient of C_n.

Notation 8.3. Often we denote by $C_\bullet$ a chain complex $(C_\bullet, d)$, leaving the differentials understood. For $i \in \mathbb{Z}$ we denote by $\mathfrak{Z}_i(C_\bullet) = \ker(d_i) \subseteq C_i$, and i-chains lying inside $\mathfrak{Z}_i(C_\bullet)$ are called *i-cycles / cycles of degree i*. We also denote $\mathfrak{B}_i(C_\bullet) = \mathrm{Im}(d_{i+1}) \subseteq \mathfrak{Z}_i(C_\bullet) \subseteq C_i$, and i-cycles lying inside $\mathfrak{B}_i(C_\bullet)$ are called *i-boundaries / boundaries of degree i.*

In this notation, we have $H_n(C_\bullet) = \mathfrak{Z}_n(C_\bullet)/\mathfrak{B}_n(C_\bullet)$.
Why are we interested in chain complexes and their homology?

- One motivation comes from the already-mentioned fact that, in general, an additive functor between abelian categories $F\colon \mathcal{A} \to \mathcal{B}$ (think of categories of modules, if you like) sends exact sequences to chain complexes that may not be exact sequences. Homology groups should measure the "failure of a chain complex from being an exact sequence", and therefore they could help keeping track of what's going on when we manipulate exact sequences with additive functors.
- Historically, chain complexes and homology were introduced in order to associate algebraic invariants to topological spaces, in order to distinguish non-homeomorphic spaces. Let X and Y be given topological spaces: if X and Y are homeomorphic, we can prove it by exhibiting a continuous bijection $X \to Y$ with continuous inverse; but if X and Y are not homeomorphic, how are we going to prove it? We usually cannot "try" one by one all maps $X \to Y$ and check that all of them are not homeomorphism! An algebraic invariant α of topological spaces is roughly supposed to associate with every topological space X an invariant $\alpha(X)$, which can be a number,

[21]Somehow, one usually says "homology group" instead of "homology module"...

a vector space, a finitely generated abelian group... in any case something easily *computable* and *comparable*; moreover we require that if X and Y are homeomorphic, then $\alpha(X)$ is "equivalent" to $\alpha(Y)$ (if they are numbers, they must be equal; if they are vector spaces or abelian groups, they must be isomorphic, and so on). An algebraic invariant can be used backwards to prove non-homeomorphism of spaces: if X and Y are two spaces and if we can compute $\alpha(X)$ and $\alpha(Y)$ and determine that they are not "equivalent", we have a proof that X and Y are not homeomorphic.
- In a similar way, chain complexes and homology are used to attach algebraic invariants to other complicated mathematical objects, like (non-commutative) groups and algebraic varieties.

Example 8.4. Consider the short exact sequence of abelian groups $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{[-]_2} \mathbb{Z}/2$ as an (exact) chain complex

$$\dots \longrightarrow C_2 = 0 \longrightarrow C_1 = \mathbb{Z} \xrightarrow{\cdot 2} C_0 = \mathbb{Z} \xrightarrow{[-]_2} C_{-1} = \mathbb{Z}/2 \longrightarrow C_{-2} = 0 \longrightarrow \dots$$

We can apply the additive functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ and obtain (after a little work) the chain complex

$$\dots \longrightarrow C'_2 = 0 \longrightarrow C'_1 = \mathbb{Z}/2 \xrightarrow{0} C'_0 = \mathbb{Z}/2 \xrightarrow{\mathrm{Id}_{\mathbb{Z}/2}} C'_{-1} = \mathbb{Z}/2 \longrightarrow C'_{-2} = 0 \longrightarrow \dots$$

The homology groups of $C'_\bullet$ can be computed as follows:

- $H_{-1}(C'_\bullet) = Z_{-1}(C'_\bullet)/B_{-1}(C'_\bullet) \cong (\mathbb{Z}/2)/(\mathbb{Z}/2) \cong 0$, i.e. $C'_\bullet$ is exact at C'_{-1};
- $H_0(C'_\bullet) = Z_0(C'_\bullet)/B_0(C'_\bullet) \cong 0/0 \cong 0$, i.e. $C'_\bullet$ is exact at C'_0;
- $H_1(C'_\bullet) = Z_1(C'_\bullet)/B_1(C'_\bullet) \cong (\mathbb{Z}/2)/0 \cong \mathbb{Z}/2$, and in particular $C'_\bullet$ is not exact at C'_1;
- for all other n, the homology group $H_n(C'_\bullet)$ is a subquotient of the zero module $C_n = 0$, and hence it vanishes as well.

The homology group $H_1(C'_\bullet)$ measures the failure of $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ preserving the exactness of the SES $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{[-]_2} \mathbb{Z}/2$.

Example 8.5. For $n \geq 0$ denote by Δ^n the standard n-simplex:

$$\Delta^n = \big\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0, \dots, x_n \geq 0\,;\, x_0 + \dots x_n = 1\big\} \subset \mathbb{R}^{n+1}.$$

Let X be a topological space; then we can define, for all $n \geq 0$, the set $\mathcal{S}_n(X)$ of all continuous maps $\sigma\colon \Delta^n \to X$, and the associated free abelian group $S_n(X) = \bigoplus_{\sigma \in \mathcal{S}_n(X)} \mathbb{Z}$. For $n < 0$ we also let $S_n(X)$ be the zero abelian group.
For $n \geq 1$ and for all $0 \leq i \leq n$ we define $d^{n,i}\colon \Delta^{n-1} \to \Delta^n$ as the continuous map $(x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$ inserting a "0" in position i. Every continuous map $\sigma\colon \Delta^n \to X$ gives rise to a map $d_{n,i}\sigma := (\sigma \circ d^{n,i})\colon \Delta^{n-1} \to X$. We thus get, for all $0 \leq i \leq n$, a map of sets $d_{n,i}\colon \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)$.
We can now define a map $d_{n,i}\colon S_n(X) \to S_{n-1}$ by setting on basis elements $\iota_\sigma(1) \mapsto \iota_{d_{n,i}\sigma}(1)$. Finally, we take an alternating sum of the previous, and define

$$d_n = d_{n,0} - d_{n,1} + d_{n,2} - \dots + (-1)^n d_{n,n}\colon S_n(X) \to S_{n-1}X.$$

For $n \leq 0$ we also let d_n be the zero map. And now three miracles occur.

- The collections $(S_n(X))_{n \in \mathbb{Z}}$, together with the maps $d_n\colon S_n(X) \to S_{n-1}(X)$, gives rise to a chain complex.

- As you may have noticed, the abelian groups $S_n(X)$ tend to be huge: they have a generator for each continuous map $\Delta^n \to X$; nevertheless, the homology groups $H_i(S_\bullet(X))$ are often small and, most importantly, computable.
- It should be apparent that if there is a homeomorphism $X \cong Y$ between two spaces, then we get bijections of sets $\mathcal{S}_n(X) \cong \mathcal{S}_n(Y)$ and thus isomorphisms of abelian groups $S_n(X) \cong S_n(Y)$, so we can in fact identify the two chain complexes $(S_\bullet(X), d)$ and $(S_\bullet(Y), d)$. In particular, we have that for all $i \in \mathbb{Z}$, the abelian groups $H_i(S_\bullet(X))$ and $H_i(S_\bullet(Y))$ are isomorphic. Viceversa, suppose that X and Y are two spaces, and suppose that for some i both abelian groups $H_i(S_\bullet(X))$ and $H_i(S_\bullet(Y))$ can be computed, they are finitely generated abelian groups, but they have different lists of cyclic summands: then $H_i(S_\bullet(X))$ and $H_i(S_\bullet(Y))$ are *not isomorphic*, and hence X and Y are *not homeomorphic*.

Exercise 8.6. Let X be a topological space and let G be a (discrete) group. Suppose that G acts on the space X by homeomorphism. Prove the following:

- G acts on each set $\mathcal{S}_n(X)$;
- each $\mathbb{Z}$-module $S_n(X)$ can be upgraded to a $\mathbb{Z}[G]$-module;
- each homology group $H_n(S_\bullet(X))$ is also naturally a $\mathbb{Z}[G]$-module.

It can happen that two spaces X and Y both have an action of G, and are homeomorphic, but we find it difficult to find a G-*equivariant* homeomorphism $X \to Y$, and we start thinking that it just does not exist. Then the abelian groups $H_n(X)$ and $H_n(Y)$ are going to be isomorphic for all n, but perhaps we are lucky enough to find some n such that $H_n(X)$ and $H_n(Y)$ are not isomorphic *as* $\mathbb{Z}[G]$-*modules*; this would be a proof that X and Y are not G-*equivariantly* homeomorphic.

Example 8.7. Here is an example of a chain complex of abelian groups, together with the subgroups of cycles, boundaries, and the homology groups. All dots represent infinite sequences of zero abelian groups and zero maps between them, so they can be safely neglected.

$$\begin{array}{cccccccc}
\dots C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 \dots \\
\dots 0 \longrightarrow & \mathbb{Z} \xrightarrow{[3\cdot -]_6} & \mathbb{Z}/6 \xrightarrow{[2\cdot -]_6} & \mathbb{Z}/6 \xrightarrow{0} & \mathbb{Z} \xrightarrow{\cdot 5} & \mathbb{Q} \xrightarrow{[-]_{\mathbb{Z}}} & \mathbb{Q}/\mathbb{Z} \longrightarrow & 0 \dots \\
 & Z_6 = 2\mathbb{Z} & Z_5 = \langle [3]_6 \rangle & Z_4 = \mathbb{Z}/6 & Z_3 = 0 & Z_2 = \mathbb{Z} & Z_1 = \mathbb{Q}/\mathbb{Z} & \\
 & B_6 = 0 & B_5 = \langle [3]_6 \rangle & B_4 = \langle [2]_6 \rangle & B_3 = 0 & B_2 = 5\mathbb{Z} & B_1 = \mathbb{Q}/\mathbb{Z} & \\
 & H_6 \cong \mathbb{Z} & H_5 \cong 0 & H_4 \cong \mathbb{Z}/3 & H_3 \cong 0 & H_2 \cong \mathbb{Z}/5 & H_1 \cong 0. &
\end{array}$$

8.2. **Chain maps.**

Definition 8.8. Let $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$ be two chain complexes in ${}_R\mathrm{Mod}$. An R-linear *chain map* $f\colon C_\bullet \to D_\bullet$ is a collection of R-linear maps $f_i\colon C_i \to D_i$ such that the following diagram commutes for all $i \in \mathbb{Z}$:

$$\begin{array}{ccc} C_i & \xrightarrow{d_i^C} & C_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ D_i & \xrightarrow{d_i^D} & D_{i-1}. \end{array}$$

An isomorphism of chain complex is an R-linear chain map f such that each f_i is an R-linear isomorphism.

Given R-linear chain maps $C_\bullet \xrightarrow{f} D_\bullet$ and $D_\bullet \xrightarrow{g} E_\bullet$, their composition $f \circ g$ is the R-linear chain map $C_\bullet \to E_\bullet$ obtained as the collection of R-linear maps $f_i \circ g_i \colon C_i \to E_i$: check that this is indded again an R-linear chain map.

We thus obtain a category $\mathrm{Ch}({}_R\mathrm{Mod})$ with objects being chain complexes in ${}_R\mathrm{Mod}$, and morphisms being R-linear chain maps. We shall also abbreviate ${}_R\mathrm{Ch} = \mathrm{Ch}({}_R\mathrm{Mod})$ and $\mathrm{Ch}_R = \mathrm{Ch}(\mathrm{Mod}_R)$.

Note that if $f, g \colon C_\bullet \to D_\bullet$ are R-linear chain maps, then their sum $f + g$, given in degree i by $f_i + g_i \colon C_i \to D_i$, is again an R-linear chain map. Check that in this way the category ${}_R\mathrm{Ch}$ becomes enriched over abelian groups.

Note also that if $C_\bullet$ and $D_\bullet$ are objects in ${}_R\mathrm{Ch}$, then we can define $C_\bullet \oplus D_\bullet$ as the chain complex whose i-chains are $C_i \oplus D_i$, and whose i^{th} differential is given by the map $C_i \oplus D_i \to C_{i-1} \oplus D_{i-1}$ mapping $C_i \to C_{i-1}$ by the i^{th} differential of $C_\bullet$, and $D_i \to D_{i-1}$ by the i^{th} differential of $D_\bullet$. Check that $C_\bullet \oplus D_\bullet$ is both a categorical product and a categorical coproduct of $C_\bullet$ and $D_\bullet$ in ${}_R\mathrm{Ch}$.

Check also that the zero chain complex $0_\bullet$, all of whose R-modules are the zero module, is an initial and terminal object in ${}_R\mathrm{Ch}$.

We obtain (after checking a lot of details) that ${}_R\mathrm{Ch}$ is an additive category. In general, for an abelian category $\mathcal{A}$, one can define a new category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$, and this category is an additive category.

We have already invoked in Example 8.5 the principle that if two chain complexes are isomorphic, we can just identify them and thus get identifications of their homology groups. To make this precise, we need the following construction.

Let $f \colon (C_\bullet, d^C) \to (D_\bullet, d^D)$ be an R-linear chain map. Then the following happens, for all $n \in \mathbb{Z}$:

- the map $f_i \colon C_i \to D_i$ restricts to an R-linear map $\mathfrak{Z}_i(C_\bullet) \to \mathfrak{Z}_i(D_\bullet)$: indeed, if $x \in \mathfrak{Z}_i(C_\bullet) \subseteq C_i$, then $(x)d_i^C = 0$, hence $0 = ((x)d_i^C)f_{i-1} = ((x)f_i)d_i^D$, implying that $f_i(x) \in \mathfrak{Z}_i(D_\bullet)$;
- the map $f_i \colon \mathfrak{Z}_i(C_\bullet) \to \mathfrak{Z}_i(D_\bullet)$ further restricts to an R-linear map $\mathfrak{B}_i(C_\bullet) \to \mathfrak{B}_i(D_\bullet)$: check it!

As a consequence, the composition $\mathfrak{Z}_i(C_\bullet) \xrightarrow{f_i} \mathfrak{Z}_i(D_\bullet) \to H_i(D_\bullet)$ vanishes on $\mathfrak{B}_i(C_\bullet)$, and hence induces an R-linear map out of the quotient $H_i(C_\bullet) \to H_i(D_\bullet)$.

Definition 8.9. Let $f \colon C_\bullet \to D_\bullet$ be an R-linear chain map. We denote by $H_n(f) \colon H_n(C_\bullet) \to H_n(D_\bullet)$ the induced R-linear map between homology groups.

We obtain, for each $n \in \mathbb{Z}$, a functor $H_n(-) \colon {}_R\mathrm{Ch} \to {}_R\mathrm{Mod}$, sending $C_\bullet \mapsto H_n(C_\bullet)$ and sending $(f \colon C_\bullet \to D_\bullet) \mapsto (H_n(f) \colon H_n(C_\bullet) \to H_n(D_\bullet)$.

Check that $H_n(-)$ is also an additive functor: for instance $H_n(0_\bullet) \cong 0$ and $H_n(C_\bullet \oplus D_\bullet) \cong H_n(C_\bullet) \oplus H_n(D_\bullet)$.

Often the map $H_n(f)$ is written simply as f_*. The "$_*$" is a common way to mean that the new map is constructed in a functorial way from the old map.

Example 8.10. If $f\colon C_\bullet \to D_\bullet$ is an isomorphism, then $H_n(f)\colon H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism. However, the converse does not hold: consider for example the following chain map of chain complexes over $\mathbb{Z}$:

$$\begin{array}{ccccccc} \dots C_3 = 0 & \longrightarrow & C_2 = \mathbb{Z} & \xrightarrow{\mathrm{Id}_{\mathbb{Z}}} & C_1 = \mathbb{Z} & \longrightarrow & C_0 = 0 \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots D_3 = 0 & \longrightarrow & D_2 = 0 & \longrightarrow & D_1 = 0 & \longrightarrow & D_0 = 0 \dots \end{array}$$

Then both $C_\bullet$ and $D_\bullet$ are exact sequences, that is, $H_n(C_\bullet) \cong H_n(D_\bullet) \cong 0$ for all $n \in \mathbb{Z}$; it follows that $H_n(f)$ is an isomorphism between zero groups for all n. Yet $C_\bullet$ and $D_\bullet$ are not isomorphic chain complexes.

Example 8.11. If $\phi\colon X \to Y$ is a continuous map of topological spaces, then it induces a map of chain complexes $S_\bullet(X) \to S_\bullet(Y)$, which is an isomorphism of chain complexes in the special case in which ϕ is a homeomorphism.
In turn, ϕ induces a map of homology groups $H_n(S_\bullet(X)) \to H_n(S_\bullet(Y))$, which is an isomorphism of abelian groups in the special case in which ϕ is a homeomorphism.

8.3. ${}_R\mathrm{Ch}$ **as an abelian category.** In constructing the functor $H_n(-)$ we have noticed that if we have an R-linear chain map $f\colon (C_\bullet, d^C) \to (D_\bullet, d^D)$ then f_i sends $\mathfrak{Z}_i(C_\bullet)$ inside $\mathfrak{Z}_i(D_\bullet)$ and $\mathfrak{B}_i(C_\bullet)$ inside $\mathfrak{B}_i(D_\bullet)$. One can use a similar argument to prove the following:

- for all $i \in \mathbb{Z}$, the differential $d_i^C\colon C_i \to C_{i-1}$ sends $\ker(f_i)$ inside $\ker(f_{i-1})$;
- we obtain a new chain complex $(\ker(f), d^C|_{\ker(f)})$, which is a categorical kernel of f in the category ${}_R\mathrm{Ch}$;
- for all $i \in \mathbb{Z}$, the differential $d_i^D\colon D_i \to D_{i-1}$ sends $\mathrm{Im}(f_i)$ inside $\mathrm{Im}(f_{i-1})$, and thus induces a differential $d_i^{\mathrm{coker}(f)}\colon \mathrm{coker}(f_i) \to \mathrm{coker}(f_{i-1})$;
- we obtain a new chain complex $(\mathrm{coker}(f), d^{\mathrm{coker}(f)}$, which is a categorical cokernel of f in the category ${}_R\mathrm{Ch}$;

It turns out (after checking many details) that the category ${}_R\mathrm{Ch}$ is an abelian category[22]. More generally, whenever $\mathcal{A}$ is an abelian category, then also $\mathrm{Ch}(\mathcal{A})$ is an abelian category. In the following examples we discuss what kernels, cokernels and images are in the category ${}_R\mathrm{Ch}$, and investigate also the notion of exactness.

Example 8.12. Consider an R-linear chain map $f\colon C_\bullet \to D_\bullet$; then:

- $\ker(f)$ is a sub-chain complex of $C_\bullet$, obtained by putting $\ker(f_i)$ in degree i, and by restricting the differential of $C_\bullet$;
- $\mathrm{Im}(f)$ is a sub-chain complex of $D_\bullet$, obtained by putting $\mathrm{Im}(f_i)$ in degree i, and by restricting the differential of $D_\bullet$;
- $\mathrm{coker}(f)$ is the quotient chain complex $D_\bullet/\mathrm{Im}(f)$, obtained by putting $D_i/\mathrm{Im}(f_i)$ in degree i, with the induced differential coming from the differential of $D_\bullet$;
- "f is a monomorphism" iff "$\ker(f)$ is the zero chain complex" iff "for all $i \in \mathbb{Z}$ the R-module $\ker(f_i)$ vanishes" iff "each f_i is injective"; in this case we also say that f is an injective R-linear chain map;

[22]We already mentioned that it is an additive category

- "f is an epimorphism" iff "$\operatorname{coker}(f)$ is the zero chain complex" iff "for all $i \in \mathbb{Z}$ the R-module $\operatorname{coker}(f_i)$ vanishes" iff "each f_i is surjective"; in this case we also say that f is a surjective R-linear chain map.

Example 8.13. A SES of chain complexes in ${}_R\mathrm{Mod}$ has the form

$$A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} C_\bullet,$$

where $A_\bullet$, $B_\bullet$ and $C_\bullet$ are chain complexes of left R-modules, i is an injective R-linear chain map, p is a surjective R-linear chain map, and there is an equality $\ker(p) = \mathrm{Im}(i)$ of sub-chain complexes of $B_\bullet$.
In particular, for all $j \in \mathbb{Z}$ we obtain a SES of left R-modules

$$A_j \xrightarrow{i_j} B_j \xrightarrow{p_j} C_j.$$

Example 8.14. A chain complex in ${}_R\mathrm{Ch}$ is commonly known with the name of *double complex of left R-modules*: it has the form of a diagram of left R-modules and R-linear maps

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{i+1,j+1} & \longrightarrow & C_{i+1,j} & \longrightarrow & C_{i+1,j-1} & \longrightarrow & C_{i+1,j-2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{i,j+1} & \longrightarrow & C_{i,j} & \longrightarrow & C_{i,j-1} & \longrightarrow & C_{i,j-2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{i-1,j+1} & \longrightarrow & C_{i-1,j} & \longrightarrow & C_{i-1,j-1} & \longrightarrow & C_{i-1,j-2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

such that each square commutes, each composition of two consecutive horizontal maps is the zero map, and each composition of two consecutive vertical maps is the zero map. In other words: each column is a chain complex, each sequence of horizontal arrows between two consecutive columns is a chain map, and the composition of two consecutive chain maps between columns is the zero chain map. If (and only if) each row is exact, we get an exact sequence of chain complexes, whose terms are the columns and whose maps are the chain maps between columns.

So far we have seen that both ${}_R\mathrm{Ch}$ and ${}_R\mathrm{Mod}$ are abelian categories; moreover, for all $n \in \mathbb{Z}$, we have constructed an additive functor $H_n(-)\colon {}_R\mathrm{Ch} \to {}_R\mathrm{Mod}$. The question whether $H_n(f)$ is an *exact* functor makes sense, and it would be wonderful if this were the case; unfortunately, no.

Example 8.15. Let $R = \mathbb{Z}$. Let $A_\bullet$, $B_\bullet$, $C_\bullet$ be the chain complexes characterised as follows:

- $A_n = 0$ for all $n \neq 0$, and $A_0 = \mathbb{Z}$;
- $C_n = 0$ for all $n \neq 1$, and $C_1 = \mathbb{Z}$;

- $B_n = 0$ for all $n \neq 0, 1$, and $B_1 = B_0 = \mathbb{Z}$; the differential $B_1 \to B_0$ is $\mathrm{Id}_{\mathbb{Z}}$.

Moreover, let $i\colon A_\bullet \to B_\bullet$ be the chain map characterised by $i_0 = \mathrm{Id}_{\mathbb{Z}}$, and let $p\colon B_\bullet \to C_\bullet$ be the chain map characterised by $p_1 = \mathrm{Id}_{\mathbb{Z}}$. We obtain a SES of chain complexes (columns in the following diagram)

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
A_2 = 0 & \longrightarrow & B_2 = 0 & \longrightarrow & C_2 = 0 \\
\downarrow & & \downarrow & & \downarrow \\
A_1 = 0 & \longrightarrow & B_1 = \mathbb{Z} & \xrightarrow{\mathrm{Id}_{\mathbb{Z}}} & C_1 = \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{\mathbb{Z}}} & & \downarrow \\
A_0 = \mathbb{Z} & \xrightarrow{\mathrm{Id}_{\mathbb{Z}}} & B_0 = \mathbb{Z} & \longrightarrow & C_0 = 0 \\
\downarrow & & \downarrow & & \downarrow \\
A_{-1} = 0 & \longrightarrow & B_{-1} = 0 & \longrightarrow & C_{-1} = 0 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

If we apply the functors $H_1(-)$ and $H_0(-)$ we obtain the following short sequences of abelian groups[23]

$$H_1(A_\bullet) \cong 0 \longrightarrow H_1(B_\bullet) \cong 0 \longrightarrow H_1(C_\bullet) \cong \mathbb{Z};$$

$$H_0(A_\bullet) \cong \mathbb{Z} \longrightarrow H_0(B_\bullet) \cong 0 \longrightarrow H_0(C_\bullet) \cong 0.$$

Sadly, the first sequence is not left exact, and the second is not right exact; so in general we cannot expect $H_n(-)$ to be neither a left exact nor a right exact functor (and even less, an exact functor).
However, we notice two remarkable facts:

- both short sequences are exact *in the middle*;
- the first sequence has a "$\mathbb{Z}$" too much to be exact, sitting on the right, and also the second sequence has precisely a "$\mathbb{Z}$" too much to be exact, sitting on the left.

We will explain these phenomena with the snake lemma.

8.4. **The snake lemma.**

Lemma 8.16 (Snake lemma)**.** *Let* $A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} C_\bullet$ *be a SES in* ${}_R\mathrm{Ch}$*. Then there are* natural *R-linear maps* $\partial_n\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$*, called* connecting homomorphisms*, such that the following is an exact sequence of left R-modules, called the*

[23]All other functors $H_n(-)$ give rise to the trivial SES $0 \to 0 \to 0$, so we neglect them here.

homology long exact sequence*:*

$$
\begin{array}{l}
\cdots \xrightarrow{\partial_{n+2}} H_{n+1}(A_\bullet) \xrightarrow{H_{n+1}(i)} H_{n+1}(B_\bullet) \xrightarrow{H_{n+1}(p)} H_{n+1}(C_\bullet) \\
\xrightarrow{\partial_{n+1}} H_n(A_\bullet) \xrightarrow{H_n(i)} H_n(B_\bullet) \xrightarrow{H_n(p)} H_n(C_\bullet) \\
\xrightarrow{\partial_n} H_{n-1}(A_\bullet) \xrightarrow{H_{n-1}(i)} H_{n-1}(B_\bullet) \xrightarrow{H_{n-1}(p)} H_{n-1}(C_\bullet) \\
\xrightarrow{\partial_{n-1}} \cdots
\end{array}
$$

You should appreciate how the last diagram has the form of a snake. The proof of the snake lemma is not difficult, but it is very long; I strongly recommend to finish on your own the beginning of proof that you find here. After the partial proof, I will comment on the word "natural" appearing in the statement.

Partial proof of the snake lemma. Even before constructing the maps ∂_n, we can check exactness at $H_n(B_\bullet)$ of the sequence $H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet)$.
Let therefore $x \in \mathfrak{Z}_n(B_\bullet)$ be a cycle that represents a homology class $[x] \in H_n(B_\bullet)$. Suppose that $H_n(p)\colon [x] \mapsto 0 \in H_n(C_\bullet)$. This means that $(x)p_n \in C_n$ is not only inside $\mathfrak{Z}_n(C_\bullet)$ (as the fact that p is a chain map guarantees), but also in $\mathfrak{B}_n(C_\bullet)$. That means that there exists $y \in C_{n+1}$ with $(y)d^C_{n+1} = (x)p_n$. By surjectivity of $p_{n+1}\colon B_{n+1} \to C_{n+1}$, we can find $z \in B_{n+1}$ such that $(z)p_{n+1} = y$. Let $w = d^B_{n+1}(z)$. Then we have

$$(w)p_n = ((z)d^B_{n+1})p_n = ((z)p_{n+1})d^C_{n+1} = (y)d^C_{n+1} = (x)p_n.$$

It follows that $(x-w)p_n = 0$; moreover $[x-w] = [x] \in H_n(B_\bullet)$, because $w \in \mathfrak{B}_n(B_\bullet)$. Since $A_n \to B_n \to C_n$ is a SES, there exists a *unique* $t \in A_n$ such that $(t)i_n = x - w$. Moreover we have $((t)d^A_n)i_{n-1} = ((t)i_n)d^B_n = (x-w)d^B_n = 0$, and since i_{n-1} is injective we conclude that $(t)d^A_n$ is a cycle in $\mathfrak{Z}_n(A_\bullet)$, i.e. it represents a homology class $[t] \in H_n(A_\bullet)$. Finally, we have $H_n(i)\colon [t] \mapsto [x-w]$, because $i_n\colon t \mapsto x - w$. The conclusion is that whenever $[x] \in H_n(B_\bullet)$ is a homology class in the kernel of $H_n(p)$, there is a homology class $[t] \in H_n(A_\bullet)$ with $H_n(i)\colon [t] \mapsto [x]$. This was the difficult half of checking exactness at $H_n(B_\bullet)$: the easy part is to check that $H_n(i) \circ H_n(p)$ is the zero map, but this follows from the fact that H_n is an additive functor: $H_n(i) \circ H_n(p) = H_n(i \circ p) = H_n(0) = 0$, together with the fact that $i \circ p = 0$ (the last composition is in the category ${}_R\mathrm{Ch}$, but degreewise it is a composition in the category ${}_R\mathrm{Mod}$).

Next, we define an R-linear map $\tilde{\partial}_n\colon \mathfrak{Z}_n(C_\bullet) \to H_{n-1}(A_\bullet)$. Let now $x \in \mathfrak{Z}_n(C_\bullet)$; by surjectivity of p_n we may find $y \in B_n$ with $(y)p_n = x$. Let now $z = (y)d_n^B$; we then have

$$(z)p_{n-1} = ((y)d_n^B)p_{n-1} = ((y)p_n)d_n^C = (x)d_n^C = 0,$$

because x is a cycle. It follows by exactness of $A_{n-1} \to B_{n-1} \to C_{n-1}$ that there is a unique $t \in A_{n-1}$ such that $(t)i_{n-1} = z$. Moreover we have

$$((t)d_{n-1}^A)i_{n-2} = ((t)i_{n-1})d_{n-1}^B = (z)d_{n-1}^B = ((y)d_n^B)d_{n-1}^B = 0,$$

which, together with the fact that i_{n-2} is injective, implies that $(t)d_{n-1}^A = 0$, i.e. (t) is a cycle.

We claim that the map of sets $\tilde{\partial}_n\colon \mathfrak{Z}_n(C_\bullet) \to H_{n-1}(A_\bullet)$ sending $x \mapsto [t]$ is well-defined. There was in fact a choice in the above argument, namely we choose one of the (possibly many) elements $y \in B_n$ with $(y)p_n = x$. Suppose that we had chosen another element $y' \in B_n$ with $(y')p_n = x$: what would have been the result?

Instead of z, we would have taken $z' = (y')d_n^B$, and instead of t, we would have taken the unique element $t' \in A_{n-1}$ such that $(t')i_{n-1} = z'$. However, note that $y' - y$ is in the kernel of p_n, and hence in the image of i_n. Let $w \in A_n$ be the unique element with $(w)i_n = y' - y$. We can now compute

$$((w)d_n^A)i_{n-1} = ((w)i_n)d_n^B = (y' - y)d_n^B = z' - z = (t' - t)i_{n-1},$$

and since i_{n-1} is injective, we obtain that $(w)d_n^A = t' - t$. The conclusion is: perhaps $t' \neq t$, but at least the difference of these two cycles is a boundary, and hence $[t] = [t'] \in H_{n-1}(A_\bullet)$.

Great, the map of sets $\tilde{\partial}_n\colon \mathfrak{Z}_n(C_\bullet) \to H_{n-1}(A_\bullet)$ is well-defined. Is it an R-linear map?

- If $x_1, x_2 \in \mathfrak{Z}_n(C_\bullet)$ and $y_1, y_2 \in B_n$ are elements such that $(y_1)p_n = x_1$ and $(y_2)p_n = x_2$, then $y_1 + y_2$ is a good example of an element of B_n which is sent to $x_1 + x_2$ under p_n; we can use this element to construct $\tilde{\partial}_n(x_1 + x_2)$, and obtain (with obvious meaning of the symbols)
$$\tilde{\partial}_n(x_1 + x_2) = [t_1 + t_2] = [t_1] + [t_2] = \tilde{\partial}_n(x_1) + \tilde{\partial}_n(x_2).$$
- If $x \in \mathfrak{Z}_n(C_\bullet)$ and $r \in R$, and if $y \in B_n$ is such that $(y)p_n = x$, then $r \cdot y \in B_n$ is a good example of an element in B_n which is sent to $r \cdot x$ under p_n; again we obtain, after a few simple steps, that
$$\tilde{\partial}_n(r \cdot x) = [r \cdot t] = r \cdot [t] = r \cdot \tilde{\partial}_n(x).$$

Finally, is it true that $\tilde{\partial}_n$ vanishes on the submodule $\mathfrak{B}_n(C_\bullet)$, so that it induces an R-linear map $\partial_n\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$? If $x \in \mathfrak{B}_n(C_\bullet)$, then there exists $u \in C_{n+1}$ with $(u)d_{n+1}^C = x$; by surjectivity of p_{n+1} we can find $v \in B_{n+1}$ with $(v)p_{n+1} = u$; we then have

$$((v)d_{n+1}^B)p_n = ((v)p_{n+1})d_{n+1}^C = (u)d_{n+1}^C = x,$$

so that the element $y := (v)d_{n+1}^B$ can be used to construct $\tilde{\partial}_n(x)$. But the next step is to take $z = (y)\partial_n^B = ((v)d_{n+1}^B)d_n^B = 0$, and then also $t \in A_{n-1}$ will just be 0. It follows that $\tilde{\partial}_n(x) = [0] \in H_{n-1}(A_\bullet)$.

At this point I stop. I think that it takes at least one hour to check all details of the proof (including checking that all maps are R-linear). And as I said, this is the kind of proof that it is better to do on one's own. □

About the word "natural": suppose that you have *two* SES of chain complexes, connected by R-linear chain maps as in the following diagram, which is assumed to be a commutative diagram[24]

$$\begin{array}{ccccc} A_\bullet & \xrightarrow{i} & B_\bullet & \xrightarrow{p} & C_\bullet \\ \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} \\ A'_\bullet & \xrightarrow{i'} & B'_\bullet & \xrightarrow{p'} & C'_\bullet \end{array}$$

Then combining the two long exact sequences provided by the snake lemma, and the morphisms induced in homology by the chain maps f_A, f_B and f_C, we obtain a diagram

$$\begin{array}{ccccccccc} \dots H_n(A_\bullet) & \xrightarrow{H_n(i)} & H_n(B_\bullet) & \xrightarrow{H_n(p)} & H_n(C_\bullet) & \xrightarrow{\partial_n} & H_{n-1}(A_\bullet) & \xrightarrow{H_{n-1}(i)} & H_{n-1}(B_\bullet)\dots \\ \downarrow{\scriptstyle H_n(f_A)} & & \downarrow{\scriptstyle H_n(f_B)} & & \downarrow{\scriptstyle H_n(f_C)} & & \downarrow{\scriptstyle H_{n-1}(f_A)} & & \downarrow{\scriptstyle H_{n-1}(f_B)} \\ \dots H_n(A'_\bullet) & \xrightarrow{H_n(i')} & H_n(B'_\bullet) & \xrightarrow{H_n(p')} & H_n(C'_\bullet) & \xrightarrow{\partial'_n} & H_{n-1}(A'_\bullet) & \xrightarrow{H_{n-1}(i')} & H_{n-1}(B'_\bullet)\dots \end{array}$$

The first two squares (of the one drawn) and the fourth one commute: indeed $H_n(-)$ is a functor (and so is $H_{n-1}(-)$). It would therefore be very sad if the third square didn't commute! Fortunately, it does.[25]

Example 8.17. What happens if $A_\bullet$ and $C_\bullet$ are chain complexes, and we take $B_\bullet := A_\bullet \oplus C_\bullet$? The we obtain a *split* SES of chain complexes $A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} C_\bullet$, i.e. there is an R-linear chain map $s\colon C_\bullet \to B_\bullet$ which is a section of p.
It follows that for all $n \in \mathbb{Z}$ the R-linear map $H_n(p)\colon H_n(B_\bullet) \to H_n(C_\bullet)$ is surjective: it is in fact *split surjective*, i.e. it is surjective and admits a section, namely $H_n(s)$. In a long exact sequence, if a map is surjective, then the next map is zero, and the second next map is injective. The long exact sequence of homology groups

[24]We can think of a category $SES({}_R\mathrm{Ch})$ whose objects are short exact sequences of chain complexes of left R-modules, and whose morphisms are diagrams like the given one

[25]Can you interpret ∂_n as a natural transformation between two functors $SES({}_R\mathrm{Ch}) \to {}_R\mathrm{Mod}$?

looks like

$$
\begin{array}{l}
\cdots \\
\xrightarrow{\partial_{n+2}=0} H_{n+1}(A_\bullet) \xhookrightarrow{H_{n+1}(i)} H_{n+1}(B_\bullet) \xrightarrow{H_{n+1}(p)} H_{n+1}(C_\bullet) \\
\xrightarrow{\partial_{n+1}=0} H_n(A_\bullet) \xhookrightarrow{H_n(i)} H_n(B_\bullet) \xrightarrow{H_n(p)} H_n(C_\bullet) \\
\xrightarrow{\partial_{n}=0} H_{n-1}(A_\bullet) \xhookrightarrow{H_{n-1}(i)} H_{n-1}(B_\bullet) \xrightarrow{H_{n-1}(p)} H_{n-1}(C_\bullet) \\
\xrightarrow{\partial_{n-1}=0} \cdots
\end{array},
$$

i.e. it splits into several short exact sequences (which in fact are split short exact sequences) $H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet)$. This should be no surprise, since each functor $H_n(-)$ is additive.

8.5. **Suspension, desuspension, cone.** Given a chain complex $(C_\bullet, d)$ we can define new chain complexes by shifting indices.

Notation 8.18. Let $(C_\bullet, d^C)$ be a chain complex in ${}_R\mathrm{Mod}$. For all $k \in \mathbb{Z}$ we define $\Sigma^k(C_\bullet, d^C)$ as the chain complex $(D_\bullet, d^D)$ such that $D_i = C_{i-k}$, and $d_i^D = d_{i-k}^C$. This is often called the k-fold suspension of $C_\bullet$. Another common notation for $\Sigma^k C_\bullet$ (yes, one often leaves the differential understood...) is $C_\bullet[k]$.

We note that $H_n(C_\bullet[k]) \cong H_{n-k}(C_\bullet)$. A chain map $f\colon C_\bullet \to D_\bullet$ gives rise to a chain map $\Sigma^k C_\bullet \to \Sigma^k D_\bullet$, and we can in fact regard Σ^k- as an autofunctor of the category ${}_R\mathrm{Ch}$, with inverse functor $\Sigma^{-k}-$.

Definition 8.19. Let $f\colon C_\bullet \to D_\bullet$ be an R-linear chain map. We define a new chain complex $\mathrm{Cone}(f)_\bullet$ as follows:

- we set $\mathrm{Cone}(f)_j = D_j \oplus C_{j-1}$;
- we define the differential $d_j^{\mathrm{Cone}}\colon D_j \oplus C_{j-1} \to D_{j-1} \oplus C_{j-2}$ as follows:
  - on the summand D_j we take the map $y \mapsto ((y)d_j^D, 0)$;
  - on the summand C_{j-1} we take the map $x \mapsto ((x)f_{j-1}, -(x)d_{j-1}^C)$.

The previous definition might seem awkward, but it allows us to write a SES of chain complexes starting with $D_\bullet$ and ending with $\Sigma C_\bullet = \Sigma^1 C_\bullet$:

$$D_\bullet \xrightarrow{i} \mathrm{Cone}(f)_\bullet \xrightarrow{p} \Sigma C_\bullet$$

We have indeed obvious SES of left R-modules $D_j \to \mathrm{Cone}(f)_j = D_j \oplus C_{j-1} \to \Sigma C_j = C_{j-1}$; these SESs assemble into a SES of chain complexes (check this: some squares must commute!).

At first glance it could seem that $\mathrm{Cone}(f)$ is just the direct sum of the chain complexes $D_\bullet$ and $\Sigma C_\bullet$, but this is (in general) not true! Indeed the connecting homomorphism $\partial_n\colon H_n(\Sigma C_\bullet) \to H_{n-1}(D_\bullet)$ coincides, after identifying $H_n(\Sigma C_\bullet)$ with $H_{n-1}(C_\bullet)$, with the map $(-1)^n H_{n-1}(f)\colon H_{n-1}(C_\bullet) \to H_{n-1}(D_\bullet)$.
The previous construction should be seen in the following light:

- if $f\colon C_\bullet \to D_\bullet$ is injective, we can consider the chain complex $\mathrm{coker}(f)$ and complete $C_\bullet \to D_\bullet$ to a SES of chain complexes $C_\bullet \to D_\bullet \to \mathrm{coker}(f)$, yielding a homology long exact sequence containing the morphisms $H_n(f)$;
- if $f\colon C_\bullet \to D_\bullet$ is surjective, we can consider the chain complex $\ker(f)$ and complete $C_\bullet \to D_\bullet$ to a SES of chain complexes $\ker(f) \to C_\bullet \to D_\bullet$, yielding a homology long exact sequence containing the morphisms $H_n(f)$;
- but if $f\colon C_\bullet \to D_\bullet$ is neither injective nor surjective, what do we do? We complete $C_\bullet$ and $D_\bullet$ to a SES of chain complexes $D_\bullet \to \mathrm{Cone}(f)_\bullet \to \Sigma C_\bullet$, where the last chain complex is not quite $C_\bullet$, but also very close to it; up to a sign, the maps $H_n(f)$ occur as connecting homomorphisms in the associated homology long exact sequence.

8.6. **Cochain complexes.** There is a dual notion, often used in the literature, of *cochain complex*. A cochain complex is usually denoted $(C^\bullet, \delta)$, and consists of left R-modules C^i, for all $i \in \mathbb{Z}$, together with maps $\delta^i\colon C^i \to C^{i+1}$. So the degree increases by 1 instead of decreasing by 1. The composition of two differentials is still required to be the zero map.
Clearly, given a cochain complex $(C^\bullet, \delta)$, we can define a chain complex $(C_\bullet, d)$ by setting $C_i = C^{-i}$ and $(d_i\colon C_i \to C_{i-1}) = (\delta^{-i}\colon C^{-i} \to C^{-i+1})$. So up to mirroring indices, there is not a great difference between chain and cochain complexes.
Homology groups of a cochain complex are called *cohomology groups*, and usually one denotes $H^n(C^\bullet)$ the group that, in the mirroring above, would be denoted $H_{-n}(C_\bullet)$. Everything in this section has an analogue version for cochain complexes and cohomology.

Example 8.20. Let $(C_\bullet, d)$ be a chain complex in ${}_R\mathrm{Mod}$, and let $F\colon {}_R\mathrm{Mod}^{op} \to {}_S\mathrm{Mod}$ be a contravariant, additive functor[26].
Then we obtain a *cochain complex* $F(C)^\bullet = F(C_\bullet)$: we let $F(C)^i = F(C_i$, and we let $\delta^i = F(d_{i+1})\colon F(C_i) \to F(C_{i+1})$. The fact that the functor is contravariant has the effect of changing a chain complex into a cochain complex (and viceversa).
As a specific example, consider the chain complex of $\mathbb{Z}$-modules

$$\dots C_4 = 0 \longrightarrow C_3 = \mathbb{Z} \xrightarrow{-\cdot(2,0)} C_2 = \mathbb{Z}^2 \xrightarrow{(m,n)\mapsto n} C_1 = \mathbb{Z} \xrightarrow{0} C_0 = \mathbb{Z} \longrightarrow C_{-1} = 0 \dots$$

If we apply the additive functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ we obtain a cochain complex that, after simple manipulations, reads

$$\dots D^4 = 0 \longleftarrow D^3 = \mathbb{Z} \xleftarrow[(m,n)\mapsto 2m]{} D^2 = \mathbb{Z}^2 \xleftarrow[-\cdot(0,1)]{} D^1 = \mathbb{Z} \xleftarrow[0]{} D^0 = \mathbb{Z} \longleftarrow D^{-1} = 0 \dots$$

where we wrote $D^i = \mathrm{Hom}_{\mathbb{Z}}(C_i, \mathbb{Z})$. If you want to have fun, compute the homology and cohomology groups of the two complexes: they will be finitely generated abelian groups, and you will notice a very regular behaviour of the summands "$\mathbb{Z}$", but a strange behaviour of the summands "$\mathbb{Z}/k$" with $k \geq 2$.

[26] For simplicity I keep making examples involving only abelian categories of the form ${}_R\mathrm{Mod}$ or Mod_R...

9. Chain homotopies, Hom and tensor of chain complexes

9.1. **Chain homotopies.** In this subsection we work for simplicity in the category ${}_R\mathrm{Mod}$ of left R-modules over a ring R, but the discussion can be generalised to any abelian category.

Definition 9.1. Let $f\colon C_\bullet \to D_\bullet$ be a chain map between chain complexes in ${}_R\mathrm{Mod}$. We say that f is a *quasi-isomorphism* if for all $n \in \mathbb{Z}$ the induced map $H_n(f)\colon H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism of left R-modules.

For instance, the chain map from Example 8.10 is a quasi-isomorphism:

$$\begin{array}{ccccccc}
\dots C_3 = 0 & \longrightarrow & C_2 = \mathbb{Z} & \xrightarrow{\mathrm{Id}_{\mathbb{Z}}} & C_1 = \mathbb{Z} & \longrightarrow & C_0 = 0\dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots D_3 = 0 & \longrightarrow & D_2 = 0 & \longrightarrow & D_1 = 0 & \longrightarrow & D_0 = 0\dots
\end{array}$$

Example 9.2. Let $C_\bullet$ be any chain complex in ${}_R\mathrm{Ch}$ and let $0_\bullet$ be the trivial chain complex. Then the (unique) chain map $f\colon 0_\bullet \to C_\bullet$ is a quasi-isomorphism if and only if all homology groups of $C_\bullet$ are the zero R-module, i.e. if and only if $C_\bullet$ is an exact sequence.
Of course, $C_\bullet$ being an exact sequence is also equivalent to requiring that the (unique) chain map $g\colon C_\bullet \to 0_\bullet$ is a quasi-isomorphism.

Notation 9.3. A chain complex with vanishing homology groups is also called an *acyclic* chain complex; this is equivalent to requiring that the chain complex is in fact an exact sequence.

Example 9.4. Suppose now that $C_\bullet$ is an acyclic complex, and let f and g be the chain maps from Example 9.2. Then the composition $f \circ g\colon 0_\bullet \to 0_\bullet$ is the identity chain map of $0_\bullet$ (what else could it be?), so in particular it is an isomorphism. Instead, the composition $g \circ f\colon C_\bullet \to C_\bullet$ is the zero chain map, which is not the same as the identity of $C_\bullet$.
It would be nice, however, if it was possible to "perturb" the chain map $\mathrm{Id}_{C_\bullet}\colon C_\bullet \to C_\bullet$ to the map $0\colon C_\bullet \to C_\bullet$: we could then pretend that, up to "perturbation", also $g \circ f$ is the identity of $C_\bullet$, and thus f and g are inverse isomorphisms...

The following definition gives a useful interpretation of what a "perturbation" of a chain map could be.

Definition 9.5. Let $f, g\colon C_\bullet \to D_\bullet$ be chain maps. A *chain homotopy from f to g* is a sequence $\mathcal{H} = (\mathcal{H}_i)_{i\in\mathbb{Z}}$ of R-linear maps $\mathcal{H}_i\colon C_i \to D_{i+1}$ such that for all $i \in \mathbb{Z}$ we have an equality of R-linear maps

$$g_i - f_i = (\mathcal{H}_i \circ d^D_{i+1}) + (d^C_i \circ \mathcal{H}_{i-1})\colon C_i \to D_i.$$

We write $\mathcal{H}\colon C_\bullet \to D_{\bullet+1}$ for a chain homotopy.
We say that f and g are *chain homotopic chain maps* if there is a chain homotopy between f and g. We write $f \simeq g$.
We say that f is *chain null-homotopic* if there is a chain homotopy between f and the zero map $0\colon C_\bullet \to D_\bullet$. We write $f \simeq 0$.

If f is chain homotopic to g, then difference $g - f$ is a chain map that can be written as a sum of two contributions $(\mathcal{H} \circ d^D)$ and $(d^C \circ \mathcal{H})$; in general neither of the two summands represents a chain map on its own. The fact that both

summands contain a differential "d", together with the intuition that differentials should attain "small" values, suggests that $g - f$ should attain "small" values and thus f is a "perturbation" of g.
I also hope that the notation $\mathcal{H}\colon C_\bullet \to D_{\bullet+1}$ does not lead you to the (wrong) idea that a chain homotopy $\mathcal{H}$ is a chain map from $C_\bullet$ to $\Sigma^{-1}D_\bullet$.
Since a differential sends a chain in degree i to a chain in degree $i-1$, a chain homotopy should do the opposite (increase degrees by 1), so that the composition of the two is degree-preserving.

Example 9.6. Let $C_\bullet$ be the chain complex from Example 9.2. Then we can define a chain homotopy $\mathcal{H}\colon C_\bullet \to C_{\bullet+1}$ from $\mathrm{Id}_{C_\bullet}$ to 0 by declaring $\mathcal{H}_1\colon C_1 \to C_2$ to be $\mathrm{Id}_{\mathbb{Z}}$, and all other maps $\mathcal{H}_i$ to be zero. We have indeed $\mathrm{Id}_{C_\bullet} = \mathcal{H} \circ d^C + d^C \circ \mathcal{H}$, as can be checked degreewise (only degrees 1 and 2 are interesting).

Example 9.7. Let $C_\bullet$ and $D_\bullet$ be two chain complexes, and let $\mathcal{H} = (\mathcal{H}_i)$ be *any* collection of R-linear maps $\mathcal{H}_i\colon C_i \to D_{i+1}$. For all $i \in \mathbb{Z}$ define an R-linear map $f_i\colon C_i \to D_i$ by $f_i = (\mathcal{H}_i \circ d^D_{i+1}) + (d^C_i \circ \mathcal{H}_{i-1})$. Then $f = (f_i)_{i\in\mathbb{Z}}$ is automatically a chain map $C_\bullet \to D_\bullet$. Indeed we have

$$\begin{aligned}
d^C_{i+1} \circ f_i &= d^C_{i+1} \circ \big((\mathcal{H}_i \circ d^D_{i+1}) + (d^C_i \circ \mathcal{H}_{i-1})\big)\\
&= (d^C_{i+1} \circ \mathcal{H}_i \circ d^D_{i+1}) + (d^C_{i+1} \circ d^C_i \circ \mathcal{H}_{i-1})\\
&= d^C_{i+1} \circ \mathcal{H}_i \circ d^D_{i+1}\\
&= (d^C_{i+1} \circ \mathcal{H}_i \circ d^D_{i+1}) + (\mathcal{H}_{i+1} \circ d^D_{i+2} \circ d^D_{i+1})\\
&= \big((d^C_{i+1} \circ \mathcal{H}_i) + (\mathcal{H}_{i+1} \circ d^D_{i+2}\big) \circ d^D_{i+1}\\
&= f_{i+1} \circ d^D_{i+1}.
\end{aligned}$$

Moreover $\mathcal{H}$ is, by the very definition of f, a chain homotopy from f to 0.

The previous example shows that any collection $(\mathcal{H}_i)$ of R-linear maps $\mathcal{H}_i\colon C_i \to D_{i+1}$ can arise as a chain homotopy between two chain maps. In the following we will often define chain homotopies on their own, just as sequences of R-linear maps, without the need of two chain maps to compare.

Lemma 9.8. *Let $C_\bullet$ adn $D_\bullet$ be chain complexes, and consider the abelian group $\mathrm{Hom}_{_R\mathrm{Ch}}(C_\bullet, D_\bullet)$ of all (R-linear) chain maps $C_\bullet \to D_\bullet$. Then the subset*

$$\mathrm{Hom}^{\simeq 0}_{_R\mathrm{Ch}}(C_\bullet, D_\bullet) \subseteq \mathrm{Hom}_{_R\mathrm{Ch}}(C_\bullet, D_\bullet)$$

of chain null-homotopic chain maps is in fact an abelian subgroup. In particular "being chain homotopic chain maps" is an equivalence relation on $\mathrm{Hom}_{_R\mathrm{Ch}}(C_\bullet, D_\bullet)$, coinciding with the equivalence relation "differing by a chain null-homotopic chain map".

Proof. Every chain map $f\colon C_\bullet \to D_\bullet$ is chain homotopic to itself, by using $\mathcal{H} \equiv 0$: indeed for all $i \in \mathbb{Z}$ we have $f_i - f_i = 0 = (0 \circ d^D_{i+1}) + (d^C_i \circ 0)$.
If there is a chain homotopy $\mathcal{H}$ from f to g, then $-\mathcal{H}$ is a chain homotopy from g to f: indeed for all $i \in \mathbb{Z}$ we have

$$\begin{aligned}
g_i - f_i = -(f_i - g_i) &= -((\mathcal{H}_i \circ d^D_{i+1}) + (d^C_i \circ \mathcal{H}_{i-1}))\\
&= (-\mathcal{H}_i \circ d^D_{i+1}) + (d^C_i \circ -\mathcal{H}_{i-1}),
\end{aligned}$$

using that composition of R-linear maps is $\mathbb{Z}$-bilinear.

Finally, if there are chain homotopies $\mathcal{H}$ from f to g and $\mathcal{H}'$ from g to l, then $\mathcal{H}+\mathcal{H}'$ is a chain homotopy from f to l: indeed for all $i \in \mathbb{Z}$ we have

$$\begin{aligned} l_i - f_i &= (l_i - g_i) + (g_i - f_i) \\ &= (\mathcal{H}_i \circ d^D_{i+1}) + (d^C_i \circ \mathcal{H}_{i-1}) + (\mathcal{H}'_i \circ d^D_{i+1}) + (d^C_i \circ \mathcal{H}'_{i-1}) \\ &= ((\mathcal{H}_i + \mathcal{H}'_i) \circ d^D_{i+1}) + (d^C_i \circ (\mathcal{H}_{i-1} + \mathcal{H}'_{i-1})), \end{aligned}$$

using again that composition of chain maps is $\mathbb{Z}$-bilinear. □

How do chain homotopies and composition of chain maps interrelate?

Example 9.9. Let $f, g \colon C_\bullet \to D_\bullet$ be chain maps, and let $\mathcal{H} \colon C_\bullet \to D_{\bullet+1}$ be a chain homotopy from f to g. Let moreover $l \colon D_\bullet \to E_\bullet$ be another chain map. We can then define a chain homotopy $\mathcal{H}' \colon C_\bullet \to E_{\bullet+1}$ by setting $\mathcal{H}'_i = \mathcal{H}_i \circ l_{i+1} \colon C_i \to E_{i+1}$. We then have, for all $i \in \mathbb{Z}$,

$$\begin{aligned} (g_i \circ l_i) - (f_i \circ l_i) &= (g_i - f_i) \circ l_i \\ &= \big(\mathcal{H}_i \circ d^D_{i+1} + d^C_i \circ \mathcal{H}_{i-1}\big) \circ l_i \\ &= \mathcal{H}_i \circ d^D_{i+1} \circ l_i + d^C_i \circ \mathcal{H}_{i-1} \circ l_i \\ &= \mathcal{H}_i \circ l_{i+1} \circ d^E_{i+1} + d^C_i \circ \mathcal{H}_{i-1} \circ l_i \\ &= \mathcal{H}'_i \circ d^E_{i+1} + d^C_i \circ \mathcal{H}'_{i-1}, \end{aligned}$$

where we have used, among other things, that l is a chain map. Hence $\mathcal{H}'$ is a chain homotopy from $f \circ l$ to $g \circ l$.

The previous example shows that if we perturb f up to homotopy and then compose it with l, the result is a perturbation of $f \circ l$. An analogue chack can be made for precomposition of f with a chain map. In other words, composition of chain maps induces a well-defined composition of chain homotopy classes of chain maps.

Definition 9.10. We define a $\mathbb{Z}$-linear category $K({}_R\mathrm{Mod})$ (called the *homotopy category* of $\mathrm{Ch}({}_R\mathrm{Mod})$) as follows:

- the objects of $K({}_R\mathrm{Mod})$ are chain complexes of left R-modules;
- the abelian group of morphisms $\mathrm{Hom}_{K({}_R\mathrm{Mod})}(C_\bullet, D_\bullet)$ is the quotient group $\mathrm{Hom}_{{}_R\mathrm{Ch}}(C_\bullet, D_\bullet)/\mathrm{Hom}^{\simeq 0}_{{}_R\mathrm{Ch}}(C_\bullet, D_\bullet)$.

Composition of morphisms is induced from the composition in ${}_R\mathrm{Ch}$. For a chain map $f \colon C_\bullet \to D_\bullet$ we denote by $[f]$ the corresponding morphism in $K({}_R\mathrm{Mod})$.
We define a functor $\mathbb{K} \colon {}_R\mathrm{Ch} \to K({}_R\mathrm{Mod})$, sending an object (a chain complex) to itself, and chain map to its chain homotopy class.

Example 9.11. For all $i \in \mathbb{Z}$ there is a functor $\theta_i \colon {}_R\mathrm{Mod} \to {}_R\mathrm{Ch}$ sending M to the chain complex $\theta_i(M)_\bullet$ with $\theta_i(M)_i = M$, and all other $\theta_i(M)_j = 0$. An R-linear map $f \colon M \to N$ is sent to the chain map $\theta_i(f)$ with $\theta_i(f)_i = f$, and all other maps $\theta_i(f)_j$ are clearly the zero map.
Now, suppose that $C_\bullet = \theta_i(M)$ and $D_\bullet = \theta_i(N)$ are in the image of the functor θ_i. Then every chain homotopy $\mathcal{H} \colon C_\bullet \to D_{\bullet+1}$ vanishes, whereas a chain map $C_\bullet \to D_\bullet$ is equivalent to what it does in degree i, i.e. to an R-linear map $M \to N$. It follows that we have bijections of abelian groups (induced by the various functors considered)

$$\mathrm{Hom}_R(M, N) \overset{\theta_i}{\cong} \mathrm{Hom}_{{}_R\mathrm{Ch}}(C_\bullet, D_\bullet) \overset{\mathbb{K}}{\cong} \mathrm{Hom}_{K({}_R\mathrm{Mod})}(C_\bullet, D_\bullet).$$

In other words, both functors θ_i and $\mathbb{K} \circ \theta_i$ are fully faithful.

The category $K({}_R\mathrm{Mod})$ is an additive category, and even more holds: the zero object and finite products/coproducts are *created* by the functor $\mathbb{K}$. By the word "created" we mean that $0_\bullet$ is a zero object also in $K({}_R\mathrm{Mod})$, and, for instance, $(C_\bullet \oplus D_\bullet, [\iota_{C_\bullet}], [\iota_{D_\bullet}])$ is a categorical coproduct of $C_\bullet$ and $D_\bullet$ in $K({}_R\mathrm{Mod})$.

Example 9.12. Suppose that $C_\bullet$ is a chain complex as the one of Example 9.2, i.e. such that $\mathrm{Id}_{C_\bullet}$ is chain homotopic to the zero self-map of $C_\bullet$ (it is chain null-homotopic). Then each chain map out of $C_\bullet$ or into $C_\bullet$ is chain homotopic to the zero map. As a result, $C_\bullet$ is a zero object in $K({}_R\mathrm{Mod})$.

Unfortunately, $K({}_R\mathrm{Mod})$ is in general not an abelian category: hence, although considering chain maps up to chain homotopy can have its advantages, it also comes with a price.

9.2. **Chain homotopies and induced maps in homology.** Recall that for all $n \in \mathbb{Z}$ we have a functor $H_n(-)\colon {}_R\mathrm{Ch} \to {}_R\mathrm{Mod}$. In particular, given a chain map $f\colon C_\bullet \to D_\bullet$, we obtain an R-linear map $H_n(f)\colon H_n(C_\bullet) \to H_n(D_\bullet)$. How does $H_n(f)$ change if we replace f by a chain homotopic map g? The following lemma tells us that nothing changes, and in its proof we will understand how well-designed the notion of chain homotopy of maps is.

Lemma 9.13. *Let $f, g\colon C_\bullet \to D_\bullet$ be chain homotopic maps. Then $H_n(f) = H_n(g)\colon H_n(C_\bullet) \to H_n(D_\bullet)$.*

Proof. Let $H\colon C_\bullet \to D_{\bullet+1}$ be a chain homotopy from f to g. Let $[x] \in H_n(C_\bullet)$ be a homology class, represented by a cycle $x \in \mathfrak{Z}_i(C_\bullet)$. Then $H_n(f)\colon [x] \mapsto [(x)f_i]$, whereas $H_n(g)\colon [x] \mapsto [(x)g_i]$; on the other hand we have the following chain of equalities in $H_n(D_\bullet)$:

$$\begin{aligned}[(x)g_i] - [(x)f_i] = [(x)g_i - (x)f_i] &= [((x)\mathcal{H}_i)d^D_{i+1}) + ((x)d^C_i)\mathcal{H}_{i-1}] \\ &= [((x)\mathcal{H}_i)d^D_{i+1} + 0] = 0,\end{aligned}$$

where the last equality follows from $((x)\mathcal{H}_i)d^D_{i+1} \in \mathfrak{B}_i(D_\bullet)$. It follows that $H_n(f)$ and $H_n(g)$ coincide. □

Perhaps Lemma 9.13 is the main motivation to consider at all the notion of chain homotopy equivalence: if we are only interested in the behaviour of a chain map f in homology, we can perturb f by a chain homotopy without affecting $H_n(f)$.

Notation 9.14. We say that $C_\bullet$ is a *chain null-homotopic* chain complex if $\mathrm{Id}_{C_\bullet}$ is chain homotopic to the zero self-chain map of $C_\bullet$.

If $C_\bullet$ is chain null-homotopic, then we also have that $C_\bullet$ is acyclic. To see this, let $n \in \mathbb{Z}$, and consider the homology group $H_n(C_\bullet)$. We have a sequence of equalities of R-linear self maps of $H_n(C_\bullet)$

$$\mathrm{Id}_{H_n(C_\bullet)} = H_n(\mathrm{Id}_{C_\bullet}) = H_n(0) = 0$$

where the first equality follows from $H_n(-)$ being a functor, the second follows from Lemma 9.13, and the third follows from $H_n(-)$ being an additive functor. Hence, the left R-module $H_n(C_\bullet)$ has the remarkable property that its identity coincides with the zero map: this means that $H_n(C_\bullet) = 0$.

Example 9.15. Let $R = \mathbb{Q}[x]$ and consider the chain complex $C_\bullet \in {}_{\mathbb{Q}[x]}\mathrm{Ch}$

$$\dots C_3 = 0 \longrightarrow C_2 = \mathbb{Q}[x] \xrightarrow{\cdot x} C_1 = \mathbb{Q}[x] \xrightarrow{[-]_x} C_0 = \mathbb{Q}[x]/(x) \longrightarrow C_{-1} = 0 \dots$$

The complex is acyclic (it is after all the old good SES $\mathbb{Q}[x] \to \mathbb{Q}[x] \to \mathbb{Q}[x]/x$). Yet we can prove that $\mathrm{Id}_{C_\bullet}$ is not chain null-homotopic. For this, let $\mathcal{H} = (\mathcal{H}_i)_{i\in\mathbb{Z}}$ be any chain homotopy $C_\bullet \to C_{\bullet+1}$. Then $\mathcal{H}_0\colon \mathbb{Q}[x]/(x) \to \mathbb{Q}[x]$, which is supposed to be a $\mathbb{Q}[x]$-linear map, must be the zero map. Similarly, $\mathcal{H}_{-1}\colon 0 \to \mathbb{Q}[x]/(x)$ is the zero map. It follows that $\mathcal{H}_0 \circ d_1^C + d_0^C \circ \mathcal{H}_{-1} = 0 + 0$ is not equal to the identity of $C_0 = \mathbb{Q}[x]/(x)$. In a similar way, $\mathcal{H}_1 \circ d_2^C + d_1^C \circ \mathcal{H}_0 = \mathcal{H}_1 \circ d_2^C$ cannot the identity of $C_1 = \mathbb{Q}[x]$ (as d_2^C is not surjective).

The previous example shows that a chain complex can be acyclic and yet not be chain null-homotopic. As a spoiler for the future, this essentially happens because one of the $\mathbb{Q}[x]$-modules occurring, namely $\mathbb{Q}[x]/(x)$, is not projective.
A reformulation of Lemma 9.13 is that for each $n \in \mathbb{Z}$ the functor $H_n(-)\colon {}_R\mathrm{Ch} \to {}_R\mathrm{Mod}$ can be written as a composition of two functors: the functor $\mathbb{K}\colon {}_R\mathrm{Ch} \to K({}_R\mathrm{Mod})$, followed by a functor $H_n^K(-)\colon K({}_R\mathrm{Mod}) \to {}_R\mathrm{Mod}$.

9.3. **Chain homotopy equivalences.** We can think of a chain null-homotopic chain complex $C_\bullet$ as a chain complex that, even if not isomorphic to $0_\bullet$, is for many purposes (including the homology computation) *equivalent* to $0_\bullet$. We make this idea of *equivalence* precise in the following definition.

Definition 9.16. Let $C_\bullet$ and $D_\bullet$ be chain complexes in ${}_R\mathrm{Mod}$. A chain map $f\colon C_\bullet \to D_\bullet$ is a *chain homotopy equivalence* if the morphism

$$[f] \in \mathrm{Hom}_{K({}_R\mathrm{Mod})}(C_\bullet, D_\bullet),$$

obtained by applying $\mathbb{K}$ to f, is an isomorphism in the category $K({}_R\mathrm{Mod})$.

Concretely, this means that there exists a chain map $g\colon D_\bullet \to C_\bullet$ (representing a morphism $[g]\colon C_\bullet \to D_\bullet$) such that the composite $f \circ g$ is chain homotopic to the identity of $C_\bullet$ (so that $[f] \circ [g] = [f \circ g] = [\mathrm{Id}_{C_\bullet}]$), and the composite $g \circ f$ is chain homotopic to the identity of $D_\bullet$ (so that, similarly, $[g] \circ [f] = [\mathrm{Id}_{D_\bullet}]$, and thus $[g]$ is an inverse of $[f]$).
Even more concretely, $f\colon C_\bullet \to D_\bullet$ is a chain homotopy equivalence if there exist a chain map $g\colon D_\bullet \to C_\bullet$, a chain homotopy $\mathcal{H}^C\colon C_\bullet \to C_{\bullet+1}$ and a chain homotopy $\mathcal{H}^D\colon D_\bullet \to D_{\bullet+1}$ such that $f \circ g - \mathrm{Id}_{C_\bullet} = \mathcal{H}^C \circ d^C + d^C \circ \mathcal{H}^C$ and such that $g \circ f - \mathrm{Id}_{D_\bullet} = \mathcal{H}^D \circ d^D + d^D \circ \mathcal{H}^D$.
For instance, a chain complex $C_\bullet$ is null-homotopic if and only if the inital map $0_\bullet \to C_\bullet$ and the terminal map $C_\bullet \to 0_\bullet$ are chain homotopy equivalences.
Note that if $f\colon C_\bullet \to D_\bullet$ is a chain homotopy equivalence, then $[f]\colon C_\bullet \to D_\bullet$ is an isomorphism in $K({}_R\mathrm{Mod})$, and for all $n \in \mathbb{Z}$, applying the functor $H_n^K(-)$, we obtain that also the map $H_n(f) = H_n^K([f])\colon H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism of R-modules.

Example 9.17. Consider the following chain map $f\colon C_\bullet \to D_\bullet$

$$\begin{array}{ccccccc}
\dots C_2 = 0 & \longrightarrow & C_1 = \mathbb{Z} & \xrightarrow{\cdot 7} & C_0 = \mathbb{Z} & \longrightarrow & C_{-1} = 0 \dots \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0 = [-]_7} & & \downarrow{\scriptstyle f_{-1}} \\
\dots D_2 = 0 & \longrightarrow & D_1 = 0 & \longrightarrow & D_0 = \mathbb{Z}/7 & \longrightarrow & D_{-1} = 0 \dots
\end{array}$$

We can compute the homology groups of $C_\bullet$ and $D_\bullet$, and find out that the only non-vanishing homology groups are $H_0(C_\bullet) \cong H_0(D_\bullet) \cong \mathbb{Z}/7$; moreover the map $H_0(f)$ is an isomorphism, and it follows that f is a quasi-isomorphism, i.e. it induces isomorphism between all pairs of homology groups.
However f is not a chain homotopy equivalence. To see this, note that every chain map $g\colon D_\bullet \to C_\bullet$ vanishes, as in particular every $\mathbb{Z}$-linear map $g_0\colon D_0 \to C_0$ must vanish. But the zero map $0\colon D_\bullet \to C_\bullet$ does not induce an isomorphism on H_0, and hence it is not a quasi-isomorphism and even less a chain homotopy equivalence. It follows that, in $K({}_R\mathrm{Mod})$, there is no isomorphism $D_\bullet \to C_\bullet$; but then there cannot be an isomorphism $C_\bullet \to D_\bullet$ either, so $[f]$ is not an isomorphism in $K({}_R\mathrm{Mod})$, so f is not a chain homotopy equivalence.

9.4. **Mapping cylinder.** Let $f\colon C_\bullet \to D_\bullet$ be an R-linear chain map of left R-chain complexes. We can define a new left R-chain complex $\mathrm{Cyl}(f)_\bullet$ as follows:

- we set $\mathrm{Cyl}(f)_i = C_i \oplus D_i \oplus \bar{C}_{i-1}$, where $\bar{C}_{i-1}$ is isomorphic to C_{i-1} as a left R-module, and we use the notation "$\bar{C}$" only to distinguish this direct summand of $\mathrm{Cyl}(f)_i$ from the equal summand occurring in $\mathrm{Cyl}(f)_{i-1}$;
- we set $d_i^{\mathrm{Cyl}}\colon \mathrm{Cyl}(f)_i \to \mathrm{Cyl}(f)_{i-1}$ to be the map with the following restrictions:
  - on the summand C_i we take the map $d_i^C\colon C_i \to C_{i-1}$;
  - on the summand D_i we take the map $d_i^D\colon D_i \to D_{i-1}$;
  - on the summand $\bar{C}_{i-1}$ we take the map sending

$$x \in \bar{C}_{i-1} \mapsto \Big(-x\,,\, (x)f_{i-1}\,,\, -(x)d_{i-1}^C \Big) \in C_{i-1} \oplus D_{i-1} \oplus \bar{C}_{i-2}.$$

The definition of $\mathrm{Cyl}(f)_\bullet$ may seem awkward, but this chain complex is designed to have the following properties.

(1) We an inclusion of chain complexes $i_C\colon C_\bullet \hookrightarrow \mathrm{Cyl}(f)_\bullet$, including C_i as a summand into $\mathrm{Cyl}(f)_i$.
(2) We an inclusion of chain complexes $i_D\colon D_\bullet \hookrightarrow \mathrm{Cyl}(f)_\bullet$, including D_i as a summand into $\mathrm{Cyl}(f)_i$. Composing with f, we obtain a map of chain complexes $f \circ i_D\colon C_\bullet \to \mathrm{Cyl}(f)_\bullet$.
(3) We have a chain map $\rho\colon \mathrm{Cyl}(f)_\bullet \to D_\bullet$ defined as follows: $\rho_i\colon \mathrm{Cyl}(f)_i \to D_i$ restricts to f_i on C_i, restricts to Id_{D_i} on D_i, and is the zero map on $\bar{C}_{i-1}$.
(4) The composition $i_D \circ \rho\colon D_\bullet \to D_\bullet$ is the identity of $D_\bullet$; as a consequence, the composition $f \circ i_D \circ \rho$ is the chain map f.
(5) We have a chain homotopy $\mathcal{H}\colon \mathrm{Cyl}(f)_\bullet \to \mathrm{Cyl}(f)_{\bullet+1}$ defined as follows: $\mathcal{H}_i\colon \mathrm{Cyl}(f)_i \to \mathrm{Cyl}(f)_{i+1}$ sends $C_i \to \bar{C}_i$ by the map Id_{C_i}, and is the zero map on the summands D_i and on $\bar{C}_{i-1}$.
(6) $\mathcal{H}$ is a chain homotopy between $\mathrm{Id}_{\mathrm{Cyl}(f)_\bullet}$ and $\rho \circ i_D$. In particular ρ is a chain homotopy equivalence. In the homotopy category $K({}_R\mathrm{Mod}$ we have $[\rho] = [i_D]^{-1}$, i.e. $[\rho]$ is an isomorphism. Moreover $[i_C] \circ [\rho] = [f]$.

From the point of view of the homotopy category $K({}_R\mathrm{Mod})$, $D_\bullet$ and $\mathrm{Cyl}(f)_\bullet$ are isomorphic objects; from the point of view of ${}_R\mathrm{Ch}$, however, there is a big difference: i_C is certainly an injective map $C_\bullet \to \mathrm{Cyl}(f)_\bullet$, whereas f need not be injective. We can now redefine the mapping cone $\mathrm{Cone}(f)_\bullet$ as $\mathrm{coker}(C_\bullet \overset{i_C}{\hookrightarrow} \mathrm{Cyl}(f)_\bullet)$. We obtain a LES of homology groups

$$\dots H_n(C_\bullet) \xrightarrow{H_n(i_C)} H_n(\mathrm{Cyl}(f)_\bullet) \longrightarrow H_n(\mathrm{Cone}(f)_\bullet) \xrightarrow{\partial_n} H_{n-1}(C_\bullet) \dots$$

We can now identify each homology group $H_n(\mathrm{Cyl}(f)_\bullet)$ with $H_n(D_\bullet)$ using the isomorphism $H_n(\rho)$; we thus get a LES in which the morphisms $H_n(f)$ show up

$$\dots H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \longrightarrow H_n(\mathrm{Cone}(f)_\bullet) \xrightarrow{\partial_n} H_{n-1}(C_\bullet) \dots$$

Exercise 9.18. For a chain map $f\colon C_\bullet \to D_\bullet$, show the following:

- f is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is acyclic (use the homology LES);
- f is a chain homotopy equivalence if and only if $\mathrm{Cone}(f)$ is null-homotopic.

9.5. **Tensor product of chain complexes.** Let $C_\bullet$ be a chain complex in Mod_R, and let $D_\bullet$ be a chain complex in ${}_R\mathrm{Mod}$. We would like to define a new chain complex $(C \otimes_R D)_\bullet$ in ${}_{\mathbb{Z}}\mathrm{Mod}$ by combining the tensor products $C_i \otimes_R D_j$. This is done in two steps.
In the first step we define a double complex $E_{\bullet,\bullet}$. We set $E_{i,j} = C_i \otimes_R D_j$. The differential $d'_{i,j}\colon E_{i,j} \to E_{i-1,j}$ is given by the map of abelian groups $d_i^C \otimes_R \mathrm{Id}_{D_j}$, whereas the differential $d''_{i,j}\colon E_{i,j} \to E_{i,j-1}$ is given by the map of abelian groups $\mathrm{Id}_{C_i} \otimes_R d_j^D$. You can check that the composition $d' \circ d'$ vanishes, as well as $d'' \circ d''$; moreover d' and d'' commute. Hence $(E_{\bullet,\bullet}, d', d'')$ is a double complex (a chain complex in chain complexes in ${}_R\mathrm{Mod}$).
Now we would like to convert a double complex of abelian groups into a single chain complex of abelian groups.

Definition 9.19. Let $(E_{\bullet,\bullet}, d', d'')$ be a double complex in ${}_{\mathbb{Z}}\mathrm{Mod}$[27]. We define the associated *total chain complex*, denoted $(\mathrm{Tot}(E)_\bullet, d)$, as follows:

- $\mathrm{Tot}(E)_n := \bigoplus_{i+j=n} E_{i,j}$; denote by $\iota_{i,j}\colon E_{i,j} \to \mathrm{Tot}(E)_n$ the canonical inclusion of the summand $E_{i,j}$, for all $i+j=n$;
- the differential $d_n\colon \mathrm{Tot}(E)_n \to \mathrm{Tot}(E)_{n-1}$ is defined on the summand $E_{i,n-i}$ of $\mathrm{Tot}(E)_n$ as $\iota_{i-1,n-i} \circ d'_{i,n-i} + (-1)^i \iota_{i,n-i-1} \circ d''_{i,n-i}$.

Informally, the differential of $\mathrm{Tot}(E)_\bullet$, restricted to a summand $E_{i,j}$ in degree $i+j$, is written $d'_{i,j} + (-1)^i d''_{i,j}$. The sign $(-1)^i$ is there to ensure that $d \circ d = 0$, i.e. we have indeed defined a chain complex (and not just a sequence of interrelated modules).
We can apply the previous construction to the double complex constructed above. The result is a chain complex in abelian groups $(C \otimes_R D)_\bullet$; for $x \otimes y \in C_i \otimes_R D_i$ we can write the differential $d(x \otimes y)$ as

$$d(x \otimes y) = d'(x \otimes y) + (-1)^i d''(x \otimes y) = (d_i^C(x)) \otimes y + (-1)^i x \otimes ((y) d_j^D);$$

a formula like the previous is called a “Leibniz rule”, as it is similar to the formula, apprently due to Leibniz, expressing the derivative of a product of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ as $\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx} f(x)\right) g(x) + f(x) \left(\frac{d}{dx} g(x)\right)$. In our case the product is replaced by the tensor product, and the derivative by the chain differential. Moreover there is an additional sign $(-1)^i$.

9.6. **Hom of chain complexes.** There is a similar construction taking as input two chain complexes $C_\bullet$ and $D_\bullet$ in ${}_R\mathrm{Ch}$ and giving as output a chain complex in abelian groups $\mathrm{Hom}_R(C, D)_\bullet$.

[27]Or in another abelian category admitting infinite direct sums...

The first step creates a double complex $E_{\bullet,\bullet}$ of abelian groups. We set $E_{i,j} = \mathrm{Hom}_R(C_{-i}, D_j)$. The differential $d'_{i,j}\colon E_{i,j} \to E_{i-1,j}$ is given by the map of abelian groups $\mathrm{Hom}_R(d^C_{-i+1}), D_j)\colon \mathrm{Hom}_R(C_{-i}, D_j) \to \mathrm{Hom}_R(C_{-i+1}, D_j)$ obtained applying the contravariant functor $\mathrm{Hom}_R(-, D_j)$ to the differential d^C_{-i+1}. Similarly, the differential $d''_{i,j}\colon E_{i,j} \to E_{i,j-1}$ is given by the map of abelian groups $\mathrm{Hom}_R(C_i, d^D_j)$. The situation is a bit more complicated than with the tensor product, as the functor $\mathrm{Hom}_R(-, D_j)$ is contravariant, so it transforms chain complexes into cochain complexes; in order to get chain complexes again, we have to mirror the indices.
Now we use the following alternative construction to convert a double complex into a single chain complex.

Definition 9.20. Let $(E_{\bullet,\bullet}, d', d'')$ be a double complex in ${}_{\mathbb{Z}}\mathrm{Mod}$[28]. We define the associated *alternative total chain complex*, denoted $(\widehat{\mathrm{Tot}}(E)_\bullet, d)$, as follows:

- $\widehat{\mathrm{Tot}}(E)_n := \prod_{i+j=n} E_{i,j}$; denote by $\pi_{i,j}\colon \widehat{\mathrm{Tot}}(E)_n \to E_{i,j}$ the canonical projection on the factor $E_{i,j}$, for all $i + j = n$;
- the differential $d_n\colon \widehat{\mathrm{Tot}}(E)_n \to \widehat{\mathrm{Tot}}(E)_{n-1}$ is defined by declaring its (i, j)-coordinates, for all $i + j = n - 1$: i.e., we determine a map d_n with target $\prod_{i+j=n-1} E_{i,j}$ by declaring how its postcompositions $\pi_{i,j} \circ d_n$, with targets $E_{i,j}$, behave, for each $i + j = n - 1$. We set[29]
$$\pi_{i,j} \circ d_n = d'_{i+1,j} \circ \pi_{i+1,j} + (-1)^i d''_{i,j+1} \circ \pi_{i,j+1}.$$

As you see, Definition 9.20 is analogue to Definition 9.19, but it uses products instead of coproducts; therefore we have to use the universal property of product also to define the new differential, instead of the universal property of coproduct.
If we apply the construction to the double complex constructed above, we obtain a chain complex in abelian groups $\mathrm{Hom}_R(C, D)_\bullet$. It is instructive to do the following exercise.

Exercise 9.21. Prove the following.

- A chain in $\mathrm{Hom}_R(C, D)_0$ is the datum of a family $f = (f_i)_{i\in\mathbb{Z}}$ of R-linear maps $f_i\colon C_i \to D_i$.
- A family $f = (f_i)_{i\in\mathbb{Z}}$ of R-linear maps $f_i\colon C_i \to D_i$ is a cycle (it lies in $\mathfrak{Z}_0(\mathrm{Hom}_R(C, D)_\bullet)$) if and only if it is a chain map $f\colon C_\bullet \to D_\bullet$.
- A chain in $\mathrm{Hom}_R(C, D)_0$ is a chain homotopy $\mathcal{H}\colon C_\bullet \to D_{\bullet+1}$, i.e. the datum of a family $\mathcal{H} = (\mathcal{H}_i)_{i\in\mathbb{Z}}$ of R-linear maps $\mathcal{H}_i\colon C_i \to D_{i+1}$.
- Saying that two chain maps $f, g\colon C_\bullet \to D_\bullet$ are connected by a chain homotopy $\mathcal{H}\colon C_\bullet \to D_{\bullet+1}$ is equivalent to saying that $d(\mathcal{H}) = g - f$, i.e. the two cycles f and g differ by a boundary in $\mathfrak{B}(\mathrm{Hom}_R(C, D)_\bullet)_0$.
- Conclude: $H_0(\mathrm{Hom}_R(C, D)_\bullet)$ is isomorphic to $\mathrm{Hom}_{K({}_R\mathrm{Mod})}(C_\bullet, D_\bullet)$.

Example 9.22. Suppose that $D_\bullet = \theta_0(R)$, i.e. $D_0 = R$, all other $D_i = 0$, and all differentials of $D_\bullet$ are zero. Then $\mathrm{Hom}(C, D)_\bullet$ is isomorphic to the chain complex

$$\ldots \mathrm{Hom}_R(C_{-(i+1)}, R) \to \mathrm{Hom}_R(C_{-i}, R) \to \mathrm{Hom}_R(C_{-(i-1)}, R) \to \mathrm{Hom}_R(C_{-(i-2)}, R) \ldots,$$

[28]Or in another abelian category admitting infinite products...

[29]Some of you pointed out that the coefficient "$(-1)^i$" should be replaced by "$(-1)^n$" in the following formula. I think that in both cases one obtains a chain complex (i.e. the double iteration of a differential is the zero map); moreover I think that the two chain complexes obtained (by putting "$(-1)^i$" or "$(-1)^n$" in the formula) should be isomorphic chain complexes.

which is the mirror of the cochain complex obtained from $C_\bullet$ by applying the contravariant, additive functor $\mathrm{Hom}_R(-,R)$. Compare with the second part of Example 8.20.

10. Projective resolutions

We fix a ring R and focus on the category ${}_R\mathrm{Mod}$ of left R-modules. We will focus in the entire section for simplicity on projective resolutions; there is a dual notion of injective resolution, and we will discuss it quickly at the end.

10.1. **Finitely presented modules.**

We start this section with a discussion about finitely presented R-modules.

Definition 10.1. Recall Subsection 2.3. A left R-module M is *finitely generated* if there is a finite set I_0 and a surjective R-linear map $g_0 \bigoplus_{i\in I} R \to M$, i.e. there is an exact sequence of left R-modules

$$\bigoplus_{i\in I_0} R \xrightarrow{g_0} M \longrightarrow 0.$$

Equivalently, M is finitely generated if there is a finite subset $\mathcal{S} \subset M$ such that $\mathrm{Span}_R(\mathcal{S}) = M$.

We say that M is *finitely presented* if there are finite sets I_0 and I_1 and an exact sequence of left R-modules

$$\bigoplus_{i\in I_1} R \xrightarrow{g_1} \bigoplus_{i\in I_0} R \xrightarrow{g_0} M \longrightarrow 0.$$

Equivalently, M is finitely presented if there exists a finite set I_0 and a surjective R-linear map $g_0\colon \bigoplus_{i\in I_0} R \to M$ such that $\ker(g_0)$ is a finitely generated R-module.

Clearly, if M is finitely presented, then it is also finitely generated[30]. We would like now to show that if M is finitely generated, we can use *any* finite generating set of M to determine whether M is also finitely presented.

Proposition 10.2. *Let M be a finitely generated R-module, let I_0 and I_0' be finite sets, and let $g_0\colon \bigoplus_{i\in I_0} R \to M$ and $g_0'\colon \bigoplus_{i\in I_0'} R \to M$ be surjective R-linear maps. Then $\ker(g_0)$ is finitely generated over R if and only if $\ker(g_0')$ is finitely generated over R.*

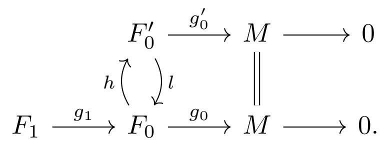
Proof. By symmetry, it suffices to assume that $\ker(g_0)$ is finitely generated over R and prove that $\ker(g_0')$ is finitely generated over R. We shorten our notation by letting $F_0 = \bigoplus_{i\in I_0} R$ and $F_0' = \bigoplus_{i\in I_0'} R$. We also fix a finite set I_1 and a surjective R-linear map $g_1\colon F_1 := \bigoplus_{i\in I_1} R \to \ker(g_0)$, witnessing that $\ker(g_0)$ is a finitely generated R-module.

We obtain a diagram with exact rows

$$\begin{array}{ccccccc} & & F_0' & \xrightarrow{g_0'} & M & \longrightarrow & 0 \\ & & & & \| & & \\ F_1 & \xrightarrow{g_1} & F_0 & \xrightarrow{g_0} & M & \longrightarrow & 0 \end{array}$$

[30]As Kaif showed to you, if M is projective and finitely generated, then it is also finitely presented. We will not use this fact in the present discussion.

Now we can use that F_0 is a free R-module, and in particular is projective, together with the fact that g_0' is surjective: there exists an R-linear map $h\colon F_0 \to F_0'$ such that $g_0 = h \circ g_0'\colon F_0 \to M$. Similarly, we can find an R-linear map $l\colon F_0' \to F_0$ such that $g_0' = l \circ g_0$. We obtain a new diagram



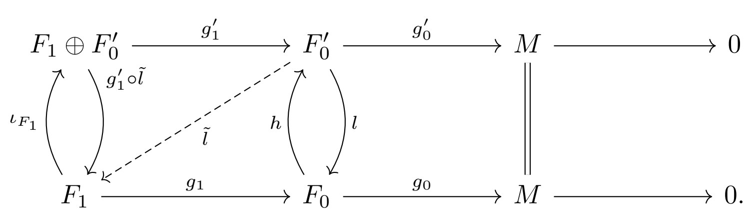
Selecting *either h or l*, we obtain a commutative square in the last diagram; but h and l are not inverse of each other (so we don't have a "commutative bigon").
In particular, the map h restricts to a map $\ker(g_0) \to \ker(g_0')$, and the map l restricts to a map $\ker(g_0') \to \ker(g_0)$.
Consider now the map $\mathrm{Id}_{F_0'} - l \circ h\colon F_0' \to F_0'$: we claim that it takes values in the submodule $\ker(g_0')$. Indeed, for all $x \in F_0'$, we can compute

$$((x)(\mathrm{Id}_{F_0'} - l \circ h))g_0' = (x)g_0' - (x)l \circ h \circ g_0' = (x)g_0' - (x)l \circ g_0 = (x)g_0' - (x)g_0' = 0.$$

We can now define a map $g_1'\colon F_1 \oplus F_0' \to \ker(g_0')$ by taking the map $g_1 \circ h$ on the summand F_1, and by taking the map $\mathrm{Id}_{F_0'} - l \circ h$ on the summand F_0'. And now the miracle occurs: the map g_1' hits $\ker(g_0')$ surjectively! To see this, let $y \in \ker(g_0')$. Then $(y)l \in \ker(g_0)$, so there exists $z \in F_1$ with $(z)g_1 = (y)l$. We can then write

$$y = (y - (y)l \circ h) + (y)l \circ h = (y)(\mathrm{Id}_{F_0'} - l \circ h) + (z)g_1 \circ h = (y, z)g_1'.$$

Since $F_1 \oplus F_0'$ is a finitely generated free R-module, we conclude that $\ker(g_0')$ is finitely generated. The proof is completed, but let us also extend the diagram above as follows, where rows are exact, and some squares and triangles (which ones?) are commutative



The map $\tilde{l}\colon F_0' \to F_1$ is a choice of lift of $l\colon F_0 \to \ker(g_0)$ along the surjective map $g_1\colon F_1 \to \ker(g_0)$: this lift exists again because F_0' is projective. In fact, we could have first fixed such a lift, and then have taken $z = (y)\tilde{l}$ in the argument before. $\square$

Example 10.3. Let $R = \prod_{i \in \mathbb{N}} \mathbb{Q}$ be the ring from the first multiple choice test, and consider $M = R/\bigoplus_{i \in \mathbb{N}} \mathbb{Q}$. Then M is finitely generated, in fact it is 1-generated; but the kernel of the canonical, surjective, R-linear map $R \to M$ is $\bigoplus_{i \in \mathbb{N}} \mathbb{Q} \subset R$, which is *not* finitely generated.

Using Proposition 10.2, the conclusion is that M is not finitely presented, not even if we replace $R \to M$ by another surjective map $F_0 \to M$ from a finitely generated free R-module.

The discussion so far is hopefully going to make you think "Oh, but this is so similar to something we just saw before!" during the rest of the section.

10.2. **Definition of projective resolution.** Let M be an R-module. The first step in presenting M is to choose a surjective R-linear map $g_0\colon F_0 \to M$ from a free R-module F_0. The second step is to choose a surjective R-linear map $g_1\colon F_1 \to \ker(g_0)$ with source another free R-module F_1: this is however precisely the first step of presenting the module $\ker(g_0)$!
We can now complete the presentation of $\ker(g_0)$, by choosing a surjective map $g_2\colon F_2 \to \ker(g_1)$ with source yet a new free module F_2: and again, we can continue in this fashon forever, obtaining an exact sequence of R-modules

$$\dots F_3 \xrightarrow{g_3} F_2 \xrightarrow{g_2} F_1 \xrightarrow{g_1} F_0 \xrightarrow{g_0} M \longrightarrow 0 \dots$$

We can now do the following somewhat artificial thing: remove M from the above sequence, and put a 0 instead. This has the somewhat sad effect of breaking the exactness and giving us a chain complex $F_\bullet$ reading

$$\dots F_3 \xrightarrow{g_3} F_2 \xrightarrow{g_2} F_1 \xrightarrow{g_1} F_0 \xrightarrow{g_0} 0 \dots$$

The good news is that now M can be recovered as $H_0(F_\bullet)$, and, most importantly, all other homology groups $H_i(F_\bullet)$ vanish.
In some sense, removing M is a natural thing to do: also when dealing with *presenting* M, we wanted just to find a map $g_1\colon F_1 \to F_0$ and then say "M can be recovered as the cokernel of this map". In particular, M should be the output of the presentation, not one of the ingredients! Similarly, we are now making a "generalised presentation" of M, keeping track of all successive kernels that arise at each step: the result is the chain complex $F_\bullet$, which is a "generalised presentation" (a.k.a. *free resolution*) of M.

Definition 10.4. Let M be a left R-module. A **projective** resolution of M is a couple $(P_\bullet, \epsilon)$, where $P_\bullet$ is a chain complex in left R-modules, and $\epsilon\colon P_0 \to M$ is an R-linear map, called *augmentation*, such that the following hold:

- for all $i \in \mathbb{Z}$ the R-module P_i is **projective**;
- "$P_\bullet$ is concentrated in degrees ≥ 0", i.e. $P_i = 0$ for all $i < 0$;
- for all $i \neq 0$ the homology group $H_i(P_\bullet)$ vanishes;
- the composition $d_1^P \circ \epsilon\colon P_1 \to M$ is the zero map, and the map $H_0(P_\bullet) = \operatorname{coker}(d_1^P) \to M$ induced by ϵ is an R-linear isomorphism.

A **projective** resolution $(P_\bullet, f)$ has *finite length* $\ell \geq 0$ if $P_\ell \neq 0$ and $P_i = 0$ for all $i > \ell$.

In the previous definition the word "**projective**" is emphasized: if you replace this word by the word "flat" or "free", you obtain the notions of flat and free resolutions of an R-module M. There is also a notion of "injective resolution", but it is somewhat different and we will see it later; replacting "**projective**" by "injective" in Definition 10.4 would lead to a quite useless definition instead.
The length of a resolution should be thought of as a measure of how complicated it is: the shorter the length, the happier we are.

It is somehow common also to consider ϵ as a surjective map $P_0 \to M$ such that $P_1 \to P_0 \to M \to 0$ is exact, and write a projective resolution as an exact sequence

$$\dots P_3 \xrightarrow{d_3^P} P_2 \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \dots ;$$

The problem with this is that it now seems that M lives in degree -1, whereas it would be more natural to have M in degree 0. The best solution is to consider ϵ as giving rise to a chain map

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \dots, \end{array}$$

and requiring that this chain map is a quasi-isomorphism $P_\bullet \to \theta_0(M)_\bullet$ (see Example 9.11).

Example 10.5. Let M be a projective R-module. Then a projective resolution of M is the pair $(\theta_0(M)_\bullet, \mathrm{Id}_M)$, where $\theta_0(M)_\bullet \in {}_R\mathrm{Ch}$ is the chain complex with M in degree 0 and zero in all other degrees:

$$\dots \to 0 \to 0 \to M \to 0 \dots .$$

with augmentation $\mathrm{Id}_M \colon M \to M$. We have that $H_0(\theta_0(M)_\bullet)$ is canonically isomorphic to M. This resolution has length 0.

Example 10.6. The chain complex from Example 8.10 is a free resolution of the zero module 0 over the ring $\mathbb{Z}$. Clearly, also $0_\bullet$ is a free resolution, and in fact it is a simpler one.

Example 10.7. Let $R = \mathbb{Z}/6$ and $M = \mathbb{Z}/3$. We saw that M is projective over R, so a projective resolution of M is given by $(\theta_0(M)_\bullet, \mathrm{Id}_M)$ as in Example 10.5. However this is not a free resolution of M.
A free resolution of M is, for instance, the following:

$$\dots P_4 = \mathbb{Z}/6 \xrightarrow{\cdot 2} P_3 = \mathbb{Z}/6 \xrightarrow{\cdot 3} P_2 = \mathbb{Z}/6 \xrightarrow{\cdot 2} P_1 = \mathbb{Z}/6 \xrightarrow{\cdot 3} P_0 = \mathbb{Z}/6 \to 0 \dots ,$$

together with the augmentation $\mathbb{Z}/6 \to \mathbb{Z}/3$ given by $[n]_6 \mapsto [n]_3$. Note that the previous is a resolution of infinite length: in fact it is a good exercise to prove that there is no resolution of $\mathbb{Z}/3$ over $\mathbb{Z}/6$ whose length is finite and whose terms are *finitely generated, free* $\mathbb{Z}/6$-modules.[31]

The previous example shows that for concrete computations it can be convenient to use generic projective resolutions, instead of only free ones. The fact that ${}_R\mathrm{Mod}$ has enough projectives implies that each R-module M admits some projective resolution.

Example 10.8. Let $R = \mathbb{F}$ be a field. Then every $\mathbb{F}$-vector space is already projective, so it admits a projective resolution of length 0.

[31]Hint: multiplying and dividing powers of 6, one never gets the number 3...

Example 10.9. Let R be a PID, and let M be any R-module. Then we can fix an R-linear surjection $g_0 \colon F_0 \to M$ with some free R-module as source. Then by Theorem 5.10 the R-module $\ker(g_0) \subset F_0$ is free, as it is a submodule of a free R-module: we can call it F_1, define $d_1 \colon F_1 \to F_0$ to be the inclusion, define $F_i = 0$ for all $i \neq 0, 1$ and define all differentials $d_i \colon F_i \to F_{i-1}$ to be the zero map, for $i \neq 1$. We obtain in this way a free resolution $(F_\bullet, \bar{g}_0)$ of length 1 of M, where $\bar{g}_0 \colon H_0(F_\bullet) = \mathrm{coker}(d_1) \to M$ is the isomorphism induced by g_1. As a very concrete example, for $k \geq 2$, the chain complex

$$\dots P_2 = 0 \longrightarrow P_1 = \mathbb{Z} \xrightarrow{\cdot k} P_0 = \mathbb{Z} \longrightarrow 0 \dots$$

together with the augmentation $[-]_k \colon \mathbb{Z} \to \mathbb{Z}/k$, gives a length-1 projective resolution of $\mathbb{Z}/k$ over $\mathbb{Z}$.

The previous example implies, for instance, that also $\mathbb{Q}$ has a projective (in fact free) resolution over $\mathbb{Z}$ of length 1; but writing down explicitly such a resolution seems to be quite a mess!

Example 10.10. Let p be a prime number and let $R = \mathbb{Z}/p^2$. Let $M = \mathbb{Z}/p$, and use the map of rings $R \to M$ to make M into an R-module. A projective (in fact, free) resolution of M is the following chain complex $P_\bullet$

$$\dots P_4 = \mathbb{Z}/p^2 \xrightarrow{\cdot p} P_3 = \mathbb{Z}/p^2 \xrightarrow{\cdot p} P_2 = \mathbb{Z}/p^2 \xrightarrow{\cdot p} P_1 = \mathbb{Z}/p^2 \xrightarrow{\cdot p} P_0 = \mathbb{Z}/p^2 \to 0 \dots,$$

together with the augmentation $\mathbb{Z}/p^2 \to \mathbb{Z}/p$ given by $[n]_{p^2} \mapsto [n]_p$. Note that this resolution has infinite length; in fact it can be proved (and hopefully we will do it at some point!) that in this case M admits no projective resolution over R of finite length.
Note that the same $M = \mathbb{Z}/p$, when considered as a $\mathbb{Z}$-module, would admit a projective resolution of length 1 over $\mathbb{Z}$.

Example 10.11. Let $R = \mathbb{F}[x]/(x^n)$ for some field $\mathbb{F}$ and some $n \geq 2$, and let $M = \mathbb{F}[x]/(x^k)$ for some $1 \leq k \leq n-1$. Then M admits a projective (in fact free) resolution over $\mathbb{F}[x]/(x^n)$ given by

$$\begin{aligned}
&\dots P_6 = \mathbb{F}[x]/(x^n) \xrightarrow{\cdot x^{n-k}} P_5 = \mathbb{F}[x]/(x^n) \xrightarrow{\cdot x^k} \\
&\quad P_4 = \mathbb{F}[x]/(x^n) \xrightarrow{\cdot x^{n-k}} P_3 = \mathbb{F}[x]/(x^n) \xrightarrow{\cdot x^k} \\
&\quad P_2 = \mathbb{F}[x]/(x^n) \xrightarrow{\cdot x^{n-k}} P_1 = \mathbb{F}[x]/(x^n) \xrightarrow{\cdot x^k} \\
&\quad P_0 = \mathbb{F}[x]/(x^n) \longrightarrow 0 \dots,
\end{aligned}$$

together with the augmentation $\mathbb{F}[x]/(x^n)) \to \mathbb{F}[x]/(x^k)$ given by $[f(x)]_{x^n} \mapsto [f(x)]_{x^k}$. Compare with Example 10.10 in the case $n = 2$ and $k = 1$.

Example 10.12. Let $R = \mathbb{Z}[x]/(x^n - 1)$ for some $n \geq 2$, and let $M = \mathbb{Z}[x]/(x-1) \cong \mathbb{Z}$. Then M admits a projective (in fact free) resolution over R given by

$$\begin{aligned}
\dots P_6 = \mathbb{Z}[x]/(x^n-1) &\xrightarrow[\cdot(x-1)]{\cdot(1+x+\cdots+x^{n-1})} P_5 = \mathbb{Z}[x]/(x^n-1) \\
\to P_4 = \mathbb{Z}[x]/(x^n-1) &\xrightarrow[\cdot(x-1)]{\cdot(1+x+\cdots+x^{n-1})} P_3 = \mathbb{Z}[x]/(x^n-1) \\
\to P_2 = \mathbb{Z}[x]/(x^n-1) &\xrightarrow[\cdot(x-1)]{\cdot(1+x+\cdots+x^{n-1})} P_1 = \mathbb{Z}[x]/(x^n-1) \\
\to P_0 = \mathbb{Z}[x]/(x^n-1) &\longrightarrow 0 \dots,
\end{aligned}$$

together with the augmentation $\mathbb{Z}[x]/(x^n-1)) \to \mathbb{Z}[x]/(x-1)$ given by $[f(x)]_{x^n-1} \mapsto [f(x)]_{x-1}$. Compare with Examples 10.10 and 10.11.

The previous example is important because $\mathbb{Z}[x]/(x^n - 1)$ is isomorphic to the group ring $\mathbb{Z}[C_n]$, where C_n is the finite cyclic group on n elements.[32]. The group homology of C_n is defined precisely using this ring and the corresponding $\mathbb{Z}[C_n]$-module $\mathbb{Z}$, as we will (hopefully) see!

10.3. **Projective resolutions and R-linear maps.** Suppose that $(P_\bullet, \epsilon)$ is a projective resolution of a left R-module M, and $(Q_\bullet, \upsilon)$ is a projective resolution of another left R-module N. Suppose moreover that $f : M \to N$ is an R-linear map. Since we can exhibit M as $H_0(P_\bullet)$ and N as $H_0(N_\bullet)$, it would be great to exhibit f as $H_0(\tilde{f})$, for some chain map $\tilde{f} : P_\bullet \to Q_\bullet$. In this subsection we prove that such a $\tilde{f}$ exists. We start by writing a diagram with exact rows

$$\begin{array}{ccccccccccc}
\dots P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle f} & & \\
\dots Q_3 & \xrightarrow{d_3^Q} & Q_2 & \xrightarrow{d_2^Q} & Q_1 & \xrightarrow{d_1^Q} & Q_0 & \xrightarrow{\upsilon} & N & \longrightarrow & 0.
\end{array}$$

And now be ready, because we will use the hypothesis that our modules P_i are projective! Since P_0 is projective, and since $\upsilon : Q_0 \to N$ is surjective, we can find an R-linear map $\tilde{f}_0 : P_0 \to Q_0$ such that the square in the following diagram commutes

$$\begin{array}{ccccccccccc}
\dots P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\
\dots Q_3 & \xrightarrow{d_3^Q} & Q_2 & \xrightarrow{d_2^Q} & Q_1 & \xrightarrow{d_1^Q} & Q_0 & \xrightarrow{\upsilon} & N & \longrightarrow & 0.
\end{array}$$

The map $d_1^Q : Q_1 \to Q_0$ is not surjective, but it is surjective when considered as a map $Q_1 \to \ker(\upsilon)$. On the other hand, the map $d_1^P \circ \tilde{f}_0 : P_1 \to Q_0$ has image contained in $\ker(\upsilon)$: indeed we have an equality of maps $P_1 \to M$

$$d_1^P \circ \tilde{f}_0 \circ \upsilon = d_1^P \circ \epsilon \circ f = 0 \circ f = 0.$$

[32]Clearly C_n is just another name of $\mathbb{Z}/n$, but it would be somehow ugly to write $\mathbb{Z}[\mathbb{Z}/k]$...

We can then use that P_1 is projective and find an R-linear map $\tilde{f}_1\colon P_1 \to Q_1$ such that both squares in the following diagram commute

$$\begin{array}{ccccccccccc} \dots P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \dots Q_3 & \xrightarrow{d_3^Q} & Q_2 & \xrightarrow{d_2^Q} & Q_1 & \xrightarrow{d_1^Q} & Q_0 & \xrightarrow{\upsilon} & N & \longrightarrow & 0. \end{array}$$

Now the projective module P_2 enters the scene: again one can check that the composition $d_2^P \circ \tilde{f}_2\colon P_2 \to Q_1$ has image in $\ker(d_1^Q)$, and there is a surjective map $d_2^Q\colon Q_2 \to \ker(d_1^Q)$: nothing is better to ensure the existence of an R-linear map $\tilde{f}_2\colon P_2 \to Q_2$ such that all three squares commute

$$\begin{array}{ccccccccccc} \dots P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}_2} & & \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \dots Q_3 & \xrightarrow{d_3^Q} & Q_2 & \xrightarrow{d_2^Q} & Q_1 & \xrightarrow{d_1^Q} & Q_0 & \xrightarrow{\upsilon} & N & \longrightarrow & 0. \end{array}$$

One can continue forever in this way. Using the axiom of choice, if you like, one finds a chain map $\tilde{f} = (\tilde{f}_i)_{i\in\mathbb{Z}}$: strictly speaking, one also has to set $\tilde{f}_i = 0$ for $i < 0$.

Exercise 10.13. Check that $H_0(\tilde{f})$ coincides with f, up to identifying $H_0(P_\bullet)$ with M by the map induced by ϵ, and $H_0(Q_\bullet)$ with N by the map induced by υ.

That was great! But at each step we have chosen a map $\tilde{f}_i$. The amazing part is still to come: another sequence of choices would have produced another chain map $\check{f}\colon P_\bullet \to Q_\bullet$, but then $\tilde{f}$ and $\check{f}$ are automatically chain homotopy equivalent. Let's prove this fact. We start with a diagram with exact rows

$$\begin{array}{ccccccccccc} \dots P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ {\scriptstyle \check{f}_3}\downarrow\downarrow{\scriptstyle \tilde{f}_3} & & {\scriptstyle \check{f}_2}\downarrow\downarrow{\scriptstyle \tilde{f}_2} & & {\scriptstyle \check{f}_1}\downarrow\downarrow{\scriptstyle \tilde{f}_1} & & {\scriptstyle \check{f}_0}\downarrow\downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \dots Q_3 & \xrightarrow[d_3^Q]{} & Q_2 & \xrightarrow[d_2^Q]{} & Q_1 & \xrightarrow[d_1^Q]{} & Q_0 & \xrightarrow[\upsilon]{} & N & \longrightarrow & 0. \end{array}$$

Consider the map $(\check{f}_0 - \tilde{f}_0)\colon P_0 \to Q_0$: it lands inside $\ker(\upsilon)$, indeed we have an sequence of equalities of maps $P_0 \to M$

$$(\check{f}_0 - \tilde{f}_0) \circ \upsilon = \check{f}_0 \circ \upsilon - \tilde{f}_0 \circ \upsilon = \epsilon - \epsilon = 0.$$

Again, $\ker(\upsilon)$ is hit surjectively by the map $d_1^Q\colon Q_1 \to \ker(\upsilon)$; we can thus define an R-linear map $\mathcal{H}_0\colon P_0 \to Q_1$ such that $(\check{f}_0 - \tilde{f}_0) = \mathcal{H}_0 \circ d_1^Q\colon P_0 \to Q_0$. We add $\mathcal{H}_0$ to our diagram, but pay attention, because only some of the squares (and in fact, none of the triangles) are known to commute

$$\begin{array}{ccccccccccc} \dots P_3 & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ {\scriptstyle \check{f}_3}\downarrow\downarrow{\scriptstyle \tilde{f}_3} & & {\scriptstyle \check{f}_2}\downarrow\downarrow{\scriptstyle \tilde{f}_2} & & {\scriptstyle \check{f}_1}\downarrow\downarrow{\scriptstyle \tilde{f}_1} & \overset{\mathcal{H}_0}{\swarrow} & {\scriptstyle \check{f}_0}\downarrow\downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \dots Q_3 & \xrightarrow[d_3^Q]{} & Q_2 & \xrightarrow[d_2^Q]{} & Q_1 & \xrightarrow[d_1^Q]{} & Q_0 & \xrightarrow[\upsilon]{} & N & \longrightarrow & 0. \end{array}$$

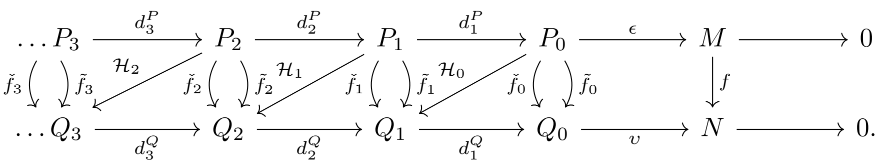
Now it is the turn of P_1: we consider the map $(\check{f}_1 - \tilde{f}_1 - d_1^P \circ \mathcal{H}_0)\colon P_1 \to Q_1$, and again some magic happens: this map lands inside $\ker(d_1^Q)$! To see this, we have a sequence of equalities of maps $P_1 \to Q_0$

$$\begin{aligned}(\check{f}_1 - \tilde{f}_1 - d_1^P \circ \mathcal{H}_0) \circ d_1^Q &= (\check{f}_1 - \tilde{f}_1) \circ d_1^Q - d_1^P \circ \mathcal{H}_0 \circ d_1^Q \\ &= (\check{f}_1 - \tilde{f}_1) \circ d_1^Q - d_1^P \circ (\check{f}_0 - \tilde{f}_0) = 0,\end{aligned}$$

where the last step uses that $\check{f} - \tilde{f}$ is a chain map. Again, we can use the surjection $d_2^Q \colon Q_2 \to \ker(d_1^Q)$ and get an R-linear map $\mathcal{H}_1 \colon P_1 \to Q_2$ such that $(\check{f}_1 - \tilde{f}_1 - d_1^P \circ \mathcal{H}_0) = \mathcal{H}_1 \circ d_2^Q \colon P_1 \to Q_1$. Reassembling, we obtain the equality

$$\check{f}_1 - \tilde{f}_1 = d_1^P \circ \mathcal{H}_0 + \mathcal{H}_1 \circ d_2^Q.$$

It starts looking like something familiar, right? We can continue in the same way constructing, one after the other, R-linear maps $\mathcal{H}_i \colon P_i \to Q_{i+1}$, obtaining a diagram as follows:



We can also set $\mathcal{H}_i$ to be the zero map for $i < 0$. The result is a chain homotopy $\mathcal{H} \colon P_\bullet \to Q_{\bullet+1}$ between $\tilde{f}$ and $\check{f}$.

10.4. **Consequences of the work done.** So far we have seen three important things:

- each left R-module M admits some projective resolution $P_\bullet$;
- each R-linear map $f \colon M \to N$ admits some lift $\tilde{f}$ as a chain map between two fixed projective resolutions $P_\bullet$ of M and $Q_\bullet$ of N;
- the lift $\tilde{f}$ is unique up to chain homotopy.

We use these observations to define a "projective resolution functor", with source ${}_R\mathrm{Mod}$, and with target... not quite ${}_R\mathrm{Ch}$, but the slightly smaller category $K({}_R\mathrm{Mod})$.

Definition 10.14. We define a functor $\mathfrak{P} \colon {}_R\mathrm{Mod} \to K({}_R\mathrm{Mod})$ and a natural transformation $[\epsilon^{\mathfrak{P}}] \colon \mathfrak{P} \Rightarrow \mathbb{K} \circ \theta_0$ as follows.[33]
First, we use a "powerful enough version of the axiom of choice" and choose for every left R-module M a projective resolution $\mathfrak{P}(M)_\bullet = (\mathfrak{P}(M)_\bullet, \epsilon^{\mathfrak{P}}(M))$ of M. The behaviour of the functor on objects is thus defined, and also the natural transformation, which is associates with every $M \in {}_R\mathrm{Mod}$ the chain homotopy class of the chain map $\epsilon_M^{\mathfrak{P}} \colon \mathfrak{P}(M)_\bullet \to \theta_0(M)_\bullet$.
Given an R-linear map $f \colon M \to N$, we define $\mathfrak{P}(f)$ to be $[\tilde{f}] \colon \mathfrak{P}(M)_\bullet \to \mathfrak{P}(N)_\bullet$ for any choice of chain map $\tilde{f} \colon \mathfrak{P}(M)_\bullet \to \mathfrak{P}(N)_\bullet$ constructed as above. The above argument shows precisely that $[\tilde{f}]$ only depends on f (and on the objects $\mathfrak{P}(M)_\bullet$ and $\mathfrak{P}(N)_\bullet$, which have already been fixed), and it also shows that $[\epsilon^{\mathfrak{P}}]$ is indeed a natural transformation.

[33] I replaced the old notation with this new one, where "$\mathfrak{P}$" recalls the first letter of the word "projective" and later "$\mathfrak{I}$" recalls the first letter of the word "injective". The old notation "$\mathfrak{L}$" and "$\mathfrak{R}$", inspired by the words "left" and "right", can lead to confusion when considering left and right derived functors in the covariant and contravariant case.

About the "powerful enough version of the axiom of choice": strictly speaking, this is a true issue, since the objects of ${}_R\mathrm{Mod}$ form a class, and not a set. There are two ways out of this problem:

- either we agree to work not with all left R-modules, but only those whose underlying set is a subset of some big-enough-for-our-purposes set $\mathscr{S}$, putting also a bound on the cardinality of allowed R-modules, in such a way that we can still study efficiently the allowed R-modules;
- or we define $\mathfrak{P}(M)$ to be the brute-force free resolution: we define $\mathfrak{P}(M)_0 = \bigoplus_{m\in M} R$; we define $\epsilon_M^{\mathfrak{P}}\colon \mathfrak{P}(M)_0 \to M$ to be the map adjoint to the identification of sets $M = M$; we define $\mathfrak{P}(M)_1 = \bigoplus_{m\in\ker(\epsilon_M^{\mathfrak{P}})} R$; we define $d_1^{\mathfrak{P},M}\colon \mathfrak{P}(M)_1 \to \ker(\epsilon_M^{\mathfrak{P}}) \subset \mathfrak{P}(M)_0$ as the map adjoint to the identification of sets $\ker(\epsilon_M^{\mathfrak{P}}) = \ker(\epsilon_M^{\mathfrak{P}})$; and for $i \geq 2$ we define recursively $\mathfrak{P}(M)_i$ to be the free R-module on the set $\ker(d_{i-1}^{\mathfrak{P},M})$, and we define $d_1^{\mathfrak{P},M}\colon \mathfrak{P}(M)_i \to \ker(d_{i-1}^{\mathfrak{P},M}) \subset \mathfrak{P}(M)_{i-1}$ as the map adjoint to the identification of sets $\ker(d_{i-1}^{\mathfrak{P},M}) = \ker(d_{i-1}^{\mathfrak{P},M})$. This allows us to avoid completely the axiom of choice.

Example 10.15. Let us prove that the functor $\mathfrak{P}$ is additive. Let therefore $f, g\colon M \to N$ be two R-linear maps, and let $\tilde{f}, \tilde{g}\colon \mathfrak{P}(M)_\bullet \to \mathfrak{P}(N)_\bullet$ be R-linear chain maps lifting f, g, constructed as above. Then $\tilde{f} + \tilde{g}$ is a good example of a chain map $\mathfrak{P}(M)_\bullet \to \mathfrak{P}(N)_\bullet$ lifting $f + g$; we conclude that $\mathfrak{P}(f + g) = [\tilde{f} + \tilde{g}] = [\tilde{f}] + [\tilde{g}] = \mathfrak{P}(f) + \mathfrak{P}(g)$.

Exercise 10.16. In Definition 10.14 a choice was made: for every object $M \in {}_R\mathrm{Mod}$, a projective resolution of M was chosen. Suppose now that someone else comes about and gives you another assignment $M \mapsto (\mathfrak{P}'(M)_\bullet, (\epsilon^{\mathfrak{P}})')$ of a projective resolution for each R-module M; then you can use these data to construct a new functor $\mathfrak{P}'$ with a natural transformation $(\epsilon')^{\mathfrak{P}}$.

- construct for all M a chain homotopy equivalence $\tilde{\mathrm{Id}}_M\colon \mathfrak{P}(M)_\bullet \to \mathfrak{P}'(M)_\bullet$ by lifting Id_M;
- prove that the collection of maps $[\mathrm{Id}_M]$ for $M \in {}_R\mathrm{Mod}$ assemble into a natural equivalence of functors $\mathfrak{P} \Rightarrow \mathfrak{P}'$.

We note that the functor $\mathfrak{P}$ does not "need" the entire category $K({}_R\mathrm{Mod})$: after all only some of the objects are used, namely those chain complexes of projective modules that are concentrated in non-negative degrees and have vanishing H_i for $i > 0$. We will restrict our category $K({}_R\mathrm{Mod})$ to a slightly bigger category than required, which is convenient for many applications.

Definition 10.17. We denote by $D^+({}_R\mathrm{Mod})$ the full subcategory of $K({}_R\mathrm{Mod})$ spanned by objects of the form $(P_\bullet, d^P)$ such that:

- each P_i is a projective R-module;
- there is $N > 0$ such that $P_i = 0$ for all $i < -N$ (we say that $P_\bullet$ is *bounded below*).

The category $D^+({}_R\mathrm{Mod})$ is known as the *bounded below derived category of (the abelian category)* ${}_R\mathrm{Mod}$.

Why do we require "bounded below" instead of the condition that we actually are using, namely "$P_i = 0$ for $i < 0$"? In simple words: the shift functors $\Sigma^k\colon {}_R\mathrm{Ch} \to$

${}_R\mathrm{Ch}$ induce shift functors $\Sigma^k \colon K({}_R\mathrm{Mod}) \to K({}_R\mathrm{Mod})$, and we are happier if these functors are available also on our subcategory $D^+({}_R\mathrm{Mod})$, even for k negative.
Why do we allow chain complexes $P_\bullet$ having possibly several non-trivial homology R-modules? The construction seem only to need those chain complexes with only H_0 (possibly) non-vanishing... In simple words: we are happier if for each morphism $[f] \colon P_\bullet \to Q_\bullet$ in $D^+({}_R\mathrm{Mod})$, for each representative f of $[f]$ the mapping cone $\mathrm{Cone}(f)$ is still an object in $D^+({}_R\mathrm{Mod})$. You can easily see, using homology LES, that allowing mapping cones quickly gives rise to chain complexes of projective modules which are bounded below, but have several non-vanishing homology groups.
These are oversimplified explanations! If you want more, continue studying homological algebra in the future.

10.5. **A glimpse into injective resolutions.** One can dually define an injective resolution of an R-module M to be a chain complex $(I_\bullet)$ concentrated in degrees ≤ 0, together with a quasi-isomorphism η *from* $\theta_0(M)_\bullet$:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots, \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & I_0 & \xrightarrow{d_0^I} & I_{-1} & \xrightarrow{d_{-1}^I} & I_{-2} & \xrightarrow{d_{-2}^I} & I_{-3} & \xrightarrow{d_{-3}^I} & \dots
\end{array}$$

Since, if one has to choose, one usually prefers indices ≥ 0 rather than indices ≤ 0, it is common to write an injective resolution as a cochain complex

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots, \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{\delta_I^0} & I^1 & \xrightarrow{\delta_I^1} & I^2 & \xrightarrow{\delta_I^2} & I^3 & \xrightarrow{\delta_I^3} & \dots
\end{array}$$

An injective resolution of M can be constructed using that ${}_R\mathrm{Mod}$ has enough injectives: one fixes an injection $\eta \colon M \hookrightarrow I^0$ into some injective R-module I^0; one then injects $\mathrm{coker}(\eta)$ into a new injective R-module I^1, and calls δ_I^0 the composite $I^0 \to \mathrm{coker}(\eta) \hookrightarrow I^1$; one then injects $\mathrm{coker}(\delta_I^0)$ into a new injective R-module I^2, and calls δ_I^1 the composite $I^1 \to \mathrm{coker}(\delta_I^0) \hookrightarrow I^2$; and so on.
An R-linear map $f \colon M \to N$ gives rise to a well-defined cochain homotopy class of cochain maps $\tilde{f}$ between any fixed injective resolutions of M and N: the proof of this is the same as in the projective case, but with all arrows reversed.
What is a cochain homotopy? Given cochain complexes $C^\bullet$ and $D^\bullet$, a cochain homotopy $\mathcal{H} \colon C^\bullet \to D^{\bullet-1}$ is a sequence of R-linear maps $(\mathcal{H}^i)_{i\in\mathbb{Z}}$; for two cochain maps $f, g \colon C^\bullet \to D^\bullet$ we say that $\mathcal{H}$ is a cochain homotopy from f to g if $g - f = \delta_C \circ \mathcal{H} + \mathcal{H} \circ \delta_D$. As you see, under interpreting $C^\bullet$ as $C_{-\bullet}$, this is not really a new definition.
As a result, one can also define an "injective resolution functor" $\mathfrak{I} \colon {}_R\mathrm{Mod} \to K({}_R\mathrm{Mod})$, assigning to each R-module an injective resolution of it. This functor comes equipped with a natural transformation $[\eta^{\mathfrak{I}}] \colon \mathbb{K} \circ \theta_0 \Rightarrow \mathfrak{I}$.
Finally, one can define a *bounded above derived category associated with the abelian category* ${}_R\mathrm{Mod}$ as the full subcategory of $K({}_R\mathrm{Mod})$ on objects that are bounded above chain complexes of injective R-modules.

10.6. **Left derived functors, covariant case.** Let $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be an additive functor[34]. We already saw that F need not be exact. We however use F to define new functors ${}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$, which are supposed to encapsulate, all together, how much F fails from being exact.

Definition 10.18. Let $n \in \mathbb{Z}$. We define the additive functor $\mathbb{L}_nF\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ as the composite of the additive functors

$$ {}_R\mathrm{Mod} \xrightarrow{\mathfrak{P}} K({}_R\mathrm{Mod}) \xrightarrow{K(F)} K({}_S\mathrm{Mod}) \xrightarrow{H_n^K(-)} {}_S\mathrm{Mod}. $$

Wait a moment: we have just seen the additive functor $\mathfrak{P}\colon {}_R\mathrm{Mod} \to K({}_R\mathrm{Mod})$ (landing actually in $D^+({}_R\mathrm{Mod})$, but never mind for now); last time we saw how the functors $H_n(-)\colon {}_R\mathrm{Ch} \to {}_R\mathrm{Mod}$ factor through the homotopy category $K({}_R\mathrm{Mod})$, giving rise to (additive) functors $H_n^K(-)\colon K({}_R\mathrm{Mod}) \to {}_R\mathrm{Mod}$; but what is "$K(F)$" now supposed to be?

We have already noticed that if F is an additive functor, then it sends chain complexes to chain complexes; since F sends commuting squares to commuting squares, then F also sends chain maps to chain maps. So we have an induced functor $F\colon {}_R\mathrm{Ch} \to {}_S\mathrm{Ch}$.

Now comes the new remark. If $f, g\colon C_\bullet \to D_\bullet$ are chain maps and $\mathcal{H}\colon C_\bullet \to D_{\bullet+1}$ is a chain homotopy from f to g, then we have, for all $i \in \mathbb{Z}$, the equality of maps in ${}_R\mathrm{Mod}$

$$ g_i - f_i = d_i^C \circ \mathcal{H}_{i-1} + \mathcal{H}_i \circ d_{i+1}^D \colon C_i \to D_i. $$

Applying F we obtain the equality of maps in ${}_S\mathrm{Mod}$

$$ F(g_i) - F(f_i) = F(d_i^C) \circ F(cH_{i-1}) + F(\mathcal{H}_i) \circ F(d_{i+1}^D)\colon F(C_i) \to F(D_i). $$

But this really means that $F(\mathcal{H})$ is a chain homotopy from $F(f)$ to $F(g)$, which are chain maps between the chain complexes $(F(C_\bullet), F(d^C)) \to (F(D_\bullet), F(d^D))$. The result is that an additive functor F maps chain homotopic chain maps to chain homotopic chain maps, and thus induces a functor $K(F)\colon K({}_R\mathrm{Mod}) \to K({}_S\mathrm{Mod})$.

Example 10.19. For $n < 0$ the functor $\mathbb{L}_nF$ is the zero functor: indeed for all R-module M we have that $\mathfrak{P}(M)_\bullet$ is concentrated in non-negative degrees, hence the same holds for $F(\mathfrak{P}(M)_\bullet)$ (which is the same as the object $K(F)(\mathfrak{P}(M)_\bullet)$).

Example 10.20. Let M be a projective R-module. Then $\theta_0(M)_\bullet$ is a projective resolution of M, and we can suppose by Exercise 10.16 that $\mathfrak{P}(M)_\bullet = \theta_0(M)_\bullet$ is just M concentrated in degree 0. Applying F we obtain $\theta_0(F(M))_\bullet$, i.e. $F(M)$ concentrated in degree 0. Applying $H_n^K(-)$ we obtain either $F(M)$ (if $n = 0$), or 0 (if $n \neq 0$).

That is, $\mathbb{L}_nF(M) = 0$ for $n \neq 0$, and $\mathbb{L}_0F(M) \cong F(M)$.

Example 10.21. Let F be an exact functor. An exact functor has the following property: if $C_\bullet$ is a chain complex, then $F(C_\bullet)$ is also a chain complex (so far F additive would suffice), and moreover $H_n(F(C_\bullet))$ is canonically isomorphic to $F(H_n(C_\bullet))$. In fact F also sends kernels to kernels, images to images, cokernels to cokernels...

[34] You can replace the target by any abelian category, and the source by any abelian category with enough projectives

Now we have, for all $n \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{L}_n F(M) = H_n^K(K(F)(\mathfrak{P}(M)_\bullet)) &\cong H_n(F(\mathfrak{P}(M)_\bullet)) \\ &\cong F(H_n(\mathfrak{P}(M)_\bullet)) \cong F(H_n(\theta_0(M)_\bullet)),\end{aligned}$$

and the last term is isomorphic to M for $n = 0$, and vanishes for $n \neq 0$.

Example 10.22. Suppose that $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ is *right exact*. Then for all $M \in {}_R\mathrm{Mod}$ the sequence

$$\mathfrak{P}(M)_1 \xrightarrow{d_1^{\mathfrak{P}(M)}} \mathfrak{P}(M)_0 \xrightarrow{\epsilon_M^{\mathfrak{P}}} M \to 0$$

is exact. Applying F we obtain that the sequence

$$F(\mathfrak{P}(M)_1) \xrightarrow{F(d_1^{\mathfrak{P}(M)})} F(\mathfrak{P}(M)_0) \xrightarrow{F(\epsilon_M^{\mathfrak{P}})} F(M) \to 0$$

is also exact. Removing $F(M)$, we obtain the last chunk of the complex $F(\mathfrak{P}(M)_\bullet) = K(F)(\mathfrak{P}(M)_\bullet)$, reading

$$F(\mathfrak{P}(M)_1) \xrightarrow{F(d_1^{\mathfrak{P}(M)})} F(\mathfrak{P}(M)_0) \to 0$$

and we can compute $\mathbb{L}_0 F(M) \cong \operatorname{coker}(F(d_1^{\mathfrak{P}(M)})) \cong F(M)$.
Nice! The functor $\mathbb{L}_0 F$ just coincides with the old functor F! The functors $\mathbb{L}_n F$ for $n > 0$ will then measure how much F fails from being left exact.

The previous example is the reason why, usually, one defines the left derived functors $\mathbb{L}_n F$ only under the additional assumption that F is right exact: in principle one can give the definition without this extra assumption, but in practice, if F is not right exact, the sequence of functors $\mathbb{L}_0 F, \mathbb{L}_1 F, \dots$ does not seem to be strongly related to the original functor F.

10.7. **Right derived functors, covariant case.** In an analogous way one can define right derived functor, by using injective resolution instead of projective resolutions. Let $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be an additive functor.

Definition 10.23. Let $n \in \mathbb{Z}$. We define the additive functor $\mathbb{R}_n F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ as the composite of the additive functors

$${}_R\mathrm{Mod} \xrightarrow{\mathfrak{I}} K({}_R\mathrm{Mod}) \xrightarrow{K(F)} K({}_S\mathrm{Mod}) \xrightarrow{H_n^K(-)} {}_S\mathrm{Mod}.$$

We also use the notation $\mathbb{R}^n F = \mathbb{R}_{-n} F$, which is very useful when working in cohomological notation.

The only difference is that we now use $\mathfrak{I}$, i.e. the functor given by injective resolutions, instead of $\mathfrak{P}$. The first consequence is that $\mathbb{R}_n F$ vanishes for $n > 0$ (whereas $\mathbb{L}_n F$ vanishes for $n < 0$): for this reason one usually writes $\mathbb{R}^{-n} F$ for $\mathbb{R}_n F$.
Again, we have the following:

- if M is injective, then $\mathbb{R}_0 F(M) \cong F(M)$ and $\mathbb{R}_n F(M) = 0$ for all $n \neq 0$;
- if F is exact, then $\mathbb{R}_0 F$ is naturally isomorphic to F, and $\mathbb{R}_n F$ is the zero functor for all $n \neq 0$;
- if F is left exact, then $\mathbb{R}_0 F$ is naturally isomorphic to F.

We will deal after Christmas with left and right derived contravariant functors (and it will be fun to distinguish what is left from what is right).

11. Derived contravariant functors, Tor and Ext

We saw how to define, for an additive functor $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ and for all $n \in \mathbb{Z}$, the two additive functors $\mathbb{L}_n F$ and $\mathbb{R}_n F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$. There were mainly three ideas involved.

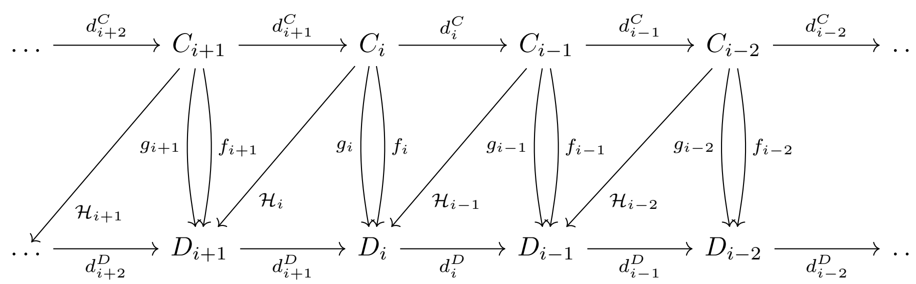
- An additive functor $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ induces a functor $F^{\mathrm{Ch}}\colon {}_R\mathrm{Ch} \to {}_S\mathrm{Ch}$, sending the chain complex $C_\bullet$ of left R-modules to the chain complex $F(C_\bullet)$ of left S-modules, and sending the chain map $f\colon C_\bullet \to D_\bullet$, represented schematically by

$$\begin{array}{ccccccc} \dots C_{i+1} & \xrightarrow{d^C_{i+1}} & C_i & \xrightarrow{d^C_i} & C_{i-1} & \xrightarrow{d^C_{i-1}} & C_{i-2}\dots \\ \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_{i-2}} \\ \dots D_{i+1} & \xrightarrow{d^D_{i+1}} & D_i & \xrightarrow{d^D_i} & D_{i-1} & \xrightarrow{d^D_{i-1}} & D_{i-2}\dots \end{array}$$

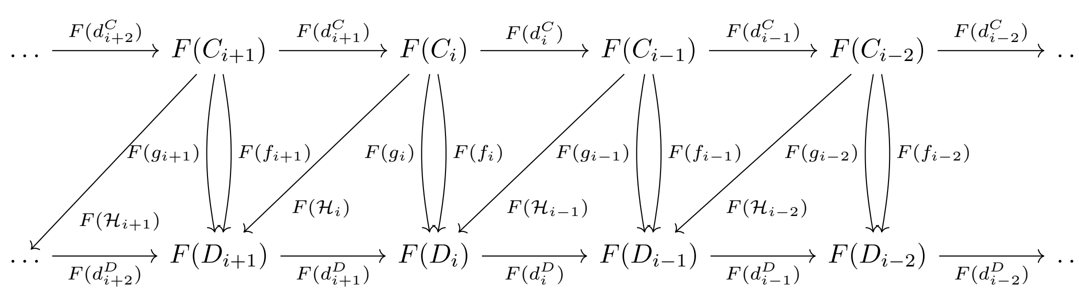
to the chain map $F(f)\colon F(C_\bullet) \to F(D_\bullet)$, represented schematically by

$$\begin{array}{ccccccc} \dots F(C_{i+1}) & \xrightarrow{F(d^C_{i+1})} & F(C_i) & \xrightarrow{F(d^C_i)} & F(C_{i-1}) & \xrightarrow{F(d^C_{i-1})} & F(C_{i-2})\dots \\ \downarrow{\scriptstyle F(f_{i+1})} & & \downarrow{\scriptstyle F(f_i)} & & \downarrow{\scriptstyle F(f_{i-1})} & & \downarrow{\scriptstyle F(f_{i-2})} \\ \dots F(D_{i+1}) & \xrightarrow{F(d^D_{i+1})} & F(D_i) & \xrightarrow{F(d^D_i)} & F(D_{i-1}) & \xrightarrow{F(d^D_{i-1})} & F(D_{i-2})\dots \end{array}$$

Moreover, F^{Ch} sends chain homotopic chain maps to chain homotopic chain maps: indeed a chain homotopy $\mathcal{H}$ between $f, g\colon C_\bullet \to D_\bullet$, represented schematically by



is sent by F to the chain homotopy $F(\mathcal{H})$ between $F(f)$ and $F(g)$, represented schematically by



We obtain therefore an induced functor $K(F)\colon K({}_R\mathrm{Mod}) \to K({}_S\mathrm{Mod})$, which is still an additive functor (between additive categories).

- There are functors $\mathfrak{P}, \mathfrak{I} \colon {}_R\mathrm{Mod} \to K({}_R\mathrm{Mod})$: the functor $\mathfrak{P}$ takes as input a left R-module M and gives a projective resolution $\mathfrak{P}(M)_\bullet$ of M; the functor $\mathfrak{I}$ gives an injective resolution, which we treat either as a chain complex concentrated in non-positive degrees, or as a cochain complex concentrated in non-negative degrees. In both cases an R-linear map $f \colon M \to N$ is sent to the homotopy class of a lift of f to a chain map between the resolutions of M and N. The functors $\mathfrak{P}$ and $\mathfrak{I}$ are additive. Their definition involves a choice (how to resolve each module), but up to a canonical natural isomorphism both functors are intrinsecally defined.
- There is an additive functor $H_n^K \colon K({}_S\mathrm{Mod}) \to {}_S\mathrm{Mod}$, sending a chain complex to its n^{th} homology group, and a chain homotopy class of chain maps to the induced map in homology.

We then defined $\mathbb{L}_n F = H_n^K \circ K(F) \circ \mathfrak{P}$ and $\mathbb{R}_n F = H_n^K \circ K(F) \circ \mathfrak{I}$, which are compositions of additive functors, hence additive functors.
Concretely, it is often already quite interesting just to determine, for an object $M \in {}_R\mathrm{Mod}$, the isomorphism type of the S-module $\mathbb{L}_n F(M)$, respectively $\mathbb{R}_n F(M)$: in this case it suffices to choose any projective, respectively injective, resolution of M, then apply the functor F and thus obtain a new chain complex, and finally compute the n^{th} homology group[35].

Example 11.1. Consider the functor $F = - \otimes_{\mathbb{Z}} \mathbb{Z}/3 \colon {}_{\mathbb{Z}}\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, and consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}$, $M_2 = \mathbb{Z}/4$ and $M_3 = \mathbb{Z}/6$. We want to compute, for all $n \in \mathbb{Z}$ and for $i = 1, 2, 3$, the $\mathbb{Z}$-modules $\mathbb{L}_n F(M_i)$.
A projective resolution of M_1 is given by $\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \ldots$, with $\mathbb{Z}$ in degree 0. Applying F we obtain the chain complex $\cdots \to 0 \to 0 \to \mathbb{Z}/3 \to 0 \ldots$, with $\mathbb{Z}/3$ in degree 0. All homology groups of the latter complex vanish, except H_0, which is isomorphic to $\mathbb{Z}/3$. Hence $\mathbb{L}_0 F(\mathbb{Z}) \cong \mathbb{Z}/3$, and $\mathbb{L}_n F(\mathbb{Z}) \cong 0$ for all $n \neq 0$.

A projective resolution of M_2 is given by $\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \to 0 \ldots$, with $\mathbb{Z}$ in degrees 0 and 1. Applying F we obtain the chain complex $\cdots \to 0 \to \mathbb{Z}/3 \xrightarrow{\cdot 4} \mathbb{Z}/3 \to 0 \ldots$, with $\mathbb{Z}/3$ in degrees 0 and 1. Multiplication by 4 induces an isomorphism $\mathbb{Z}/3 \xrightarrow{\cong} \mathbb{Z}/3$, and therefore all homology groups of the latter chain complex vanish. Hence $\mathbb{L}_n F(\mathbb{Z}/4) \cong 0$ for all $n \in \mathbb{Z}$.

A projective resolution of M_2 is given by $\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z} \to 0 \ldots$, with $\mathbb{Z}$ in degrees 0 and 1. Applying F we obtain the chain complex $\cdots \to 0 \to \mathbb{Z}/3 \xrightarrow{\cdot 6} \mathbb{Z}/3 \to 0 \ldots$, with $\mathbb{Z}/3$ in degrees 0 and 1. Multiplication by 6 induces the zero map $\mathbb{Z}/3 \to \mathbb{Z}/3$. Since all differentials in the last chain complex are the zero map, all homology groups coincide with the corresponding chain groups. Hence $\mathbb{L}_n F(\mathbb{Z}/6) \cong 0$ for all $n \neq 0, 1$, and both $\mathbb{L}_0 F(\mathbb{Z}/6)$ and $\mathbb{L}_0 F(\mathbb{Z}/6)$ are isomorphic to $\mathbb{Z}/3$.

We note that in all three cases of Example 11.1 there is an isomorphism $\mathbb{L}_0 F(M_i) \cong F(M_i)$: this is no surprise, as the functor F considered there is a right exact functor (see Example 10.22).

Example 11.2. Let F, M_1, M_2 and M_3 be as in Example 11.1. We want to compute, for all $n \in \mathbb{Z}$ and for $i = 1, 2, 3$, the $\mathbb{Z}$-modules $\mathbb{R}^n F(M_i)$.

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[35]If you consider an injective resolution as giving rise to a cochain complex, then you will write $\mathbb{R}^{-n} F$ for the functor $\mathbb{R}_n F$ and compute the $-n^{\text{th}}$ cohomology group. Do not forget that a minus sign has to be adjoined every time you change the position up/down of the indices!

An injective resolution of M_1 is given by $\dots 0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \to \dots$, with $\mathbb{Q}$ in cohomological degree 0 and $\mathbb{Q}/\mathbb{Z}$ in cohomological degree 1 (i.e. in homological degree -1). Applying F we obtain the zero cochain complex $\dots 0 \to 0 \to 0 \to 0 \to \dots$. To see this, recall that for a $\mathbb{Z}$-module N we have in general that $N \otimes_{\mathbb{Z}} \mathbb{Z}/3$ is isomorphic to the quotient $N/3N$, where $3N$ is the submodule of multiples of 3 in N. Since both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, we have $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/3 \cong \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3 \cong 0$. It thus follows that $\mathbb{R}^n F(\mathbb{Z}) \cong 0$ for all $n \in \mathbb{Z}$.
Clearly, the previous argument works also for M_2 and M_3, and in fact for any other $\mathbb{Z}$-module M: any injective resolution $I^\bullet$ of M consists of injective $\mathbb{Z}$-modules I^i, and we saw that a $\mathbb{Z}$-module is injective if and only if it is divisible; however, if each I^i is divisible (and in particular if it is 3-divisible, i.e. each $x \in I^i$ can be written as $3 \cdot y$ for some $y \in I^i$), then we also have $I^i \otimes_{\mathbb{Z}} \mathbb{Z}/3 \cong 0$ for all $i \in \mathbb{Z}$. It follows that for all $n \in \mathbb{Z}$ the group $\mathbb{R}^n F(M)$ vanishes, as it coincides with the n^{th} cohomology group of the 0 cochain complex.
In other words, for all $n \in \mathbb{Z}$ the functor $\mathbb{R}^n F \colon {}_{\mathbb{Z}}\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is the zero functor.

In principle, we want to use left and right derived functors $\mathbb{L}_n F$ and $\mathbb{R}^n F$ in order to understand better the functor F itself; in practice, there is no guarantee that these derived functors shed any light, and Example 11.2 shows in fact that it doesn't quit help to right derive the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/3$. In fact, it is common to consider right derived functor of a left exact functor, and left derived functors of a right exact functor: in this case there is a direct connection between F and the 0^{th} derived functor, and as we will see, this connection "propagates" to the other derived functors, derived on the side where F is not exact.
It can be interesting to note also the following, regarding which hypotheses are used in which part of the construction:

- to define the homotopy category $K(\mathcal{C})$ one only needs $\mathcal{C}$ to be an additive category[36]: one then has a notion of chain complex and of chain homotopy; correspondingly, any additive functor $F \colon \mathcal{C} \to \mathcal{D}$ between additive categories gives rise to an additive functor $K(F) \colon K(\mathcal{C}) \to K(\mathcal{D})$;
- to define homology groups $H_n(C_\bullet)$ of a chain complex in $\mathcal{C}$, one needs $\mathcal{C}$ to be an abelian category;
- to define the functor $\mathfrak{P}$ one needs to work with an abelian category with enough projectives; whereas for $\mathfrak{I}$ one needs an abelian category with enough injectives.

In conclusion, if $\mathcal{C}$ is an abelian category with enough projectives (respectively, enough injectives), $\mathcal{D}$ is an abelian category, and $F \colon \mathcal{C} \to \mathcal{D}$ is an additive functor, one can define left (respectively right) derived functors $\mathbb{L}_n F$ (respectively $\mathbb{R}^n F$) for all $n \in \mathbb{Z}$.

11.1. **Contravariant derived functors.** If $F \colon {}_R\mathrm{Mod}^{op} \to {}_S\mathrm{Mod}$ is an additive *covariant* functor, we have defined the composition $H_n^K \circ K(F) \circ \mathfrak{P}$ to be the "n^{th} *left* derived functor". What is the meaning of the word "left"? The traditional convention is to write exact sequences and chain/cochain complexes with maps/differentials going from left to right. This has the effect that, if $M \in {}_R\mathrm{Mod}$, then $K(F) \circ \mathfrak{P}(M)$ looks like

$$\dots F(\mathfrak{P}(M)_2) \longrightarrow F(\mathfrak{P}(M)_1) \longrightarrow F(\mathfrak{P}(M)_0) \longrightarrow 0 \dots$$

[36]Actually, even $\mathbb{Z}$-linear suffices!

i.e. it is a chain complex of left S-modules concentrated in non-negative degrees; referring to the diagram, it is concentrated on the *left* half-line of degree starting at the degree 0. This is the reason why we call $H_n^K \circ K(F) \circ \mathfrak{P}$ the "n^{th} *left* derived functor". One may object that a more meaningful and intrinsic terminology could be something like "n^{th} projectively derived functor"; the problem is that, when working with contravariant functors, one has to recall that a projective object in an abelian category $\mathcal{C}$ is really an injective object of $\mathcal{C}^{op}$, and a new source of confusion can arise. It is then better, or at least tradition, to fix a convention ("arrows of complexes are always drawn from left to right") and work with such a convention. Note that in fact we already used this convention when defining what a "right exact" or "left exact" functor is; for our purposes, it will only be important that we use twice the same convention.
Consider now an additive, contravariant functor $F\colon {}_R\mathrm{Mod}^{op} \to {}_S\mathrm{Mod}$ and let $n \in \mathbb{Z}$. We have an induced functor $K(F)$ between homotopy categories, but in this case $K(F)\colon K({}_R\mathrm{Mod})^{op} \to K({}_S\mathrm{Mod})$ is a contravariant functor as well, and thus it transforms chain complexes into cochain complexes[37].
For $n \in \mathbb{Z}$ we can then define $\mathbb{L}_n F$ as the composition of additive functors

$$ {}_R\mathrm{Mod}^{op} \xrightarrow{\mathfrak{I}^{op}} K({}_R\mathrm{Mod})^{op} \xrightarrow{K(F)} K({}_S\mathrm{Mod}) \xrightarrow{H_n^K} {}_S\mathrm{Mod} $$

Why do we use now injective resolutions instead of projective ones (as in the covariant case) to define a left derived functor? Our convention should be that, after resolving a left R-module M and applying F to the resolution, we obtain a chain complex concentrated in non-negative degrees (left half-linea of degrees). This is exactly what happens with this definition: given M, we first obtain a cochain complex

$$ \dots 0 \longrightarrow \mathfrak{I}(M)^0 \longrightarrow \mathfrak{I}(M)^1 \longrightarrow \mathfrak{I}(M)^2 \longrightarrow \dots, $$

and then, applying F, we transform the previous into a chain complex

$$ \dots \longrightarrow F(\mathfrak{I}(M)^2) \longrightarrow F(\mathfrak{I}(M)^1) \longrightarrow F(\mathfrak{I}(M)^0) \longrightarrow 0 \dots. $$

Notice that the last chain complex, i.e. the one obtained *after* applying F, is concentrated in non-negative degrees. It follows that $\mathbb{L}_n F$ is the zero functor for $n < 0$, exactly as in the covariant case.
In a similar way, for F contravariant and $n \in \mathbb{Z}$, we define $\mathbb{R}_n F$ as the composition

$$ {}_R\mathrm{Mod}^{op} \xrightarrow{\mathfrak{P}^{op}} K({}_R\mathrm{Mod})^{op} \xrightarrow{K(F)} K({}_S\mathrm{Mod}) \xrightarrow{H_n^K} {}_S\mathrm{Mod}. $$

As in the covariant case, we have that $\mathbb{R}_n F$ is the zero functor for $n > 0$; it is then common to write $\mathbb{R}^n F$ for $\mathbb{R}_{-n} F$, in order to use positive indices when describing interesting things.

Example 11.3. Let M be a projective R-module. As in Example 10.20, $\theta_0(M)_\bullet$ is a projective resolution of M, and we may assume $\mathfrak{P}(M)_\bullet = \theta_0(M)_\bullet$. Applying F we obtain $\theta_0(F(M))_\bullet$. Applying $H_n^K(-)$ we obtain either $F(M)$ (if $n = 0$), or 0 (if $n \neq 0$). That is, $\mathbb{R}_n F(M) = 0$ for $n \neq 0$, and $\mathbb{R}_0 F(M) \cong M$.

[37]If you want to avoid the use of cochain complexes, here the crucial remark we will need is that a contravariant functor transforms chain complexes concentrated in non-negative degrees into chain complexes concentrated in non-positive degrees

Example 11.4. Let F be an exact contravariant functor: F is assumed to send exact sequences to exact sequence, where directions of arrows are changed: more precisely, an exact contravariant functor from ${}_R\mathrm{Mod}$ to ${}_S\mathrm{Mod}$ is an exact covariant functor from the abelian category ${}_R\mathrm{Mod}^{op}$ to the category ${}_S\mathrm{Mod}$.
A contravariant exact functor has the following property: if $C_\bullet$ is a chain complex, then $F(C_\bullet)$ is also a chain complex (so far F additive would suffice), and moreover $H_n(F(C_\bullet))$ is canonically isomorphic to $F(H_{-n}(C_\bullet))$. In fact F also sends kernels to cokernels, images to images, cokernels to kernels (in the categorical sense). Now we have, for all $n \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{L}_n F(M) = H_n^K(K(F)(\mathfrak{I}(M)_\bullet)) &\cong H_n(F(\mathfrak{I}(M)_\bullet)) \\ &\cong F(H_{-n}(\mathfrak{I}(M)_\bullet)) \cong F(H_{-n}(\theta_0(M)_\bullet)),\end{aligned}$$

and the last term is isomorphic to $F(M)$ for $n = 0$, and vanishes for $n \neq 0$.

Example 11.5. Suppose that F is a *left exact* contravariant functor from ${}_R\mathrm{Mod}$ to ${}_S\mathrm{Mod}$. Then for all $M \in {}_R\mathrm{Mod}$ the sequence

$$\mathfrak{P}(M)_1 \overset{d_1^{\mathfrak{P}(M)}}{\to} \mathfrak{P}(M)_0 \overset{\epsilon_M^{\mathfrak{P}}}{\to} M \to 0$$

is exact. Applying F we obtain that the sequence

$$0 \to F(M) \overset{F(\epsilon_M^{\mathfrak{P}})}{\to} F(\mathfrak{P}(M)_0) \overset{F(d_1^{\mathfrak{P}(M)})}{\to} F(\mathfrak{P}(M)_1)$$

is also exact. Removing $F(M)$, we obtain the first chunk of the cochain complex $F(\mathfrak{P}(M)_\bullet) = K(F)(\mathfrak{P}(M)_\bullet)$, reading

$$0 \to F(\mathfrak{P}(M)_0) \overset{F(d_1^{\mathfrak{P}(M)})}{\to} F(\mathfrak{P}(M)_1)$$

and we can compute $\mathbb{R}_0 F(M) \cong \ker(F(d_1^{\mathfrak{P}(M)})) \cong F(M)$.
Nice! The functor $\mathbb{R}_0 F$ just coincides with the old functor F! The functors $\mathbb{R}_n F$ for $n < 0$ will then measure how much F fails from being right exact. It is common to use non-negative indices and thus to say that $\mathbb{R}^0 F$ is naturally isomorphic to F, and that the functors $R^n F$ measure how much F fails from being right exact.

The following example is an adaptation of Example 11.1

Example 11.6. Consider the functor $F = \mathrm{Hom}_{\mathbb{Z}}(-;\mathbb{Z}/3)$ as a contravariant, additive functor from ${}_{\mathbb{Z}}\mathrm{Mod}$ to ${}_{\mathbb{Z}}\mathrm{Mod}$, and consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}$, $M_2 = \mathbb{Z}/4$ and $M_3 = \mathbb{Z}/6$. We want to compute, for all $n \in \mathbb{Z}$ (but in fact, for $n \geq 0$) and for $i = 1, 2, 3$, the $\mathbb{Z}$-modules $\mathbb{R}^n F(M_i)$.
A projective resolution of M_1 is given by $\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \dots$, with $\mathbb{Z}$ in degree 0. Applying F we obtain the chain complex $\dots 0 \to \mathbb{Z}/3 \to 0 \to 0 \to \dots$, with $\mathbb{Z}/3$ in degree 0. All homology groups of the latter complex vanish, except H_0, which is isomorphic to $\mathbb{Z}/3$. Hence $\mathbb{R}^0 F(\mathbb{Z}) = \mathbb{R}_0 F(\mathbb{Z}) \cong \mathbb{Z}/3$, and $\mathbb{R}^n F(\mathbb{Z}) \cong 0$ for all $n \neq 0$.
A projective resolution of M_2 is given by $\cdots \to 0 \to \mathbb{Z} \overset{\cdot 4}{\to} \mathbb{Z} \to 0 \dots$, with $\mathbb{Z}$ in degrees 0 and 1. Applying F we obtain the chain complex $\dots 0 \to \mathbb{Z}/3 \overset{\cdot 4}{\to} \mathbb{Z}/3 \to 0 \to \dots$, with $\mathbb{Z}/3$ in degrees 0 and -1. Multiplication by 4 induces an isomorphism $\mathbb{Z}/3 \overset{\cong}{\to} \mathbb{Z}/3$, and therefore all homology groups of the latter chain complex vanish. Hence $\mathbb{R}^n F(\mathbb{Z}/4) \cong 0$ for all $n \in \mathbb{Z}$.

A projective resolution of M_2 is given by $\dots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z} \to 0 \dots$, with $\mathbb{Z}$ in degrees 0 and 1. Applying F we obtain the chain complex $\dots 0 \to \mathbb{Z}/3 \xrightarrow{\cdot 6} \mathbb{Z}/3 \to 0 \to \dots$, with $\mathbb{Z}/3$ in degrees 0 and -1. Multiplication by 6 induces the zero map $\mathbb{Z}/3 \to \mathbb{Z}/3$. Since all differentials in the last chain complex are the zero map, all homology groups coincide with the corresponding chain groups. Hence $\mathbb{R}^n F(\mathbb{Z}/6) = \mathbb{R}_{-n}(\mathbb{Z}/6) \cong 0$ for all $n \neq 0, 1$, and both $\mathbb{R}^1 F(\mathbb{Z}/6)$ and $\mathbb{R}^0 F(\mathbb{Z}/6)$ are isomorphic to $\mathbb{Z}/3$.

We note that the fact that $F = \mathrm{Hom}_{\mathbb{Z}}(-; \mathbb{Z}/3)$ is a left exact, contravariant functor immediately predicts that, in all three cases, we have isomorphisms $\mathbb{R}^0 F(M_i) \cong F(M_i)$.

11.2. **The horseshoe lemma and the LES of derived functors.** If $M' \to M \to M''$ is a SES in ${}_R\mathrm{Mod}$ and $F \colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ is a (for simplicity) covariant, additive functor, then in general $F(M') \to F(M) \to F(M'')$ is not a SES. If however F is right exact, then two things simultaneously happen:

- by definition, $F(M') \to F(M) \to F(M'') \to 0$ is exact;
- by Example 10.22, the functor F coincides with the left derived functor $\mathbb{L}_0 F$.

We can then apply $\mathbb{L}_0 F$ to the SES $M' \to M \to M''$ and obtain an exact sequence $\mathbb{L}_0 F(M') \to \mathbb{L}_0 F(M) \to \mathbb{L}_0 F(M'') \to 0$. In general the map $\mathbb{L}_0 F(M') \to \mathbb{L}_0 F(M)$ is not injective; however there is a natural, surjective map onto the kernel of $\mathbb{L}_0 F(M') \to \mathbb{L}_0 F(M)$ with source... $\mathbb{L}_1 F(M'')$! We will prove this and a more general statement in this subsection. The upshot will be that when F is right exact, it is convenient to consider F as $\mathbb{L}_0 F$, and then to consider $\mathbb{L}_0 F$ as one of the functors $\mathbb{L}_n F$ for varying n: each single functor $\mathbb{L}_n F$ is in general not exact, but all together these functors give rise to a mild form of exactness, which is anyway useful for applications[38].

Lemma 11.7. *Let $M' \xrightarrow{i} M \xrightarrow{p} M''$ be a SES in ${}_R\mathrm{Mod}$ (or in an abelian category with enough projectives), and fix projective resolutions $(P'_\bullet, \epsilon')$ of M' and $(P''_\bullet, \epsilon'')$ of M''. Then there exists a projective resolution $(P_\bullet, \epsilon)$ of M fitting into a diagram of left R-modules as follows, in which each row is exact, each column is a SES, and each square commutes:*

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & P'_3 & \xrightarrow{d_3^{P'}} & P'_2 & \xrightarrow{d_2^{P'}} & P'_1 & \xrightarrow{d_1^{P'}} & P'_0 & \xrightarrow{\epsilon'} & M' & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \tilde{i}_3} & & \downarrow{\scriptstyle \tilde{i}_2} & & \downarrow{\scriptstyle \tilde{i}_1} & & \downarrow{\scriptstyle \tilde{i}_0} & & \downarrow{\scriptstyle i} & & \downarrow & & \\
\dots & \longrightarrow & P_3 & \xrightarrow{d_3^{P}} & P_2 & \xrightarrow{d_2^{P}} & P_1 & \xrightarrow{d_1^{P}} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \tilde{p}_3} & & \downarrow{\scriptstyle \tilde{p}_2} & & \downarrow{\scriptstyle \tilde{p}_1} & & \downarrow{\scriptstyle \tilde{p}_0} & & \downarrow{\scriptstyle p} & & \downarrow & & \\
\dots & \longrightarrow & P''_3 & \xrightarrow{d_3^{P''}} & P''_2 & \xrightarrow{d_2^{P''}} & P''_1 & \xrightarrow{d_1^{P''}} & P''_0 & \xrightarrow{\epsilon''} & M'' & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

We will see the proof of Lemma 11.7 in the next lecture.

The chain maps $\tilde{i}$ and $\tilde{p}$ make $P'_\bullet \xrightarrow{\tilde{i}} P_\bullet \xrightarrow{\tilde{p}} P''_\bullet$ into a SES of chain complexes; most importantly, for all $i \in \mathbb{Z}$, we have a SES of left R-modules $P'_i \to P_i \to P''_i$; since P''_i is a projective left R-module, the SES $P'_i \to P_i \to P''_i$ is split [39].

[38] and is anyway the best available on the market, so we should be content with it!

[39] Here, as usual, we consider projective resolutions as deleted exact sequences, so that, for instance, $P'_\bullet$ is 0 and not M' in degree -1; thus this statement is obvious for $i < 0$

We can assume that our functor $\mathfrak{P}\colon {}_R\mathrm{Mod} \to K({}_R\mathrm{Mod})$ sends M', M, M'' precisely to $P'_\bullet, P_\bullet, P''_\bullet$. We can then apply $K(F)$, where F is our additive functor fixed at the beginning of the subsection, and land in $K({}_S\mathrm{Mod})$. And now the miracle occurs: in spite of the fact that $K(F)$ is additive but possibly not exact (and in spite of the fact that the categories $K({}_R\mathrm{Mod})$ and $K({}_S\mathrm{Mod})$ are not even abelian categories, but only additive categories), the chain complexes $F(P'_\bullet)$, $F(P_\bullet)$ and $F(P''_\bullet)$ fit into a SES in the abelian category ${}_S\mathrm{Ch}$, namely $F(P'_\bullet) \overset{F(\tilde{i})}{\to} F(P_\bullet) \overset{F(\tilde{p})}{\to} F(P''_\bullet)$: to check this, we have to check that we have a SES of left S-modules in each degree, and this is true because, since $P'_i \to P_i \to P''_i$ is a *split* SES in ${}_R\mathrm{Mod}$, we also have $F(P'_i) \to F(P_i) \to F(P''_i)$ is a (split) SES in ${}_S\mathrm{Mod}$.
We can now apply the functors H^K_n with target ${}_S\mathrm{Mod}$, and obtain all evaluations of left derived functors $\mathbb{L}_nF$ at M', M and M''. But now the snake lemma applies and tells us that the left S-modules $\mathbb{L}_nF(M')$, $\mathbb{L}_nF(M)$ and $\mathbb{L}_nF(M'')$ fit all together into a long exact sequence. This creates an unexpected connection between different left derived functors (but in fact already the snake lemma creates a connection between homology groups in different degrees).

We remark that $P'_\bullet \overset{\tilde{i}}{\to} P_\bullet \overset{\tilde{p}}{\to} P''_\bullet$ from the horseshoe lemma is in general *not* a *split* SES in the abelian category ${}_R\mathrm{Ch}$: even if there is no difficulty in finding degreewise a section $P''_i \to P_i$, it is not always possible to choose these sections so that they give a *chain* map $P''_\bullet \to P_\bullet$.
We summarise the previous discussion in the following two theorems, in which we also put the generalisation to contravariant functors and/or left exact functors.

Theorem 11.8. *Let $M' \overset{i}{\to} M \overset{p}{\to} M''$ be a SES in ${}_R\mathrm{Mod}$, and let F be a covariant additive functor from ${}_R\mathrm{Mod}$ to ${}_S\mathrm{Mod}$; then there are natural S-linear maps $\partial^{\mathbb{L}}_n\colon \mathbb{L}_nF(M'') \to \mathbb{L}_{n-1}F(M')$ and $\partial^n_{\mathbb{R}}\colon \mathbb{R}^nF(M'') \to \mathbb{R}^{n+1}F(M')$ giving rise to long exact sequences of left S-modules*

$$\dots \xrightarrow{\partial^{\mathbb{L}}_3} \mathbb{L}_2F(M') \xrightarrow{\mathbb{L}_2(i)} \mathbb{L}_2F(M) \xrightarrow{\mathbb{L}_2(p)} \mathbb{L}_2F(M'') \xrightarrow{\partial^{\mathbb{L}}_2}$$
$$\mathbb{L}_1F(M') \xrightarrow{\mathbb{L}_1(i)} \mathbb{L}_1F(M) \xrightarrow{\mathbb{L}_1(p)} \mathbb{L}_1F(M'') \xrightarrow{\partial^{\mathbb{L}}_1}$$
$$\mathbb{L}_0F(M') \xrightarrow{\mathbb{L}_0(i)} \mathbb{L}_0F(M) \xrightarrow{\mathbb{L}_0(p)} \mathbb{L}_0F(M'') \to 0;$$

$$0 \to \mathbb{R}^0F(M') \xrightarrow{\mathbb{R}^0(i)} \mathbb{R}^0F(M) \xrightarrow{\mathbb{R}^0(p)} \mathbb{R}^0F(M'') \xrightarrow{\partial^0_{\mathbb{R}}}$$
$$\mathbb{R}^1F(M') \xrightarrow{\mathbb{R}^1(i)} \mathbb{R}^1F(M) \xrightarrow{\mathbb{R}^1(p)} \mathbb{R}^1F(M'') \xrightarrow{\partial^1_{\mathbb{R}}}$$
$$\mathbb{R}^2F(M') \xrightarrow{\mathbb{R}^2(i)} \mathbb{R}^2F(M) \xrightarrow{\mathbb{R}^2(p)} \mathbb{R}^2F(M'') \xrightarrow{\partial^2_{\mathbb{R}}} \dots$$

If F is right exact, the first sequence ends with $\mathbb{L}_1F(M'') \to F(M') \to F(M) \to F(M'') \to 0$; If F is left exact, the second sequence begins with $0 \to F(M') \to F(M) \to F(M'') \to \mathbb{R}^1F(M')$.

Theorem 11.9. *Let $M' \xrightarrow{i} M \xrightarrow{p} M''$ be a SES in ${}_R\mathrm{Mod}$, and let F be a contravariant additive functor from ${}_R\mathrm{Mod}$ to ${}_S\mathrm{Mod}$; then there are natural S-linear maps $\partial_n^{\mathbb{L}} \colon \mathbb{L}_nF(M') \to \mathbb{L}_{n-1}F(M'')$ and $\partial_{\mathbb{R}}^n \colon \mathbb{R}^nF(M') \to \mathbb{R}^{n+1}F(M'')$ giving rise to long exact sequences of left S-modules*

$$\dots \xrightarrow{\partial_3^{\mathbb{L}}} \mathbb{L}_2F(M'') \xrightarrow{\mathbb{L}_2(p)} \mathbb{L}_2F(M) \xrightarrow{\mathbb{L}_2(i)} \mathbb{L}_2F(M') \xrightarrow{\partial_2^{\mathbb{L}}} \mathbb{L}_1F(M'') \xrightarrow{\mathbb{L}_1(p)} \mathbb{L}_1F(M) \xrightarrow{\mathbb{L}_1(i)} \mathbb{L}_1F(M') \xrightarrow{\partial_1^{\mathbb{L}}} \mathbb{L}_0F(M'') \xrightarrow{\mathbb{L}_0(p)} \mathbb{L}_0F(M) \xrightarrow{\mathbb{L}_0(i)} \mathbb{L}_0F(M') \to 0;$$

$$0 \to \mathbb{R}^0F(M'') \xrightarrow{\mathbb{R}^0(p)} \mathbb{R}^0F(M) \xrightarrow{\mathbb{R}^0(i)} \mathbb{R}^0F(M') \xrightarrow{\partial_{\mathbb{R}}^0} \mathbb{R}^1F(M'') \xrightarrow{\mathbb{R}^1(p)} \mathbb{R}^1F(M) \xrightarrow{\mathbb{R}^1(i)} \mathbb{R}^1F(M') \xrightarrow{\partial_{\mathbb{R}}^1} \mathbb{R}^2F(M'') \xrightarrow{\mathbb{R}^2(p)} \mathbb{R}^2F(M) \xrightarrow{\mathbb{R}^2(i)} \mathbb{R}^2F(M') \xrightarrow{\partial_{\mathbb{R}}^2} \dots$$

If F is right exact, the first sequence ends with $\mathbb{L}_1F(M') \to F(M'') \to F(M) \to F(M') \to 0$; If F is left exact, the second sequence begins with $0 \to F(M'') \to F(M) \to F(M') \to \mathbb{R}^1F(M'')$.

Note that Theorem 11.9 reduces to Theorem 11.8 after replacing the SES $M' \xrightarrow{i} M \xrightarrow{p} M''$ in ${}_R\mathrm{Mod}$ by the SES $M'' \xrightarrow{i^{op}} M \xrightarrow{p^{op}} M$ in the abelian category ${}_R\mathrm{Mod}^{op}$.

Exercise 11.10. Formulate and prove the version of the horseshoe lemma which uses injective resolutions instead of projective resolutions, getting a horseshoe oriented as "$\sqsubset$". This version of the lemma is needed in proving the parts of Theorems 11.8 and 11.9 dealing with right derived functors.

11.3. **Tor.** Let R be a ring. For a right R-module M we saw that the functor $M \otimes_R - \colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is right exact.

Definition 11.11. For $n \in \mathbb{Z}$ we define $\mathrm{Tor}_n^R(M, -) \colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ to be the n^{th} left derived functor $\mathbb{L}_n(M \otimes_R -)$.

The name "Tor" is an abbreviation of the word "torsion", and we will see next time the explanation of this name.

Example 11.12. Let $\mathbb{F}$ be a field, let $R = \mathbb{F}[x, y]$, and consider $M = \mathbb{F}[x, y]/(x)$ and $N = \mathbb{F}[x, y]/(y)$ as R-modules. We want to compute $\mathrm{Tor}_n^R(M, N)$ for all n.

A projective resolution of N over R is given by $\dots 0 \to \mathbb{F}[x,y] \xrightarrow{\cdot y} \mathbb{F}[x,y] \to 0 \dots$. Applying the functor $M \otimes_R -$, we obtain the chain complex $\dots 0 \to \mathbb{F}[x,y]/x \xrightarrow{\cdot y} \mathbb{F}[x,y]/x \to 0 \dots$. Computing homology, we obtain that $\mathrm{Tor}_0^R(M,N) = \mathbb{F}[x,y]/(x,y) \cong M \otimes_R N$, whereas $\mathrm{Tor}_1^R(M,N)$ vanishes, since the map $\cdot y$ is injective from $\mathbb{F}[x,y]/(x)$ to itself.

One can also consider a left R-module M and the associated right exact functor $- \otimes_R M \colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$.

Definition 11.13. For $n \in \mathbb{Z}$ we define $\check{\mathrm{Tor}}_n^R(-,M) \colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$ to be the n^{th} left derived functor $\mathbb{L}_n(- \otimes_R M)$.

The notation "$\check{\mathrm{Tor}}$" is non-standard, and we will soon get rid of it: we will in fact prove the following theorem.

Theorem 11.14. *For all $M \in \mathrm{Mod}_R$ and $N \in {}_R\mathrm{Mod}$ and for all $n \in \mathbb{Z}$ there is an isomorphism of abelian groups* $\mathrm{Tor}_n^R(M,N) \cong \check{\mathrm{Tor}}_n^R(M,N)$ which is natural both in M and in N.

11.4. **Ext.** Let R be a ring. For a left R-module N we saw that the functor $\mathrm{Hom}_R(-,N) \colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is contravariant and left exact.

Definition 11.15. For $n \in \mathbb{Z}$ we define $\mathrm{Ext}_R^n(-,N) \colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$ to be the right derived functor $\mathbb{R}^n(\mathrm{Hom}_R(-,N))$.

The name "Ext" is an abbreviation of the word "extension", and we will see next time the explanation of this name.

Example 11.16. Let $\mathbb{F}$ be a field, let $R = \mathbb{F}[x,y]$, and consider $M = \mathbb{F}[x,y]/(x)$ and $N = \mathbb{F}[x,y]/(y)$ as R-modules. We want to compute $\mathrm{Ext}_R^n(M,N)$ for all n.
A projective resolution of N over R is given by $\dots 0 \to \mathbb{F}[x,y] \xrightarrow{\cdot y} \mathbb{F}[x,y] \to 0 \dots$, with non-vanishing terms in degrees 0 and 1. Applying the contravariant functor $\mathrm{Hom}_R(-,N)$, we obtain the chain complex $\dots 0 \to \mathbb{F}[x,y]/x \xrightarrow{\cdot y} \mathbb{F}[x,y]/x \to 0 \dots$, with non-vanishing terms in degrees 0 and -1. Hence we already expect only $\mathrm{Ext}_R^0(M,N)$ and $\mathrm{Ext}_R^1(M,N) = \mathbb{L}_{-1}(\mathrm{Hom}_R(-,N))(M)$ possibly not to vanish.
Computing homology, we obtain that $\mathrm{Ext}_R^1(M,N) = \mathbb{F}[x,y]/(x,y)$, whereas $\mathrm{Ext}_R^0(M,N)$ vanishes, since the map $\cdot y$ is injective from $\mathbb{F}[x,y]/(x)$ to itself. Note that $\mathrm{Ext}_R^0(M,N) = 0 = \mathrm{Hom}_R(M,N)$, as it should be.

One can also consider a left R-module M and the left exact, covariant functor $\mathrm{Hom}_R(M,-) \colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$.

Definition 11.17. For $n \in \mathbb{Z}$ we define $\check{\mathrm{Ext}}_R^n(M,-) \colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ to be the functor $\mathbb{R}^n(\mathrm{Hom}_R(M,-))$.

The notation "$\check{\mathrm{Ext}}$" is non-standard, and we will soon get rid of it: we will in fact prove the following theorem.

Theorem 11.18. *For all $M,N \in {}_R\mathrm{Mod}$ and for all $n \in \mathbb{Z}$ there is an isomorphism of abelian groups* $\mathrm{Ext}_R^n(M,N) \cong \check{\mathrm{Ext}}_R^n(M,N)$ which is natural both in M and in N.

The meaning of the emphasized naturality statements in Theorems 11.14 and 11.18 will be also explained in the future.

12. Proof of horseshoe lemma, examples of Tor and Ext

12.1. **Proof of Lemma 11.7.** First, note that if the statement holds, then for each $n \geq 0$ we have a SES $P'_n \to P_n \to P''_n$ which must be split since P''_n is projective. It is therefore no harm defining $P_n := P'_n \oplus P''_n$, and setting $\tilde{i}_n \colon P'_n \to P'_n \oplus P''_n$ and $\tilde{p}_n \colon P'_n \oplus P''_n \to P''_n$ to be the natural inclusion and projection.
We work recursively, and start from the diagram

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & P'_3 & \xrightarrow{d_3^{P'}} & P'_2 & \xrightarrow{d_2^{P'}} & P'_1 & \xrightarrow{d_1^{P'}} & P'_0 & \xrightarrow{\epsilon'} & M' & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & & & & & \downarrow{\scriptstyle \tilde{i}_0} & & \downarrow{\scriptstyle i} & & \downarrow & & \\
 & & & & & & & & P'_0 \oplus P''_0 & \overset{\epsilon}{\dashrightarrow} & M & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & & & & & \downarrow{\scriptstyle \tilde{p}_0} & & \downarrow{\scriptstyle p} & & \downarrow & & \\
\dots & \longrightarrow & P''_3 & \xrightarrow{d_3^{P''}} & P''_2 & \xrightarrow{d_2^{P''}} & P''_1 & \xrightarrow{d_1^{P''}} & P''_0 & \xrightarrow{\epsilon''} & M'' & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Our first aim is to define a surjective R-linear map $\epsilon \colon P_0 = P'_0 \oplus P''_0 \to M$ making the previous diagram commute. We let $\epsilon|_{P'_0} = i \circ \epsilon'$, which is in fact a forced assignment if we want the top square to commute. We then use that P''_0 is projective and lift the map $\epsilon'' \colon P''_0 \to M''$ along the surjective map $p \colon M \to M''$: we declare the resulting map to be $\epsilon|_{P''_0}$. By this assignment we obtain a commutative diagram whose columns are SESs

$$\begin{array}{ccc}
P'_0 & \xrightarrow{\epsilon'} & M' \\
\downarrow{\scriptstyle \tilde{i}_0} & & \downarrow{\scriptstyle i} \\
P'_0 \oplus P''_0 & \xrightarrow{\epsilon} & M \\
\downarrow{\scriptstyle \tilde{p}_0} & & \downarrow{\scriptstyle p} \\
P''_0 & \xrightarrow{\epsilon''} & M''
\end{array}$$

And now we use a nice trick: we add zeroes on left and right in order to obtain a SES of chain complexes[40]

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P'_0 & \xrightarrow{\epsilon'} & M' & \longrightarrow & 0\dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{i}_0} & & \downarrow{\scriptstyle i} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P'_0 \oplus P''_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0\dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{p}_0} & & \downarrow{\scriptstyle p} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P''_0 & \xrightarrow{\epsilon''} & M'' & \longrightarrow & 0\dots
\end{array}$$

The snake lemma provides a long exact sequence of homology groups. In particular we obtain an exact sequence

$$0 \longrightarrow \ker(\epsilon') \longrightarrow \ker(\epsilon) \longrightarrow \ker(\epsilon'') \longrightarrow \operatorname{coker}(\epsilon') \longrightarrow \operatorname{coker}(\epsilon) \longrightarrow \operatorname{coker}(\epsilon'') \longrightarrow 0,$$

[40]These chain complexes have very little to do with the projective resolutions we starter with, but still more than nothing

by neglecting all homology groups that are utterly zero. We now recall that ϵ' and ϵ'' are surjective, hence $\mathrm{coker}(\epsilon') = \mathrm{coker}(\epsilon'') = 0$; it follows by exactness that also $\mathrm{coker}(\epsilon) = 0$, i.e. ϵ is automatically surjective (this is one of the things we wanted to check). Moreover the above LES reduces to a SES $\ker(\epsilon') \to \ker(\epsilon) \to \ker(\epsilon'')$.
The next step is to find a map d_1^P making the following diagram commute

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & P_3' & \xrightarrow{d_3^{P'}} & P_2' & \xrightarrow{d_2^{P'}} & P_1' & \xrightarrow{d_1^{P'}} & P_0' & \xrightarrow{\epsilon'} & M' & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & & & \downarrow{\scriptstyle \tilde{i}_1} & & \downarrow{\scriptstyle \tilde{i}_0} & & \downarrow{\scriptstyle i} & & \downarrow & & \\
 & & & & & & P_1' \oplus P_1'' & \overset{d_1^P}{\dashrightarrow} & P_0' \oplus P_0'' & \xrightarrow{\epsilon} & M & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & & & \downarrow{\scriptstyle \tilde{p}_0} & & \downarrow{\scriptstyle \tilde{p}_0} & & \downarrow{\scriptstyle p} & & \downarrow & & \\
\dots & \longrightarrow & P_3'' & \xrightarrow{d_3^{P''}} & P_2'' & \xrightarrow{d_2^{P''}} & P_1'' & \xrightarrow{d_1^{P''}} & P_0'' & \xrightarrow{\epsilon''} & M'' & \longrightarrow & 0 & \longrightarrow & \dots,
\end{array}$$

with the additional requirement that the image of d_1^P is $\ker(\epsilon)$. We define $d_1^P|_{P_1'} = d_1^{P'} \circ \tilde{i}_0$ and we lift $d_1^{P''}: P_1'' \to \ker(\epsilon'')$ along the surjective map $\ker(\epsilon) \to \ker(\epsilon'')$ to obtain a map $P_1'' \to \ker(\epsilon)$, that we declare to be $d_1^P|_{P_1''}$. We then consider the SES of chain complexes

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P_1' & \xrightarrow{d_1^{P'}} & \ker(\epsilon') & \longrightarrow & 0\dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{i}_1} & & \downarrow{\scriptstyle \tilde{i}_0} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P_1' \oplus P_1'' & \xrightarrow{d_1^{P}} & \ker(\epsilon) & \longrightarrow & 0\dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{p}_1} & & \downarrow{\scriptstyle \tilde{p}_0} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P_1'' & \xrightarrow{d_1^{P''}} & \ker(\epsilon'') & \longrightarrow & 0\dots
\end{array}$$

and from the corresponding LES of homology groups we deduce, similarly as above, that $d_P^1 \to \ker(\epsilon)$ is surjective, and that there is a SES $\ker(d_1^{P'}) \to \ker(d_1^P) \to \ker(d_1^{P''})$. We continue constructing all differentials d_n^P in this way: at each step we guarantee that all squares commute, and each d_n^P is constructed so as to have image contained in (and in fact coinciding with) $\ker(d_{n-1}^P)$.
Note that the first diagram in the proof of Lemma 11.7 has a shape "$\sqsupset$" that reminds the shape of a horseshoe.

12.2. **Naturality of Tor and Ext in the fixed module.** Let R be a ring, let M and M' be right R-modules, and let $f: M \to M'$ be an R-linear map. We want use f to compare, for all $n \in \mathbb{Z}$, the functors $\mathrm{Tor}_n^R(M, -)$ and $\mathrm{Tor}_n^R(M', -)$: both are functors ${}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$.
Let N be a left R-module. To compute $\mathrm{Tor}_n^R(M, N)$ and $\mathrm{Tor}_n^R(M', N)$, we first need (in both cases) to replace $N \in {}_R\mathrm{Mod}$ by the projective resolution $\mathfrak{P}(N)_\bullet$, which is a chain complex[41].
The next step is to apply the functors $M \otimes_R -$ and $M' \otimes_R -$ to the chain complex $\mathfrak{P}(N)_\bullet$, obtaining chain complexes of abelian groups $M \otimes_R \mathfrak{P}(N)_\bullet$ and $M' \otimes_R \mathfrak{P}(N)_\bullet$. The complex $M \otimes_R \mathfrak{P}(N)_\bullet$ has $M \otimes_R \mathfrak{P}(N)_i$ in degree i, and the differentials are given by tensoring the differentials of $\mathfrak{P}(N)_\bullet$ with the identity of M; similarly

[41]Technically, we consider $\mathfrak{P}(N)_\bullet$ as an object in the homotopy category $K({}_R\mathrm{Mod})$

for the complex $M' \otimes_R \mathfrak{P}(N)_\bullet$. We can now use the R-linear map $f\colon M \to M'$ to define a chain map $f \otimes_R \mathrm{Id}_{\mathfrak{P}(N)_\bullet}$: the chain map is given schematically by

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\mathrm{Id}_M\otimes_R d_4^{\mathfrak{P}}} & M\otimes_R \mathfrak{P}(N)_3 & \xrightarrow{\mathrm{Id}_M\otimes_R d_3^{\mathfrak{P}}} & M\otimes_R \mathfrak{P}(N)_2 & \xrightarrow{\mathrm{Id}_M\otimes_R d_2^{\mathfrak{P}}} & M\otimes_R \mathfrak{P}(N)_1 & \xrightarrow{\mathrm{Id}_M\otimes_R d_1^{\mathfrak{P}}} & M\otimes_R \mathfrak{P}(N)_0 & \xrightarrow{\mathrm{Id}_M\otimes_R d_0^{\mathfrak{P}}} & 0 \\
 & & \downarrow{\scriptstyle f\otimes_R \mathrm{Id}_{\mathfrak{P}(N)_3}} & & \downarrow{\scriptstyle f\otimes_R \mathrm{Id}_{\mathfrak{P}(N)_2}} & & \downarrow{\scriptstyle f\otimes_R \mathrm{Id}_{\mathfrak{P}(N)_1}} & & \downarrow{\scriptstyle f\otimes_R \mathrm{Id}_{\mathfrak{P}(N)_0}} & & \\
\dots & \xrightarrow{\mathrm{Id}_{M'}\otimes_R d_4^{\mathfrak{P}}} & M'\otimes_R \mathfrak{P}(N)_3 & \xrightarrow{\mathrm{Id}_{M'}\otimes_R d_3^{\mathfrak{P}}} & M'\otimes_R \mathfrak{P}(N)_2 & \xrightarrow{\mathrm{Id}_{M'}\otimes_R d_2^{\mathfrak{P}}} & M'\otimes_R \mathfrak{P}(N)_1 & \xrightarrow{\mathrm{Id}_{M'}\otimes_R d_1^{\mathfrak{P}}} & M'\otimes_R \mathfrak{P}(N)_0 & \xrightarrow{\mathrm{Id}_{M'}\otimes_R d_0^{\mathfrak{P}}} & 0.
\end{array}$$

We only consider the chain homotopy class of $f \otimes_R \mathrm{Id}_{\mathfrak{P}(N)_\bullet}$. We can now compute n^{th} homology, obtaining two abelian groups $\mathrm{Tor}_n^R(M,N)$ and $\mathrm{Tor}_n^R(M',N)$; but we also have a $\mathbb{Z}$-linear map between them, which we can denote by

$$\mathrm{Tor}_n^R(f,N)\colon \mathrm{Tor}_n^R(M,N) \to \mathrm{Tor}_n^R(M',N);$$

this is the map induced in homology by the chain map $f \otimes_R \mathrm{Id}_{\mathfrak{P}(N)_\bullet}$. For varying N, the maps $\mathrm{Tor}_n^R(f,N)$ assemble into a natural transformation

$$\mathrm{Tor}_n^R(f,-)\colon \mathrm{Tor}_n^R(M,-) \Rightarrow \mathrm{Tor}_n^R(M',-).$$

What we saw above is a particular example of a more general principle. Suppose that F and F' are additive functors ${}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$[42], and suppose that ϕ is a natural transformation $\phi\colon F \Rightarrow F'$. Then we can define a natural transformation $\mathbb{L}_n\phi\colon \mathbb{L}_nF \Rightarrow \mathbb{L}_nF'$ as follows: given $N \in {}_R\mathrm{Mod}$ we apply F and F' to the chain complex $\mathfrak{P}(N)_\bullet$, and we use ϕ to define a chain map as follows:

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{F(d_4^{\mathfrak{P}})} & F(\mathfrak{P}(N)_3) & \xrightarrow{F(d_3^{\mathfrak{P}})} & F(\mathfrak{P}(N)_2) & \xrightarrow{F(d_2^{\mathfrak{P}})} & F(\mathfrak{P}(N)_1) & \xrightarrow{F(d_1^{\mathfrak{P}})} & F(\mathfrak{P}(N)_0) & \xrightarrow{F(d_0^{\mathfrak{P}})} & 0 \\
 & & \downarrow{\scriptstyle \phi_{\mathfrak{P}(N)_3}} & & \downarrow{\scriptstyle \phi_{\mathfrak{P}(N)_2}} & & \downarrow{\scriptstyle \phi_{\mathfrak{P}(N)_1}} & & \downarrow{\scriptstyle \phi_{\mathfrak{P}(N)_0}} & & \\
\dots & \xrightarrow{F(d_4^{\mathfrak{P}})} & F'(\mathfrak{P}(N)_3) & \xrightarrow{F'(d_3^{\mathfrak{P}})} & F'(\mathfrak{P}(N)_2) & \xrightarrow{F'(d_2^{\mathfrak{P}})} & F'(\mathfrak{P}(N)_1) & \xrightarrow{F'(d_1^{\mathfrak{P}})} & F'(\mathfrak{P}(N)_0) & \xrightarrow{F'(d_0^{\mathfrak{P}})} & 0.
\end{array}$$

We only consider the homotopy class of the chain map $\phi_{\mathfrak{P}(N)_\bullet}$. Applying the functor H_n^K, we obtain a map $\mathbb{L}_n\phi_N\colon \mathbb{L}_nF(N) \to \mathbb{L}_NF'(N)$, and the maps $\mathbb{L}_n\phi_N$, for varying N, assemble into a natural transformation $\mathbb{L}_n\phi$ as above.

Exercise 12.1. Suppose that $f\colon M \to M'$ and $g\colon M' \to M''$ are R-linear maps between right R-modules.
Prove that the composite natural transformation $\mathrm{Tor}_n^R(f,-)\circ\mathrm{Tor}_n^R(g,-)$ coincides with $\mathrm{Tor}_n^R(f\circ g,-)$. Hint: this check has to be made objectwise, so fix $N \in {}_R\mathrm{Mod}$ and check that $\mathrm{Tor}_n^R(f,N)\circ\mathrm{Tor}_n^R(g,N)$ coincides with $\mathrm{Tor}_n^R(f\circ g,N)$, as maps of abelian groups $\mathrm{Tor}_n^R(M,N) \to \mathrm{Tor}_n^R(M'',N)$.

If now $M, M' \in {}_R\mathrm{Mod}$ are left R-modules, and $f\colon M \to M'$ is an R-linear map, we can apply the principle above to the contravariant functors $\mathrm{Hom}_R(-,M)$ and $\mathrm{Hom}_R(-,M')$, which are functors ${}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ connected by a natural transformation $\mathrm{Hom}(-,f)\colon \mathrm{Hom}_R(-,M) \Rightarrow \mathrm{Hom}_R(-,M')$. We obtain a natural transformation

$$\mathrm{Ext}_R^n(-,f)\colon \mathrm{Ext}_R^n(-,M) \Rightarrow \mathrm{Ext}_R^n(-,M').$$

Exercise 12.2. Discuss in a similar way what happens about the functors from Definitions 11.13 and 11.17 when considering a R-linear maps between two choices of fixed module.

[42]Or between two abelian categories, the first of which has enough projectives

Exercise 12.3. Let M and M' be right R-modules. Note that the functors ${}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ given by $(M \oplus M') \otimes_R -$ and $(M \otimes_R -) \oplus (M' \otimes_R -)$ are naturally isomorphic. Prove that for all $n \in \mathbb{Z}$ also the functors ${}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ given by $\mathrm{Tor}_n^R(M \oplus M', -)$ and $\mathrm{Tor}_n^R(M, -) \oplus \mathrm{Tor}_n^R(M', -)$ are naturally isomorphic. Prove similarly that $\mathrm{Ext}_R^n(-, M) \oplus \mathrm{Ext}_R^n(-, M')$ is naturally isomorphic to $\mathrm{Ext}_R^n(-, M \oplus M')$ as functors $\mathrm{Mod}_R \to \mathrm{Mod}_R$. Discuss also the other functors from Definitions 11.13 and 11.17.

12.3. **More examples of computations of Tor and Ext.**

Example 12.4. Recall Examples 10.10 and 10.11. Let p be a prime number, let $n \geq 2$ and let $1 \leq k \leq n-1$. Consider the ring $R = \mathbb{Z}/p^n$ and the R-modules $M = \mathbb{Z}/p$ and $N = \mathbb{Z}/p^k$.
Let us compute $\mathrm{Tor}_n^R(M, N)$ for all n. A projective resolution of N over R is the infinite, periodic resolution

$$\dots \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p^n \xrightarrow{\cdot p^k} \mathbb{Z}/p^n \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p^n \xrightarrow{\cdot p^k} \mathbb{Z}/p^n \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p^n \xrightarrow{\cdot p^k} \mathbb{Z}/p^n \longrightarrow 0 \dots,$$

with the right-most $\mathbb{Z}/p^n$ in degree 0, and with augmentation $[-]_{p^k} : \mathbb{Z}/p^n \to \mathbb{Z}/p^k$.
Applying the functor $M \otimes_R - = \mathbb{Z}/p \otimes_{\mathbb{Z}/p^n} -$ we obtain the chain complex

$$\dots \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p \xrightarrow{\cdot p^k} \mathbb{Z}/p \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p \xrightarrow{\cdot p^k} \mathbb{Z}/p \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p \xrightarrow{\cdot p^k} \mathbb{Z}/p \longrightarrow 0 \dots,$$

and since multiplication by a multiple of p induces the zero map on $\mathbb{Z}/p$, we have that all differentials in the last chain complex are zero. It follows that $\mathrm{Tor}_n^{\mathbb{Z}/p^n}(\mathbb{Z}/p, \mathbb{Z}/p^k) \cong \mathbb{Z}/p$ for all $n \geq 0$.
Let us now compute $\mathrm{Ext}_R^n(N, M)$ for all n. We apply the contravariant functor $\mathrm{Hom}_R(-, M) = \mathrm{Hom}_{\mathbb{Z}/p^n}(-, \mathbb{Z}/p)$ to the projective resolution of N, obtaining the cochain complex

$$\dots 0 \longrightarrow \mathbb{Z}/p \xrightarrow{\cdot p^k} \mathbb{Z}/p \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p \xrightarrow{\cdot p^k} \mathbb{Z}/p \xrightarrow{\cdot p^{n-k}} \mathbb{Z}/p \xrightarrow{\cdot p^k} \mathbb{Z}/p \xrightarrow{\cdot p^{n-k}} \dots,$$

where the left-most $\mathbb{Z}/p$ is in degree 0, and the next one is in homological degree -1, i.e. cohomological degree 1. Again all differentials vanish. It follows that $\mathrm{Ext}_{\mathbb{Z}/p^n}^n(\mathbb{Z}/p^k, \mathbb{Z}/p) \cong \mathbb{Z}/p$ for all $n \geq 0$.

Each half of the previous computation implies that every projective resolution of $\mathbb{Z}/p^k$ over $\mathbb{Z}/p^n$ must have infinite length: if by absurd there was a projective resolution of finite length $\ell \geq 0$, we could use this resolution to compute, for instance, $\mathrm{Tor}_{\ell+1}^R(M, N) \cong 0$. But we know that any two projective resolutions give up to isomorphism the same Tor groups, and the computation above gives $\mathrm{Tor}_{\ell+1}^R(M, N) \cong \mathbb{Z}/p$.
We can now make a small comparison among the rings $\mathbb{Z}/p$, $\mathbb{Z}/p^n$ (with $n \geq 2$) and $\mathbb{Z}$:

- $\mathbb{Z}/p$ is a field, hence every $\mathbb{Z}/p$-module is projective and thus admits a projective resolution of length ≤ 0 (i.e. length 0);
- $\mathbb{Z}$ is a PID, hence every $\mathbb{Z}$-module admits a projective (actually free) resolution of length ≤ 1;
- $\mathbb{Z}/p^n$ has modules that only admit projective resolutions of infinite length.

Definition 12.5. Let R be a ring and M be a left R-module. The *projective dimension* of M over R, denoted $\mathrm{pd}(M)$, is the minimum $\ell \geq 0$ such that M

admits a projective resolution over R of length ℓ; if such an ℓ does not exist, we say that M has infinite projective dimension.
The *left, projective global dimension* of the ring R, denoted $\mathrm{lpdim}(R)$, is the supremum of all projective dimensions of all left R-modules. The *right, projective global dimension* is denoted $\mathrm{rpdim}(R)$ and is defined in an analogous way, after defining the projective dimension of right R-modules.

If R is commutative (the case we are now most interested in), then left and right projective global dimensions coincide: in this case we just write $\mathrm{pdim}(R)$ for the global dimension of R. The discussion above can be summarised as follows:

- $\mathrm{pdim}(\mathbb{Z}/p) = 0$;
- $\mathrm{pdim}(\mathbb{Z}) = 1$;
- $\mathrm{pdim}(\mathbb{Z}/p^n) = \infty$ for $n \geq 2$.

Exercise 12.6. Generalise Example 12.4 and compute $\mathrm{Tor}_n^{\mathbb{Z}/p^n}(\mathbb{Z}/p^\ell, \mathbb{Z}/p^k)$ and $\mathrm{Ext}^n_{\mathbb{Z}/p^n}(\mathbb{Z}/p^\ell, \mathbb{Z}/p^k)$ for all $n \geq 2$ and all $1 \leq k, \ell \leq n-1$.

13. Balancing of Tor and Ext

Let R be a ring. We have introduced so far several functors:

- for a right R-module M, we have a functor
$$M \otimes_R -\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod},$$
with left derived functors
$$\mathrm{Tor}_n^R(M, -)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod};$$
- for a left R-module N, we have a functor
$$- \otimes_R N\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod},$$
with left derived functors
$$\check{\mathrm{Tor}}_n^R(-, N)\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod};$$
- for a right R-module M (respectively, a left R-module N) we have a contravariant functor
$$\mathrm{Hom}_R(-, M)\colon {\mathrm{Mod}_R}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$$
$$(\text{respectively, } \mathrm{Hom}_R(-, N)\colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}),$$
with right derived functors
$$\mathrm{Ext}^n_R(-, M)\colon {\mathrm{Mod}_R}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod}$$
$$(\text{respectively, } \mathrm{Ext}_R(-, N)\colon {}_R\mathrm{Mod}^{op} \to {}_{\mathbb{Z}}\mathrm{Mod});$$
- for a right R-module M (respectively, a left R-module N) we have a functor
$$\mathrm{Hom}_R(M, -)\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$$
$$(\text{respectively, } \mathrm{Hom}_R(N, -)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}),$$
with right derived functors
$$\check{\mathrm{Ext}}^n_R(M, -)\colon \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$$
$$(\text{respectively, } \check{\mathrm{Ext}}_R(N, -)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}).$$

We have already seen that an R-linear map of right R-modules $M \to M'$ induces a natural transformation $f \otimes_R -\colon M \otimes_R - \Rightarrow M' \otimes_R -$, which induces in turn a derived natural transformation $\mathrm{Tor}_n^R(f,-)\colon \mathrm{Tor}_n^R(M,-) \Rightarrow \mathrm{Tor}_n^R(M',-)$; similarly, f induces natural transformations:

- $\mathrm{Hom}_R(-,f)\colon \mathrm{Hom}_R(-,M) \Rightarrow \mathrm{Hom}_R(-,M')$, inducing in turn natural transformations $\mathrm{Ext}^n_R(-,f)\colon \mathrm{Ext}^n_R(-,M) \Rightarrow \mathrm{Ext}^n_R(-,M')$;
- $\mathrm{Hom}_R(f,-)\colon \mathrm{Hom}_R(M',-) \to \mathrm{Hom}_R(M,-)$, inducing in turn natural transformations $\check{\mathrm{Ext}}^n_R(f,-)\colon \check{\mathrm{Ext}}^n_R(M',-) \Rightarrow \check{\mathrm{Ext}}^n_R(M,-)$.

(check carefully that in this last case the natural transformation goes in the opposite direction!).
We can thus think of Tor_n^R as a functor with Mod_R as source and the functor category $\mathrm{Fun}({}_R\mathrm{Mod}, {}_{\mathbb{Z}}\mathrm{Mod})$ as target, according to the following definition.

Definition 13.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories; we define $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to be the category whose objects are functors $F\colon \mathcal{C} \to \mathcal{D}$, and whose morphisms $F \to F'$ are natural transformations $\phi\colon F \Rightarrow G$, for two functors F and F'.

Alternatively, we can use the following definition, giving an "adjoint" description of Tor_n^R.

Definition 13.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We define $\mathcal{C} \boxtimes \mathcal{D}$ to be the category whose objects are pairs (x,y) with $x \in \mathcal{C}$ and $y \in \mathcal{D}$, and whose morphisms $(x,y) \to (x',y')$ are pairs of morphisms $f\colon x \to x'$ in $\mathcal{C}$ and $g\colon y \to y'$ in $\mathcal{D}$.

We can then consider $- \otimes_R -$ as a functor $\mathrm{Mod}_R \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, and similarly both $\mathrm{Tor}_n^R(-,-)$ and $\check{\mathrm{Tor}}_n^R(-,-)$ can be considered as functors $\mathrm{Mod}_R \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$.
Similarly, we can consider $\mathrm{Hom}_R(-,-)$, $\mathrm{Ext}^n_R(-,-)$ and $\check{\mathrm{Ext}}^n_R(-,-)$ as functors ${}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, or in the case of right R-modules as functors ${\mathrm{Mod}_R}^{op} \boxtimes \mathrm{Mod}_R \to {}_{\mathbb{Z}}\mathrm{Mod}$.
We can now state Theorems 11.14 and 11.18 in a more functorial way.

Theorem 13.3. *Let R be a ring and let $n \in \mathbb{Z}$; then the following couples of functors are naturally isomorphic:*

- $\mathrm{Tor}_n^R(-,-)$ *and* $\check{\mathrm{Tor}}_n^R(-,-)$*, from* $\mathrm{Mod}_R \boxtimes {}_R\mathrm{Mod}$ *to* ${}_{\mathbb{Z}}\mathrm{Mod}$*;*
- $\mathrm{Ext}^n_R(-,-)$ *and* $\check{\mathrm{Ext}}^n_R(-,-)$*, from* ${}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod}$ *to* ${}_{\mathbb{Z}}\mathrm{Mod}$*;*
- $\mathrm{Ext}^n_R(-,-)$ *and* $\check{\mathrm{Ext}}^n_R(-,-)$*, from* ${\mathrm{Mod}_R}^{op} \boxtimes \mathrm{Mod}_R$ *to* ${}_{\mathbb{Z}}\mathrm{Mod}$.

The rest of the lecture is devoted to the proof of Theorems 11.14 and 11.18, i.e. the objectwise version of 13.3. The proof of Theorem 13.3 is then left as exercise (see Exercise 13.6). We will also mainly focus on the proof of Theorem 11.14, and only sketch the (quite analogous) proof of theorem 11.18.

13.1. **Double complexes with exact rows.** In this subsection we prove a key lemma, which at first glance has little to do with Theorems 11.14 and 11.18.

Lemma 13.4. *Let $E_{\bullet,\bullet}$ be a double complex of abelian groups*[43]*. Assume that $E_{i,j} = 0$ whenever $i < 0$ or $j < 0$ (or both). Assume further that each row of $E_{\bullet,\bullet}$*

[43]The lemma holds in any abelian category different from ${}_{\mathbb{Z}}\mathrm{Mod}$, but we will only use it in this special case

is exact, i.e. for all fixed $i \in \mathbb{Z}$ we have an exact sequence

$$\dots \xrightarrow{d''_{i,4}} E_{i,3} \xrightarrow{d''_{i,3}} E_{i,2} \xrightarrow{d''_{i,2}} E_{i,1} \xrightarrow{d''_{i,1}} E_{i,0} \longrightarrow 0 \longrightarrow \dots.$$

Then the total chain complex $\mathrm{Tot}(E)_\bullet$*, constructed according to Definition 9.19, is acyclic/exact.*

Proof. We have to prove that all homology groups of $\mathrm{Tot}(E)_\bullet$ vanish; note that $\mathrm{Tor}(E)_\bullet$ is concentrated in non-negative degrees, indeed $E_{i,j} = 0$ whenever $i+j < 0$. Hence we have to prove that $H_n(\mathrm{Tot}(E)_\bullet) = 0$ for all $n \geq 0$: very concretely, this means that each cycle in $\mathrm{Tot}(E)_\bullet$ is also a boundary.
Let $n \geq 0$ be fixed, and let $x \in \mathrm{Tot}(E)_n$ be a cycle: we have $\mathrm{Tot}(E)_n = \bigoplus_{i=0}^{n} E_{i,n-i}$, hence we can write x as a formal sum $x_0 \oplus \cdots \oplus x_n$ with $x_i \in E_{i,n-i}$, where the "$\oplus$" is to stress that these components belong to different summands $E_{i,n-i}$.
By exactness of the n^{th} row, the map $d''_{n,1} \colon E_{n,1} \to E_{n,0}$ is surjective, hence there is $y_n \in E_{n,1}$ with $d''_{n,1}(y_n) = x_n$. We can consider y_n as an element in $\mathrm{Tot}(E)_{n+1}$: we then have $d_{n+1}(y_n) = d'_{n,1}(y_n) \oplus (-1)^n d''_{n,1}(y_n)$
Recall that our aim is to prove that the cycle x is a boundary in the complex $\mathrm{Tot}(E)_\bullet$: it is equivalent to prove that the cycle $x^{(1)} := x - (-1)^n d_{n+1}(y_n)$ is a boundary.
The cycle $x^{(1)}$ of $\mathrm{Tot}(E)_\bullet$ has a new feature that x, a priori, did not have: its component in $E_{n,0}$ vanishes. We then write $x^{(1)} = x_0^{(1)} + \cdots + x_n^{(1)}$, with $x_i^{(1)} \in E_{i,n-i}$ and with $x_n^{(1)} = 0$. Now we compute the differential of $x^{(1)}$, which is supposed to vanish: the formula predicts that $d_n(x^{(1)})$ has a component in $E_{n-1,0} \subset \mathrm{Tot}(E)_{n-1}$ equal to $d'_{n,0}(x_n^{(1)}) + (-1)^{n-1} d''(x_{n-1}^{(1)})$, and using that $x_n^{(1)} = 0$ we get just $(-1)^{n-1} d''(x_{n-1}^{(1)})$. This component of $d_n(x^{(1)})$ must vanish, but this implies, by exactness of the $n-1^{\mathrm{st}}$ row of $E_{\bullet,\bullet}$, that $x_{n-1}^{(1)} \in E_{n-1,1}$ can be written as $d''_{n-1,2}(y_{n-1})$, for some $y_{n-1} \in E_{n-1,2}$.
We can then perturb $x^{(1)}$ to a new cycle $x^{(2)} := x^{(1)} - (-1)^{n-1} d_{n+1}(y_{n-1})$. Proving that $x^{(2)}$ is a boundary is equivalent to proving that $x^{(1)}$ is a boundary, but now we have a new feature: the cycle $x^{(2)}$ has trivial components in bidegrees $(n,0)$ *and* $(n-1,0)$.
The next step is to compute the component of $d_n(x^{(2)})$ in bidegree $(n-2,1)$, and conclude that $x_{n-2}^{(2)}$ is equal to $d''_{n-2}(y_{n-2})$ for some $y_{n-2} \in E_{n-2,3}$, using exactness of the $n-2^{\mathrm{nd}}$ row of $E_{\bullet,\bullet}$.
By this procedure we recursively perturb our original cycle x and obtain a sequence of cycles $x, x^{(1)}, x^{(2)}, \dots, x^{(n+1)} \in \mathrm{Tot}(E)_n$: each two consecutive cycles differ by a boundary, and the cycle $x^{(j)}$ has trivial components in the summands $E_{n-i,i}$ for $i < j$. In particular the last cycle $x^{(n+1)}$ is the zero cycle. This proves that all $x^{(j)}$ and also x are boundaries in the chain complex $\mathrm{Tot}(E)_\bullet$. □

Note that the statement of Lemma 13.4 also holds if we do not assume that all rows of $E_{\bullet,\bullet}$ are exact, but instead assume that all columns of $E_{\bullet,\bullet}$ are exact. We can also change the assumption "$E_{i,j} = 0$ whenever $i < 0$ or $j < 0$" to the assumption, depending on two fixed constants $\bar{i}, \bar{j} \in \mathbb{Z}$, "$E_{i,j} = 0$ whenever $i < \bar{i}$ or $j < \bar{j}$".
Finally, there is a dual version which we state as an exercise. To have a genuinely dual statement we use the alternative total complex $\widehat{\mathrm{Tot}}(E)_\bullet$, but you can check that in this situation each product involved in defining a degree component of

$\widehat{\mathrm{Tot}}(E)_\bullet$ is in fact a finite product, so it coincides with the corresponding direct sum. In other words, in the statement of the following exercise it is harmless to replace $\widehat{\mathrm{Tot}}(E)_\bullet$ by $\mathrm{Tot}(E)_\bullet$.

Exercise 13.5. Let $E_{\bullet,\bullet}$ be a double complex of abelian groups. Assume that $E_{i,j} = 0$ whenever $i > 0$ or $j > 0$ (or both). Assume further that each row of $E_{\bullet,\bullet}$ is exact, i.e. for all fixed $i \in \mathbb{Z}$ we have an exact sequence

$$\dots \longrightarrow 0 \longrightarrow E_{i,0} \xrightarrow{d''_{i,0}} E_{i,-1} \xrightarrow{d''_{i,-1}} E_{i,-2} \xrightarrow{d''_{i,-2}} E_{i,-3} \xrightarrow{d''_{i,-3}} \dots$$

Then the alternative total chain complex $\widehat{\mathrm{Tot}}(E)_\bullet$, constructed according to Definition 9.20, is acyclic/exact.

13.2. **Tensor product of resolutions.** Let M be a right R-module and N be a left R-module, and fix projective resolutions $(P_\bullet, \epsilon)$ of M (in Mod_R) and $(Q_\bullet, \upsilon)$ of N (in ${}_R\mathrm{Mod}$). We can create a double complex of abelian groups $P_\bullet \otimes_R Q_\bullet$ by tensoring the two projective resolutions: in bidegree (i,j) we have the abelian group $P_i \otimes_R Q_j$. Compare with the construction in Subsection 9.5. We obtain the following double complex, which we call $E^{P,Q}_{\bullet,\bullet}$ for short in the following

$$\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_4^P\otimes \mathrm{Id}_{Q_3}} & & \downarrow{\scriptstyle d_4^P\otimes \mathrm{Id}_{Q_2}} & & \downarrow{\scriptstyle d_4^P\otimes \mathrm{Id}_{Q_1}} & & \downarrow{\scriptstyle d_4^P\otimes \mathrm{Id}_{Q_0}} & & \downarrow & & \\
\dots & \xrightarrow{\mathrm{Id}_P\otimes d^Q} & P_3\otimes_R Q_3 & \xrightarrow{\mathrm{Id}_{P_3}\otimes d_3^Q} & P_3\otimes_R Q_2 & \xrightarrow{\mathrm{Id}_{P_3}\otimes d_2^Q} & P_3\otimes_R Q_1 & \xrightarrow{\mathrm{Id}_{P_3}\otimes d_1^Q} & P_3\otimes_R Q_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle d_3^P\otimes \mathrm{Id}_{Q_3}} & & \downarrow{\scriptstyle d_3^P\otimes \mathrm{Id}_{Q_2}} & & \downarrow{\scriptstyle d_3^P\otimes \mathrm{Id}_{Q_1}} & & \downarrow{\scriptstyle d_3^P\otimes \mathrm{Id}_{Q_0}} & & \downarrow & & \\
\dots & \xrightarrow{\mathrm{Id}_{P_2}\otimes d_4^Q} & P_2\otimes_R Q_3 & \xrightarrow{\mathrm{Id}_{P_2}\otimes d_3^Q} & P_2\otimes_R Q_2 & \xrightarrow{\mathrm{Id}_{P_2}\otimes d_2^Q} & P_2\otimes_R Q_1 & \xrightarrow{\mathrm{Id}_{P_2}\otimes d_1^Q} & P_2\otimes_R Q_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle d_2^P\otimes \mathrm{Id}_{Q_3}} & & \downarrow{\scriptstyle d_2^P\otimes \mathrm{Id}_{Q_2}} & & \downarrow{\scriptstyle d_2^P\otimes \mathrm{Id}_{Q_1}} & & \downarrow{\scriptstyle d_2^P\otimes \mathrm{Id}_{Q_0}} & & \downarrow & & \\
\dots & \xrightarrow{\mathrm{Id}_{P_1}\otimes d_4^Q} & P_1\otimes_R Q_3 & \xrightarrow{\mathrm{Id}_{P_1}\otimes d_3^Q} & P_1\otimes_R Q_2 & \xrightarrow{\mathrm{Id}_{P_1}\otimes d_2^Q} & P_1\otimes_R Q_1 & \xrightarrow{\mathrm{Id}_{P_1}\otimes d_1^Q} & P_1\otimes_R Q_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle d_1^P\otimes \mathrm{Id}_{Q_3}} & & \downarrow{\scriptstyle d_1^P\otimes \mathrm{Id}_{Q_2}} & & \downarrow{\scriptstyle d_1^P\otimes \mathrm{Id}_{Q_1}} & & \downarrow{\scriptstyle d_1^P\otimes \mathrm{Id}_{Q_0}} & & \downarrow & & \\
\dots & \xrightarrow{\mathrm{Id}_{P_0}\otimes d_4^Q} & P_0\otimes_R Q_3 & \xrightarrow{\mathrm{Id}_{P_0}\otimes d_3^Q} & P_0\otimes_R Q_2 & \xrightarrow{\mathrm{Id}_{P_0}\otimes d_2^Q} & P_0\otimes_R Q_1 & \xrightarrow{\mathrm{Id}_{P_0}\otimes d_1^Q} & P_0\otimes_R Q_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

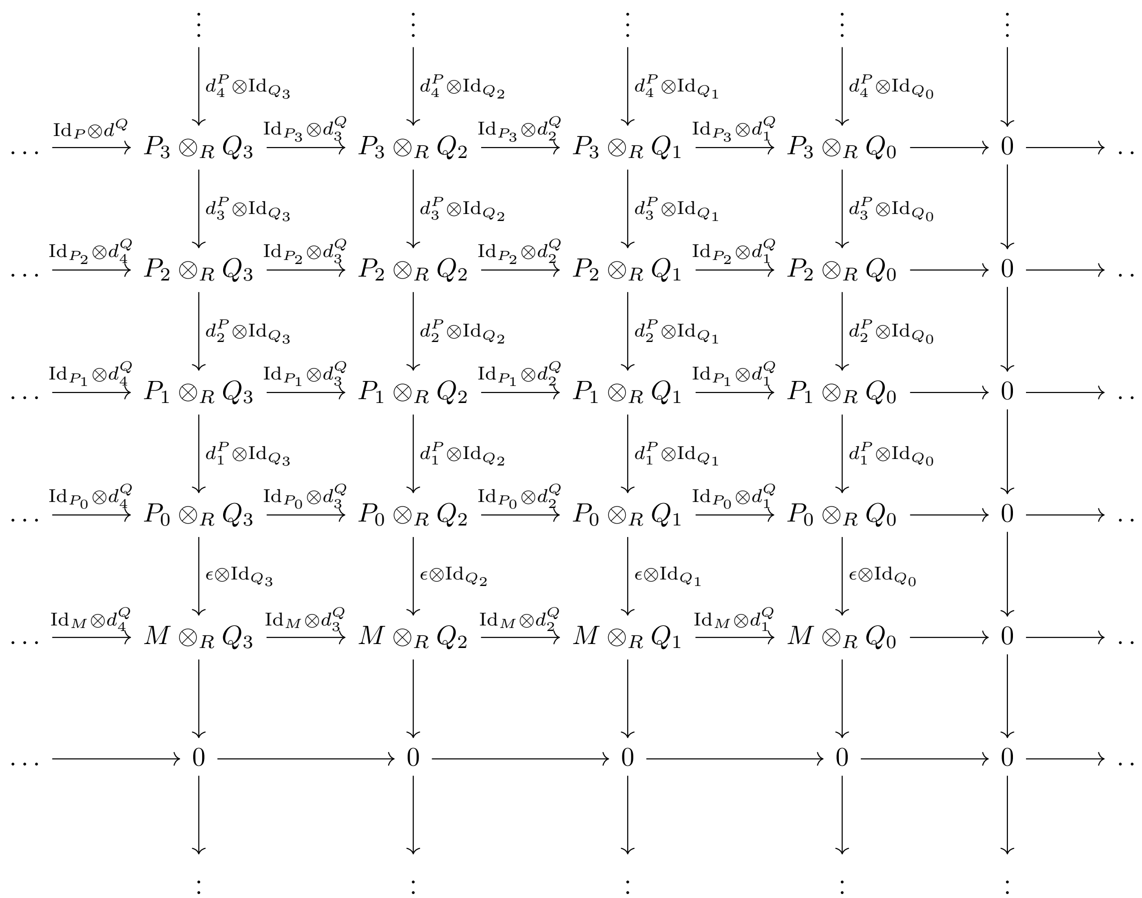
Our aim is to prove, for all $n \in \mathbb{Z}$, that $H_n(\mathrm{Tot}(E^{P,Q})_\bullet)$ is isomorphic to both $\mathrm{Tor}_n^R(M,N)$ and to $\check{\mathrm{Tor}}_n^R(M,N)$: this would imply Theorem 11.14. You will appreciate how the construction of the double complex $E^{P,Q}_{\bullet,\bullet}$ is *symmetric* in M and

N, in the sense that the two modules are treated in the same way: instead of resolving only N (as when computing $\mathrm{Tor}_n^R(M,N)$), or only M (as when computing $\check{\mathrm{Tor}}_n^R(M,N)$), we are now projectively resolving both modules, and then applying the bifunctor $-\otimes_R -$. In the following we will prove that $H_n(\mathrm{Tot}(E^{P,Q})_\bullet)$ is isomorphic to $\check{\mathrm{Tor}}_n^R(M,N)$; the other isomorphism $H_n(\mathrm{Tot}(E^{P,Q})_\bullet) \cong \mathrm{Tor}_n^R(M,N)$ is analogous.

13.3. **A short exact sequence of double complexes.** The double complex $E^{P,Q}_{\bullet,\bullet}$ can be recovered as follows. We take the tensor product of the exact sequence

$$\ldots \xrightarrow{d_3^P} P_2 \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{\epsilon} M \to 0 \ldots$$

with the chain complex $Q_\bullet$, and obtain a double complex $E^{P,M,Q}_{\bullet,\bullet}$ represented schematically as follows



The double complex $E^{P,M,Q}_{\bullet,\bullet}$ contains a sub-double-complex $E^{M,Q}_\bullet, \bullet$, spanned by the elements in bidegrees $(-1,i)$ for varying i: the double complex $E^{M,Q}_{\bullet,\bullet}$ looks like a single row, placed at height -1, and surrounded by zeroes. We have in particular an isomorphism of chain complexes

$$\mathrm{Tot}(E^{M,Q})_\bullet \cong M \otimes_R (\Sigma^{-1} Q_\bullet);$$

in particular, up to a mild shift, the homology groups of $\mathrm{Tot}(E^{M,Q})_\bullet$ are precisely the groups $\mathrm{Tor}_n^R(M,N)$, obtained by resolving N projectively and then tensoring with M.
The quotient of double complexes $E^{P,M,Q}_{\bullet,\bullet}/E^{M,Q}_{\bullet,\bullet}$ is then isomorphic to $E^{P,Q}_{\bullet,\bullet}$; if we take total chain complexes, we obtain a short exact sequence of chain complexes

$$\mathrm{Tot}(E^{M,Q})_\bullet \to \mathrm{Tot}(E^{P,M,Q})_\bullet \to \mathrm{Tot}(E^{P,Q})_\bullet$$

And now the wonderful remark: the double complex $E^{P,M,Q}_{\bullet,\bullet}$ has exact columns! Indeed the i^{th} column it is obtained by applying the exact functor $-\otimes_R Q_i$ to the exact sequence

$$\dots \xrightarrow{d_3^P} P_2 \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{\epsilon} M \to 0 \dots.$$

You should appreciate how the module N, and in particular the functor $-\otimes_R N$, just don't play a role in the previous argument. By Lemma 13.4 we then have that $\mathrm{Tot}(E^{P,M,Q})_\bullet$ is acyclic; we can then apply the snake lemma segment-wise, and obtain exact sequences

$$H_n(\mathrm{Tot}(E^{P,M,Q})_\bullet) = 0 \longrightarrow H_n(\mathrm{Tot}(E^{P,Q})_\bullet) \xrightarrow[\cong]{\partial_n} H_{n-1}(\mathrm{Tot}(E^{M,Q})_\bullet) \cong \mathrm{Tor}_n^R(M,N) \longrightarrow H_{n-1}(\mathrm{Tot}(E^{P,M,Q})_\bullet) = 0$$

This concludes the proof of Theorem 11.14.

Exercise 13.6. For $n \in \mathbb{Z}$ define a functor $\overline{\mathrm{Tor}}_n^R(-,-)\colon \mathrm{Mod}_R \boxtimes {}_R\mathrm{Mod} \to {}_\mathbb{Z}\mathrm{Mod}$ as the following composition (part of the exercise is to make sense of all categories and functors)

$$\mathrm{Mod}_R \boxtimes {}_R\mathrm{Mod} \xrightarrow{\mathfrak{P}\boxtimes\mathfrak{P}} K(\mathrm{Mod}_R) \boxtimes K({}_R\mathrm{Mod}) \xrightarrow{K(\mathrm{Tot}(-\otimes_R -))} K({}_\mathbb{Z}\mathrm{Mod}) \xrightarrow{H_n^K} {}_\mathbb{Z}\mathrm{Mod}.$$

Prove that the functor $\overline{\mathrm{Tor}}_n^R(-,-)$ is naturally isomorphic to both $\mathrm{Tor}_n^R(-,-)$ and $\check{\mathrm{Tor}}_n^R(-,-)$, by putting together, for varying M, N, the isomorphisms produced in the proof of Theorem 11.14.

13.4. **Tor and flat resolutions.** The argument of the proof of Theorem 11.14 seen in the previous section can be summarised as follows:

- we choose projective resolutions $(P_\bullet, \epsilon)$ and $(Q_\bullet, \upsilon)$ of our modules M and N;
- we construct a double complex $E^{P,Q}$ by tensoring the two resolutions;
- up to shifts, we find a short exact sequence of chain complexes involving $P_\bullet \otimes_R Q$, an acyclic chain complex, and $\mathrm{Tot}(E^{P,Q})_\bullet$;
- we apply the snake lemma.

The acyclic complex arises by applying Lemma 13.4 to a double complex $E^{P,M,Q}$ with acyclic columns; the proof of acyclicity of columns of $E^{P,M,Q}$ uses crucially that the functors $-\otimes_R Q_i$ are exact, because Q_i are projective, hence flat.
In fact, the argument would have worked if we just assume from the very beginning that $Q_\bullet$ is a flat resolution of N in ${}_R\mathrm{Mod}$. We therefore obtain this spin off proposition.

Proposition 13.7. *Let $M \in \mathrm{Mod}_R$ and $N \in {}_R\mathrm{Mod}$, let $(Q_\bullet, \upsilon)$ be a flat resolution of N; then for all $n \in \mathbb{Z}$ the n^{th} homology group of the chain complex $M \otimes_R Q_\bullet$ is isomorphic to $\mathrm{Tor}_n^R(M, N)$ (and to $\check{\mathrm{Tor}}_n^R(M, N)$ as well, by Theorem 11.14).*

Proof. Choose a projective resolution $(P_\bullet, \epsilon)$ of M, and apply the argument of the previous subsection word by word. □

It is important to stress that Proposition 13.7 does not imply that one can use flat resolutions of left R-modules to define, up to natural isomorphism, the *functor* $\mathrm{Tor}_n^R(M, -)$. A functor must be defined not only on objects, but also on morphisms. The behaviour of $\mathrm{Tor}_n^R(M, -)$ on R-linear maps $f \colon N \to N'$ is defined by lifting f to a chain map between the resolutions of N and N' respectively; the existence of such a lift, as well as its uniqueness up to chain homotopy, have been proved using that the resolutions are *projective*. Hence proposition 13.7 does not admit a genuine functorial version.

Example 13.8. Let M be an abelian group. We want to compute $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. For this, we fix a flat resolution of $\mathbb{Q}/\mathbb{Z}$, namely $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Q} \to 0 \ldots$; we then apply the functor $M \otimes_{\mathbb{Z}} -$, obtaining the chain complex

$$\ldots 0 \longrightarrow M \longrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0 \ldots .$$

We then have a description of $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ as the kernel of the map of abelian groups $M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $m \mapsto m \otimes 1$.
We can now identify $M \otimes_{\mathbb{Z}} \mathbb{Q}$ with the localisation $(\mathbb{Z} \setminus \{0\})^{-1}M$, and write the previous as the map $m \mapsto \frac{m}{1}$. An element $m \in M$ lies in the kernel of the latter map if and only if $\frac{m}{1} = \frac{0}{1}$, i.e. there exists $t \in \mathbb{Z} \setminus \{0\}$ such that $t \cdot m = 0$. This precisely means that $m \in \mathrm{tors}(M)$, see Definition 5.22.
With a little more care, we can conclude the following: $\mathrm{Tor}_1^{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic, as a functor ${}_{\mathbb{Z}}\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, to the functor tors assigning to each $\mathbb{Z}$-module its torsion submodule.

Example 13.8 is more or less the reason why we use "Tor" to designate the left derived functors of the tensor product.

13.5. **Sketch of proof of Theorem 11.18.** Let now for simplicity $M, N \in {}_R\mathrm{Mod}$, let $(P_\bullet, \epsilon)$ be a projective resolution of M, and let $(I_\bullet, \eta)$ be an injective resolution of N, i.e. $0 \to N \xrightarrow{\eta} I_0 \to I_{-1} \to I_{-2} \to \ldots$ is exact.

One can then construct two double complexes as follows. The first is called $E_{\bullet,\bullet}^{P,M,I}$ and has exact columns:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(M,I_0) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_M,d_0^I)} & \mathrm{Hom}_R(M,I_{-1}) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_M,d_{-1}^I)} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\epsilon,\mathrm{Id}_{I_0})} & & \downarrow{\scriptstyle \mathrm{Hom}(\epsilon,\mathrm{Id}_{I_{-1}})} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(P_0,I_0) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_0},d_0^I)} & \mathrm{Hom}_R(P_0,I_{-1}) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_0},d_{-1}^I)} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(d_1^P,\mathrm{Id}_{I_0})} & & \downarrow{\scriptstyle \mathrm{Hom}(d_1^P,\mathrm{Id}_{I_{-1}})} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(P_1,I_0) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_1},d_0^I)} & \mathrm{Hom}_R(P_1,I_{-1}) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_1},d_{-1}^I)} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(d_2^P,\mathrm{Id}_{I_0})} & & \downarrow{\scriptstyle \mathrm{Hom}(d_2^P,\mathrm{Id}_{I_{-1}})} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The second is called $E_{\bullet,\bullet}^{P,I,N}$ and has exact rows:
The same double complex $E^{P,I}$, represented schematically as follows, occurs as sub-double-complex of $E^{P,M,I}$ and $E^{P,I,N}$ by considering only elements in non-positive bidegrees

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(P_0,I_0) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_0},d_0^I)} & \mathrm{Hom}_R(P_0,I_{-1}) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_0},d_{-1}^I)} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(d_1^P,\mathrm{Id}_{I_0})} & & \downarrow{\scriptstyle \mathrm{Hom}(d_1^P,\mathrm{Id}_{I_{-1}})} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(P_1,I_0) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_1},d_0^I)} & \mathrm{Hom}_R(P_1,I_{-1}) & \xrightarrow{\mathrm{Hom}(\mathrm{Id}_{P_1},d_{-1}^I)} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(d_2^P,\mathrm{Id}_{I_0})} & & \downarrow{\scriptstyle \mathrm{Hom}(d_2^P,\mathrm{Id}_{I_{-1}})} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

One can then consider short exact sequences of chain complexes

$$\widehat{\mathrm{Tot}}(E^{P,I})_\bullet \to \widehat{\mathrm{Tot}}(E^{P,M,I})_\bullet \to \Sigma\mathrm{Hom}_R(M, I_\bullet);$$

$$\widehat{\mathrm{Tot}}(E^{P,I})_\bullet \to \widehat{\mathrm{Tot}}(E^{P,I,N})_\bullet \to \Sigma\mathrm{Hom}_R(P_\bullet, N).$$

Using acyclicity of the middle terms (ensured by Exercise 13.5), one obtains a sequence of isomorphisms of abelian groups, for all $n \in \mathbb{Z}$:

$$\mathrm{Ext}_R^n(M,N) \cong H_{-n}(\widehat{\mathrm{Tot}}(E_\bullet^{P,I}) \cong \check{\mathrm{Ext}}_R^n(M,N).$$

This concludes the proof of Theorem 11.18.

Example 13.9. Let us compute $\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/2)$ and $\check{\mathrm{Ext}}_1^{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/2)$. For the first computation, we use the projective resolution $\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \to 0 \dots$ of $\mathbb{Z}/2$; applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ we obtain the chain complex

$$\dots 0 \to \mathbb{Z}/2 \xrightarrow{\cdot 4} \mathbb{Z}/2 \to 0 \to \dots;$$

all differentials vanish, and in particular we get $\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/2) \cong \mathbb{Z}/2$ by considering the right copy of $\mathbb{Z}/2$ in the last complex.
For the second computation, we can embed $\mathbb{Z}/2$ into the injective module $\mathbb{Q}/\mathbb{Z}$ by sending $[1]_2 \mapsto [\frac{1}{2}]_{\mathbb{Z}}$. The cokernel of this embedding is $\mathbb{Q}/(\frac{1}{2}\mathbb{Z})$, which is isomorphic to $\mathbb{Q}/\mathbb{Z}$ as an abstract abelian group, by the map $[x]_{\frac{1}{2}\mathbb{Z}} \mapsto [2x]_{\mathbb{Z}}$.

We thus have an injective resolution $\dots 0 \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Q}/\mathbb{Z} \to 0 \to \dots$ of $\mathbb{Z}/2$; applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, -)$, and recalling that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/4$, we obtain the chain complex

$$\dots 0 \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to 0 \to \dots;$$

the image of the differential "$\cdot 2$" is the subgroup of $\mathbb{Z}/4$ generated by $[2]_4$; the cokernel, which is $\check{\mathrm{Ext}}(\mathbb{Z}/4, \mathbb{Z}/2)$, is thus isomorphic to $\mathbb{Z}/2$.

14. Ext and extensions, projective and injective dimension

In the last lecture we have seen the reason why Tor has this name: for an abelian group M, we have that $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is isomorphic to $\mathrm{tors}(M)$, the subgroup of M of torsion elements. Similarly, one can prove the following fact (which is left as exercise).

Exercise 14.1. Let $k \geq 2$ and let M be an abelian group. Then $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/k)$ is isomorphic to $M[k]$, i.e. the subgroup of M of *k-torsion elements*, where $m \in M$ is defined to be k-torsion if $k \cdot m = 0$.

One can generalise the previous to any PID, here is another exercise.

Exercise 14.2. Let R be a domain and let M be an R-module. Recall Example 7.7. Then $\mathrm{Tor}_1^R(M, \mathrm{Frac}(R)/R)$ is isomorphic to $\mathrm{tors}(M) \subset M$. If $a \neq 0$ is an element of R, then $\mathrm{Tor}_1^R(M, R/(a))$ is isomorphic to $M[a] \subset M$.

We want now to justify the name of Ext.

Definition 14.3. Let R be a ring and let A, C be left R-modules. An extension of C by A is a triple (B, i, p) consisting of a left R-module B and R-linear maps $i \colon A \to B$ and $p \colon B \to C$ such that the following is a SES

$$A \xrightarrow{i} B \xrightarrow{p} C.$$

Two extensions (B, i, p) and (B', i', p') are equivalent if there exists an R-linear isomorphism $\phi \colon B \to B'$ such that the following diagram commutes.

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \| & & \downarrow \phi & & \| \\ A & \xrightarrow{i'} & B' & \xrightarrow{p'} & C. \end{array}$$

We denote by $\mathrm{ext}(C, A)$ the set of equivalence classes of extensions of C by A, where the ring R is implicit. The equivalence class of (B, i, p) is denoted $[B, i, p]$.

In the literature it is common only to give an apparently weaker requirement in Definition 14.3, namely one declares (B, i, p) and (B', i', p') equivalent if there is an R-linear map $\phi\colon B \to B'$ making the diagram commute: one then uses the five lemma [Rot, Proposition 2.72] to conclude that the mere fact that ϕ makes the diagram commute already implies that ϕ must be an isomorphism. We will sometimes use later this fact.

Example 14.4. If (B, i, p) and (B', i', p') are equivalent extensions of C by A, there may be several maps $B \to B'$ witnessing that. Consider for instance the case in which $R = \mathbb{Z}$ and both (B, i, p) and (B', i', p') are the extension $(\mathbb{Z}/9, [3\cdot -]_9, [-]_3)$ of $\mathbb{Z}/3$ by $\mathbb{Z}/3$. Then we can surely choose ϕ to be $\mathrm{Id}_{\mathbb{Z}/9}$, but also the maps $\cdot 4$ and $\cdot 7$, which are also automorphisms of $\mathbb{Z}/9$, would make the diagram commute:

$$\begin{array}{ccccc} \mathbb{Z}/3 & \xrightarrow{[3\cdot -]_9} & \mathbb{Z}/9 & \xrightarrow{[-]_3} & \mathbb{Z}/3 \\ \| & & {\cdot 7}\downarrow\ \downarrow{\cdot 4}\ \downarrow{\mathrm{Id}_{\mathbb{Z}/9}} & & \| \\ \mathbb{Z}/3 & \xrightarrow{[3\cdot -]_9} & \mathbb{Z}/9 & \xrightarrow{[-]_3} & \mathbb{Z}/3. \end{array}$$

Example 14.5. If (B, i, p) and (B', i', p') are *split* extensions of C by A, i.e. both $A \xrightarrow{i} B \xrightarrow{p} C$ and $A \xrightarrow{i'} B' \xrightarrow{p'} C$ are split SES of left R-modules, then (B, i, p) and (B', i', p') are equivalent. Indeed one can choose sections $s\colon C \to B$ of p and $s'\colon C \to B'$ of p', and thus identify both B and B' with $A \oplus C'$. The following diagram then commutes, where all vertical maps are isomorphisms and can thus be inverted

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \| & & \uparrow{\scriptstyle i\oplus s} & & \| \\ A & \xrightarrow{\iota_A} & A \oplus C & \xrightarrow{\pi_C} & C \\ \| & & \downarrow{\scriptstyle i'\oplus s'} & & \| \\ A & \xrightarrow{i'} & B' & \xrightarrow{p'} & C. \end{array}$$

Example 14.6. Consider the following three extensions of $\mathbb{Z}/3$ by $\mathbb{Z}/3$:

$$\mathbb{Z}/3 \xrightarrow{\iota_1} \mathbb{Z}/3 \oplus \mathbb{Z}/3 \xrightarrow{\pi_2} \mathbb{Z}/3;$$

$$\mathbb{Z}/3 \xrightarrow{[3\cdot -]_9} \mathbb{Z}/9 \xrightarrow{[-]_3} \mathbb{Z}/3;$$

$$\mathbb{Z}/3 \xrightarrow{[6\cdot -]_9} \mathbb{Z}/9 \xrightarrow{[-]_3} \mathbb{Z}/3;$$

We claim that these three extensions are pairwise non-equivalent. The first extension is qualitatively different from the other two because it is split. To distinguish the second and the third extension, consider the following procedure:

- take the element $[1]_3$ in the rightmost $\mathbb{Z}/3$;
- lift $[1]_3$ to an element x in the middle group along the surjective map from the middle group to $\mathbb{Z}/3$;
- multiply x by 3 in the middle group, obtaining an element $3 \cdot x$ in the kernel of the map from the middle group to $\mathbb{Z}/3$;

- lift $3 \cdot x$ along the injective map from $\mathbb{Z}/3$ to the middle group, obtaining an element $y \in \mathbb{Z}/3$;
- return y.

No matter which x we choose, the procedure gives as output the element $y = [1]_3$ if we use the second extension, and gives as output $y = [2]_3$ if we use the third extension; thus the two extensions cannot be equivalent.

The following theorem justifies the name of Ext.

Theorem 14.7. *Let R be a ring and let A, C be left R-modules*[44]*. There is a bijection of sets* $\mathrm{ext}(C, A) \cong \mathrm{Ext}^1_R(C, A)$.

14.1. **The map ψ.** I take from [Rot, 7.2] the notation for the maps ψ and θ. In this subsection we construct, for fixed left R-modules A and C, a map of sets $\psi\colon \mathrm{ext}(C, A) \to \mathrm{Ext}^1_R(C, A)$. The second group is computed as first cohomology of the cochain complex $\mathrm{Hom}_R(P_\bullet, A)$, for a fixed projective resolution $(P_\bullet, \epsilon)$ of C.
Given an extension (B, i, p) representing an equivalence class in $\mathrm{ext}(C, A)$, we can write a diagram with exact rows

$$
\begin{array}{ccccccccccc}
\dots & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0\dots \\
 & & & & & & & & \| & & \\
\dots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0\dots
\end{array}
$$

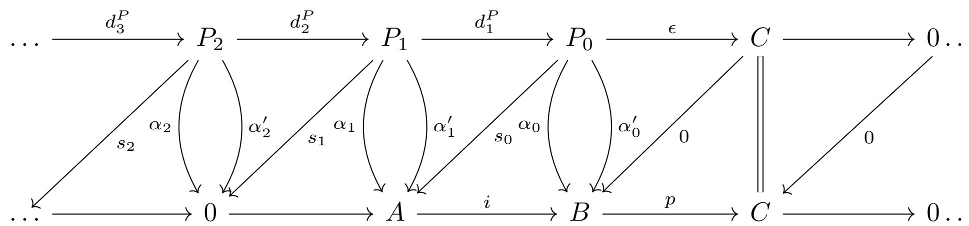
We can then use that the bottom row is exact, together with the fact that the top row contains projective modules P_i, to lift the identity of C to a chain map between $P_\bullet$ and the chain complex $\dots \to 0 \to A \to B \to 0\dots$: here the argument is the same as the one use to lift an R-linear map to a chain map between projective resolutions, and we just have to note that we only need exactness in the bottom row, but we don't need that all objects different from C in the bottom row be projective:

$$
\begin{array}{ccccccccccc}
\dots & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0\dots \\
 & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_0} & & \| & & \\
\dots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0\dots
\end{array}
$$

We now have that $\alpha_1 \in \mathrm{Hom}_R(P_1, A)$ is a cocycle in the cochain complex $\mathrm{Hom}_R(P_\bullet, A)$, namely $\mathrm{Hom}_R(d_2^P, A)(\alpha_1) = d_2^P \circ \alpha_1 = 0$, as the composition $d_2^P \circ \alpha_1$ can be replaced by the composition $P_2 \xrightarrow{\alpha_2} 0 \to A$. We have therefore that α_1 represents a class in $\mathrm{Ext}^1_R(C, A) = H^1(\mathrm{Hom}_R(P_\bullet, A))$.
We would like to define $\psi([B, i, p]) = [\alpha_1]$; however, there were some choices involved in the previous construction, namely how to lift the identity of C to the chain map α. Fortunately, again, the fact that all modules P_i are projective suffices to imply that any two different choices of lifts α and α' are chain homotopic by

[44]Of course the theorem also holds for right R-modules

some chain homotopy s:



In particular we get an equality $\alpha_1 - \alpha_1' = d_1^P \circ s_0 = \mathrm{Hom}_R(d_1^P, A)(s_0)$, so that the difference $\alpha_1 - \alpha_1'$ is a coboundary in the cochain complex $\mathrm{Hom}(P_\bullet, A)$, and in particular $[\alpha_1] = [\alpha_1'] \in \mathrm{Ext}^1_R(C, A)$.
Finally, if we replace (B, i, p) by an equivalent extension (B', i', p') of C by A, we can choose an identification of the exact sequences $\to 0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \dots$ and $\to 0 \to A \xrightarrow{i'} B' \xrightarrow{p'} C \to 0 \dots$ restricting to the identity of A and of C, and use the "same" α_0 and the *same* α_1 to lift the identity of C: here "same" means same up to identification, but *same* means same on the nose.

14.2. **The map θ.** We want now to construct a map $\theta\colon \mathrm{Ext}^1_R(C, A) \to \mathrm{ext}(C, A)$. Let therefore $\alpha_1 \in \mathrm{Hom}_R(P_1, A)$ be a cocycle representing a class $[\alpha_1] \in \mathrm{Ext}^1_R(C, A)$. Then $\alpha_1\colon P_1 \to A$ vanishes on the image of $d_2^P\colon P_2 \to P_1$ and thus induces a map $\bar\alpha_1\colon P_1/\mathrm{Im}(d_2^P) \to A$. Similarly, d_1^P induces a map $\bar d_1^P$. We obtain a diagram, whose first row is a SES

$$\begin{array}{ccccc} P_1/\mathrm{Im}(d_2^P) & \xrightarrow{\bar d_1^P} & P_0 & \xrightarrow{\epsilon} & C \\ \downarrow{\scriptstyle \bar\alpha_1} & & & & \| \\ A & & & & C \end{array}$$

We now define B to be the push-out of A and P_0 along $P_1/\mathrm{Im}(d_2^P)$ with the maps $\bar\alpha_1$ and $\bar d_1^P$, i.e. B is the colimit (in the category of left R-modules) of the diagram

$$\begin{array}{ccc} P_1/\mathrm{Im}(d_2^P) & \xrightarrow{\bar d_1^P} & P_0 \\ \downarrow{\scriptstyle \bar\alpha_1} & & \\ A & & \end{array}.$$

Concretely, B is the quotient of $A \oplus P_0$ by the submodule spanned by elements of the form $((x)\bar\alpha_1, -(x)\bar d_1^P)$ for varying $x \in P_1/\mathrm{Im}(d_2^P)$. The push-out, as every colimit, is equipped with maps from the objects of the diagram we took the colimit of; in particular we denote by $i\colon A \to B$ and by $\alpha_0\colon P_0 \to B$ two[45] of the structure maps of the push-out. Using the universal property of the push-out, we can define a map $p\colon B \to C$ by declaring the maps $0\colon A \to C$ and $\epsilon\colon P_0 \to C$, which in fact satisfy the property $\bar d_1^P \circ \epsilon = \bar\alpha_1 \circ 0$, as both composition are the zero map

[45]Strictly speaking, there is a third structure map, namely the map $P_1/\mathrm{Im}(d_2^P) \to B$; this can however be recovered from either i or α_0.

$P_1/\mathrm{Im}(d_2^P) \to C$. We thus obtain a commutative diagram

$$\begin{array}{ccccc} P_1/\mathrm{Im}(d_2^P) & \xrightarrow{\bar{d}_1^P} & P_0 & \xrightarrow{\epsilon} & C \\ \downarrow{\scriptstyle \bar{\alpha}_1} & & \downarrow{\scriptstyle \alpha_0} & & \| \\ A & \xrightarrow{i} & B & \xrightarrow{p} & C \end{array}$$

We already know that the top row is a SES, so let us check that also the bottom row is a SES:

- since $\epsilon = \alpha_0 \circ p$ is surjective, also p must be surjective;
- the composition $i \circ p \colon A \to C$ is the zero map by definition of p; viceversa, let $(x, y) \in A \oplus P_0$ and suppose that the corresponding element[46] $(x)i + (y)\alpha_0 \in B$ is in the kernel of p: this implies that $(y)\epsilon = 0$, but then exactness of the top row implies that there is $z \in P_1/\mathrm{Im}(d_2^P)$ such that $(z)\bar{d}_1^P = y$; it follows that $(x)i + (y)\alpha_0 = (x - (z)\bar{\alpha}_1)i$ is in the image of i;
- the map i is injective: if $x \in A$ is such that $(x)i = 0$, then the element $(x, 0) \in A \oplus C$ projects to zero in the quotient B, hence it is of the form $((y)\bar{\alpha}_1, -(y)\bar{d}_1^P)$ for some $y \in P_1/\mathrm{Im}(d_2^P)$; using injectivity of $\bar{d}_1^P$ we conclude that $y = 0$ and hence $x = (y)\bar{\alpha}_1 = 0$.

We would like to define $\theta([\alpha_1]) = [B, i, p]$; before doing that, we prove the following lemma.

Lemma 14.8. *Let (B', i', p') be another extension of C by A and let $\alpha'_0 \colon P_0 \to B'$ be a map such that the following diagram commutes*

$$\begin{array}{ccccc} P_1/\mathrm{Im}(d_2^P) & \xrightarrow{\bar{d}_1^P} & P_0 & \xrightarrow{\epsilon} & C \\ \downarrow{\scriptstyle \bar{\alpha}_1} & & \downarrow{\scriptstyle \alpha'_0} & & \| \\ A & \xrightarrow{i'} & B' & \xrightarrow{p'} & C. \end{array}$$

Then (B', i', p') is equivalent to the extension (B, i, p) constructed above.

Proof. We use again the universal property of B as a push-out: we can define a map $\phi \colon B \to B'$ by declaring the map $i' \colon A \to B'$ and $\alpha'_0 \colon P_0 \to B$: indeed we have $\bar{\alpha}_1 \circ i' = \bar{d}_1^P \circ \alpha'_0$ by the assumed commutative diagram. We then obtain a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \| & & \downarrow{\scriptstyle \phi} & & \| \\ A & \xrightarrow{i'} & B' & \xrightarrow{p'} & C \end{array}$$

and using the five lemma [Rot, Proposition 2.72] we conclude that ϕ is an isomorphism, hence (B', i', p') and (B, i, p) are equivalent extensions. □

In other words, the hypotheses of Lemma 14.8 imply that B', together with the structure maps i', α'_0 and $i' \circ \bar{\alpha}_1 = \alpha'_0 \circ d_1^P$, is another model for the push-out of A and P_0 along $P_1/\mathrm{Im}(d_2^P)$ and the maps $\bar{\alpha}_1$ and $\bar{d}_1^P$.

We can now give a safer definition of θ: we first define θ on cocycles and let $\theta(\alpha_1) \in \mathrm{ext}(C, A)$ be the unique equivalence class of extensions (B', i', p') of C by A for which there exists some α'_0 such that the hypotheses of Lemma 14.8 are

[46] A generic element of B can be represented in this way.

satisfied. The existence of such a class is given by the example of (B, i, p) together with α_0, and the uniqueness is given by Lemma 14.8 itself.
Does $\theta(\alpha_1) \in \mathrm{ext}(C, A)$ really only depend on $[\alpha_1]$, or does it depend, as it would seem from the construction, on the cocycle α_1 we used? In fact, it only depends on the cohomology class! If α'_1 is another cocycle such that $[\alpha_1] = [\alpha'_i] \in \mathrm{Ext}^1_R(C, A)$, then there is $s_0 \in \mathrm{Hom}(P_0, A)$ with $\alpha'_1 = \alpha_1 + \mathrm{Hom}_R(d_1^P, A)(s_0) = \alpha_1 + d_1^P \circ s_0$; we then also have an equality $\bar\alpha'_1 = \bar\alpha_1 + \bar d_1^P \circ s_0$, and the following diagram is commutative

$$\begin{array}{ccccc} P_1/\mathrm{Im}(d_2^P) & \xrightarrow{\bar d_1^P} & P_0 & \xrightarrow{\epsilon} & C \\ \downarrow{\scriptstyle \bar\alpha_1 + \bar d_1^P \circ s_0} & & \downarrow{\scriptstyle \alpha_0 + s_0 \circ i} & & \| \\ A & \xrightarrow{i} & B & \xrightarrow{p} & C. \end{array}$$

By Lemma 14.8 we obtain that B, together with i, $\alpha_0 + s_0 \circ i$ and a third map from $P_1/\mathrm{Im}(d_2^P)$ which now I don't specify, is also a model for the push-out of A and P_0 along $P_1/\mathrm{Im}(d_2^P)$ and the maps $\bar\alpha'_1$ and $\bar d_1^P$; thus (B, i, p) also represents $\theta(\alpha'_1)$, by the definition of θ on cocycles that we gave. It follows that $\theta(\alpha_1) = \theta(\alpha'_1)$, and hence θ descends to a function $\mathrm{Ext}^1_R(C, A) \to \mathrm{ext}(C, A)$, that we still call θ.

Exercise 14.9. Check that θ and ψ are inverse bijections.

14.3. **Baer sum.** You may have noticed that Theorem 14.7 establishes a bijection between the sets $\mathrm{ext}(C, A)$ and $\mathrm{Ext}^1_R(C, A)$, the second of which is however also an abelian group; we can then transfer the abelian group structure on $\mathrm{Ext}^1_R(C, A)$ to some abelian group structure on $\mathrm{ext}(C, A)$.

Example 14.10. The following is a commutative diagram with SES as rows

$$\begin{array}{ccccc} P_1/\mathrm{Im}(d_2^P) & \xrightarrow{\bar d_1^P} & P_0 & \xrightarrow{\epsilon} & C \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \epsilon \circ \iota_C} & & \| \\ A & \xrightarrow{\iota_A} & A \oplus C & \xrightarrow{\pi_C} & C. \end{array}$$

It follows that the class of the split extension $[A \oplus C, \iota_A, \pi_C]$ (which by Example 14.5 is also the class of any other split extension of C by A) corresponds, along the bijection $\mathrm{Ext}^1_R(C, A) \cong \mathrm{ext}(C, A)$, to the zero class $[0] \in \mathrm{Ext}^1_R(C, A)$.

In the following we give an alternative description of the sum operation on $\mathrm{ext}(C, A)$; we will however not prove that the following assignment $\mathrm{ext}(C, A) \times \mathrm{ext}(C, A) \to \mathrm{ext}(C, A)$ corresponds to the usual sum on $\mathrm{Ext}^1_R(C, A)$ along the bijection: the proof of this is for instance in [Rot, 7.2.1].
Let (B, i, p) and (B', i', p') be extensions of C by A, representing classes in $\mathrm{ext}(C, A)$. The first idea to combine the two extension into their "sum" is to take their direct sum: we however obtain a SES

$$A \oplus A \xrightarrow{i \oplus i'} B \oplus B' \xrightarrow{p \oplus p'} C \oplus C$$

whose external terms are $A \oplus A$ and $C \oplus C$ rather than A and C. We now use the following two tricks.
The first trick is to consider the *diagonal* copy of C contained in $C \oplus C$, which is isomorphic, as a left R-module, to the product $C \times C$: more precisely, we consider the map $\Delta\colon C \to C \times C$ given by the universal property of the product, by declaring twice the map $\mathrm{Id}_C\colon C \to C$; Δ is injective, and we consider the submodule $\Delta(C) \subset$

$C \oplus C$. We can then define $(B \oplus B')^{\Delta} \subset B \oplus B'$ as the preimage $(p \oplus p')^{-1}(\Delta(C))$; note that the image of $A \oplus A$ along $i \oplus i'$ is contained in $(B \oplus B')^{\Delta}$, for the simple reason that the composition $(p \oplus p') \circ (i \oplus i')$ is the zero map, having in particular image inside $\Delta(C)$. We thus get a SES

$$A \oplus A \xrightarrow{i \oplus i'} (B \oplus B')^{\Delta} \xrightarrow{p \oplus p'} \Delta(C) \cong C$$

The second trick is the "dual" trick. There is a map $\nabla \colon A \oplus A \to A$ given by the universal property of the direct sum, declaring twice the identity of A; ∇ is surjective, and we consider the two (isomorphic) submodules $\ker \nabla \subset A \oplus A$ and, correspondingly, $(i \oplus i')(\ker(\nabla)) \subset (B \oplus B')^{\Delta}$. We let $(B \oplus B')^{\Delta}_{\nabla}$ be the quotient $(B \oplus B')^{\Delta}/(i \oplus i')(\ker(\nabla))$; we obtain a SES

$$A \oplus A/\ker(\nabla) \cong A \xrightarrow{\overline{i \oplus i'}} (B \oplus B')^{\Delta}_{\nabla} \xrightarrow{\overline{p \oplus p'}} \Delta(C) \cong C.$$

The latter is an extension of C by A, and it turns out to represent $[B, i, p] + [B', i', p']$ under the transfered abelian group structure on $\mathrm{ext}(C, A)$ coming from $\mathrm{Ext}^1_R(C, A)$.

14.4. **Naturality of** ext**.** Recall that $\mathrm{Ext}^1_R \colon {}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is a bifunctor; using the bijections of abelian groups $\mathrm{ext}(C, A) \cong \mathrm{Ext}^1_R(C, A)$, one can tautologically define a bifunctor $\mathrm{ext} \colon {}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ which is naturally isomorphic to Ext^1_R. In the following we describe how ext looks like.
We fix left R-modules A, A', C, C' and R-linear maps $f \colon A \to A'$ and $g \colon C \to C'$. We content ourselves with giving a description of the maps $\mathrm{ext}(g, A) \colon \mathrm{ext}(C', A) \to \mathrm{ext}(C, A)$ and $\mathrm{ext}(C, f) \colon \mathrm{ext}(C, A) \to \mathrm{ext}(C, A')$ corresponding, under the identification of bifunctors $\mathrm{Ext}^1_R \cong \mathrm{ext}$, to $\mathrm{Ext}^1_R(g, \mathrm{Id}_A)$ and $\mathrm{Ext}^1_R(\mathrm{Id}_C, f)$.

Exercise 14.11. After finishing to read this subsection, prove of all following statements:

- the maps of sets $\mathrm{ext}(g, A)$ and $\mathrm{ext}(C, f)$ are well-defined;
- they correspond, under the identifications, to $\mathrm{Ext}^1_R(g, \mathrm{Id}_A)$ and $\mathrm{Ext}^1_R(\mathrm{Id}_C, f)$; in particular they are $\mathbb{Z}$-linear maps, and belong to a bifunctor

$$\mathrm{ext} \colon {}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}.$$

We start describing a map $\mathrm{ext}(C, f) \colon \mathrm{ext}(C, A) \to \mathrm{ext}(C, A')$. Given an extension (B, i, p) of C by A, we consider the push-out B' of the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow{\scriptstyle f} & & \\ A' & & \end{array},$$

and we denote by $i' \colon A' \to B'$ and by $\tilde{f} \colon B \to B'$ two of the structure maps of the push-out. This is very similar to what we did in the construction of θ in Subsection 14.2. We use the universal property of push-out and define a map $p' \colon B' \to C$ by declaring $0 \colon A' \to C$ and $p \colon B \to C$; thus we obtain a commutative diagram, whose first row is a SES

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} & & \| \\ A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C, \end{array}$$

and by a simple diagram chasing one can show that also the second row is a SES. We then have $\mathrm{ext}(C,f)\colon [B,i,p]\mapsto[B',i',p']$.
The description of the map $\mathrm{ext}(g,A)\colon \mathrm{ext}(C',A)\to\mathrm{ext}(C,A)$ is dual. Given an extension (B',i',p') of C' by A, we consider the pull-back B of the following diagram

$$\begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{p'} & C'. \end{array}$$

Concretely, the pull-back can be described as the submodule of $B'\oplus C$ containing all couples (x,y) such that $(x)p=(y)g$. We denote by $\tilde g\colon B\to B'$ and $p\colon B\to C$ two of the structure maps of the pull-back. We can then use the universal property of pull-back and define a map $i\colon A\to B$ by declaring $0\colon A\to C$ and $i'\colon A\to B'$; thus we obtain a commutative diagram, whose second row is a SES

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \| & & \downarrow{\scriptstyle \tilde g} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{i'} & B' & \xrightarrow{p'} & C', \end{array}$$

and by a simple diagram chasing one can show that also the second row is a SES. We then have $\mathrm{ext}(g,A)\colon [B',i',p']\mapsto[B,i,p]$.

14.5. **Yoneda sequences and cohomology product.** For two left R-modules A and C we have interpreted $\mathrm{Ext}^1_R(C,A)$ as the set of equivalence relations of SES $A\to B\to C$; can we do something similar for $\mathrm{Ext}^n_R(C,A)$ for $n\geq 2$?

Definition 14.12. A Yoneda sequence from A to C of length n is an exact sequence of left R-modules of the form

$$\dots 0\longrightarrow A\xrightarrow{i} B_{n-1}\xrightarrow{d^B_{n-1}} B_{n-2}\xrightarrow{d^B_{n-2}}\dots\xrightarrow{d^B_1} B_0\xrightarrow{p} C\longrightarrow 0\dots$$

We denote by $(B_\bullet,d^B,i,p)$ a generic Yoneda sequence.

We consider on Yoneda sequences the smallest equivalence relation spanned by the following basic equivalences: we regard $(B_\bullet,d^B,i,p)$ and $(B'_\bullet,d^{B'},i',p')$ to be "basic equivalent" if there exists a commutative diagram as follows, for suitable R-linear maps $\phi_0,\dots,\phi_{n-1}$:

$$\begin{array}{ccccccccccccc} \dots 0 & \longrightarrow & A & \xrightarrow{i} & B_{n-1} & \xrightarrow{d^B_{n-1}} & B_{n-2} & \xrightarrow{d^B_{n-2}} & \dots & \xrightarrow{d^B_1} & B_0 & \xrightarrow{p} & C \longrightarrow 0\dots \\ & & \| & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & & & \downarrow{\scriptstyle \phi_0} & & \| \\ \dots 0 & \longrightarrow & A & \xrightarrow{i'} & B'_{n-1} & \xrightarrow{d^{B'}_{n-1}} & B'_{n-2} & \xrightarrow{d^{B'}_{n-2}} & \dots & \xrightarrow{d^{B'}_1} & B'_0 & \xrightarrow{p'} & C \longrightarrow 0\dots \end{array}$$

You will note that we do not require the maps ϕ_i to be isomorphisms, and in particular the relation spanned by the above condition is not automatically symmetric. Nevertheless, we consider the smalles equivalence relation on Yoneda sequences which keeps track of the previous basic relations.
One can in fact characterise[47] as follows the equivalence relation: $(B_\bullet,d^B,i,p)$ and $(B'_\bullet,d^{B'},i',p')$ are equivalent if and only if there exists a third Yoneda sequence

[47]We do not give here a proof of this fact.

$(B''_\bullet, d^{B''}, i'', p'')$ and a commutative diagram

$$\begin{array}{ccccccccccccccc}
\dots 0 & \longrightarrow & A & \xrightarrow{i} & B_{n-1} & \xrightarrow{d^B_{n-1}} & B_{n-2} & \xrightarrow{d^B_{n-2}} & \dots & \xrightarrow{d^B_1} & B_0 & \xrightarrow{p} & C & \longrightarrow & 0\dots \\
 & & \| & & \uparrow{\scriptstyle \phi'_{n-1}} & & \uparrow{\scriptstyle \phi'_{n-2}} & & & & \uparrow{\scriptstyle \phi'_0} & & \| & & \\
\dots 0 & \longrightarrow & A & \xrightarrow{i''} & B''_{n-1} & \xrightarrow{d^{B''}_{n-1}} & B''_{n-2} & \xrightarrow{d^{B''}_{n-2}} & \dots & \xrightarrow{d^{B''}_1} & B''_0 & \xrightarrow{p''} & C & \longrightarrow & 0\dots \\
 & & \| & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & & & \downarrow{\scriptstyle \phi_0} & & \| & & \\
\dots 0 & \longrightarrow & A & \xrightarrow{i'} & B'_{n-1} & \xrightarrow{d^{B'}_{n-1}} & B'_{n-2} & \xrightarrow{d^{B'}_{n-2}} & \dots & \xrightarrow{d^{B'}_1} & B'_0 & \xrightarrow{p'} & C & \longrightarrow & 0\dots
\end{array}$$

Let $\mathrm{ext}^n(C, A)$ denote the set of equivalence classes $[B_\bullet, d^B, i, p]$ of Yoneda sequences of length n from A to C. In the previous discussion we have argued that $\mathrm{ext}^1(C, A)$, which is defined precisely as $\mathrm{ext}(C, A)$, is in natural bijection with $\mathrm{Ext}^1_R(C, A)$.

Example 14.13. Let $(B_\bullet, d^B, i, p)$ be a Yoneda sequence of length n from A to C, and let $(P_\bullet, \epsilon)$ be a projective resolution of C. Then we can lift the identity of C to a chain map from the complex $P_\bullet$ to the complex $\dots 0 \to A \to B_\bullet \to 0 \dots$:

$$\begin{array}{ccccccccccccccc}
\dots P_{n+1} & \xrightarrow{d^P_{n+1}} & P_n & \xrightarrow{d^P_n} & P_{n-1} & \xrightarrow{d^P_{n-1}} & P_{n-2} & \xrightarrow{d^P_{n-2}} & \dots & \xrightarrow{d^P_1} & P_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0\dots \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_{n-2}} & & & & \downarrow{\scriptstyle \alpha_0} & & \| & & \\
\dots 0 & \longrightarrow & A & \xrightarrow{i} & B_{n-1} & \xrightarrow{d^B_{n-1}} & B_{n-2} & \xrightarrow{d^B_{n-2}} & \dots & \xrightarrow{d^B_1} & B_0 & \xrightarrow{p} & C & \longrightarrow & 0\dots
\end{array}$$

A simple diagram chasing shows that $\alpha_n \in \mathrm{Hom}_R(P_n, A)$ is a cocycle of the cochain complex $\mathrm{Hom}_R(P_\bullet, A)$.
It can in fact be proved that for all $n \geq 2$ the sets $\mathrm{ext}^n(C, A)$ and $\mathrm{Ext}^n_R(C, A)$ are in natural bijection, by the assignment $[B_\bullet, d^B, i, p] \mapsto [\alpha_n]$.

The description of Ext with Yoneda sequences allows us to do one further step[48]. Indeed, let A, C, D be three left R-modules and let $(B_\bullet, d^B, i, p)$ and $(B'_\bullet, d^{B'}, i', p')$ be Yoneda sequences of lengths $n, n' \geq 1$ respectively, from A to C and from C to D respectively. We can define a Yoneda sequence of length $n + n'$ from A to D as follows

$$\dots 0 \longrightarrow A \xrightarrow{i} B_{n-1} \xrightarrow{d^B_{n-1}} \dots \xrightarrow{d^B_1} B_0 \xrightarrow{p \circ i'} B'_{n'-1} \xrightarrow{d^{B'}_{n'-1}} \dots \xrightarrow{d^{B'}_1} B'_0 \xrightarrow{p'} D \longrightarrow 0 \dots$$

This assignment descends to a product map

$$\mathrm{ext}^{n'}(D, C) \times \mathrm{ext}^n(C, A) \to \mathrm{ext}^{n+n'}(D, A).$$

Using push-outs and pull-backs in a similar way as in Subsection 14.4, one can also define product maps $\mathrm{ext}^{n'}(D, C) \times \mathrm{ext}^n(C, A) \to \mathrm{ext}^{n+n'}(D, A)$ when either n or n' is zero. One can check that these product maps are $\mathbb{Z}$-bilinear, and also that they are associative: given four R-modules A, C, D, E, any order of composition gives rise to the same map

$$\mathrm{ext}^{n''}(E, D) \times \mathrm{ext}^{n'}(D, C) \times \mathrm{ext}^n(C, A) \to \mathrm{ext}^{n+n'+n''}(E, A).$$

[48]There are also other, and probably more effective ways to define the product on Ext, this seemed to me to be the most intuitive

Finally, when $n = n' = 0$, a product map $\mathrm{ext}^{n'}(D,C)\times\mathrm{ext}^{n}(C,A)\to\mathrm{ext}^{n+n'}(D,A)$ is the same as a product map $\mathrm{Hom}_R(D,C)\times\mathrm{Hom}_R(C,A)\to\mathrm{Hom}_R(D,A)$, and in this special case we have a very natural definition to choose, namely the composition of R-linear maps! We thus obtain for all n' and n product maps $\mathrm{ext}^{n'}(D,C)\times\mathrm{ext}^{n}(C,A)\to\mathrm{ext}^{n+n'}(D,A)$, which are $\mathbb{Z}$-bilinear and satisfy associativity. We conclude with an example.

Example 14.14. Let R be an associative ring and let M be a left R-module. Recall that $\mathrm{Hom}_R(M,M)$ is an associative ring, by considering composition of R-linear maps. In fact, also

$$\mathrm{Ext}^*_R(M,M) := \bigoplus_{n\geq 0}\mathrm{Ext}^n_R(M,M)$$

has a natural structure of (graded) associative ring; the component in degree 0 is the sub-ring $\mathrm{Ext}^0_R(M,M)$, and it is isomorphic as a ring to $\mathrm{Hom}_R(M,M)$.

14.6. **Global dimensions.** Recall Definition 12.5. There is a dual definition, using injective resolutions.

Definition 14.15. Let R be a ring and M be a left R-module. The *injective dimension* of M over R, denoted $\mathrm{id}(M)$, is the minimum $\ell\geq 0$ such that M admits an injective resolution over R of length ℓ; if such an ℓ does not exist, we say that M has infinite injective dimension.
The *left, injective global dimension* of the ring R, denoted $\mathrm{lidim}(R)$, is the supremum of all injective dimensions of all left R-modules. The *right injective global dimension* is denoted $\mathrm{ridim}(R)$ and is defined in an analogous way, after defining the injective dimension of right R-modules.

In general, for an abelian category $\mathcal{C}$, if $\mathcal{C}$ has enough projectives one can define the projective global dimension $\mathrm{pdim}(\mathcal{C})$, and if $\mathcal{C}$ has enough injectives one can define the injective global dimension $\mathrm{idim}(\mathcal{C})$. The abelian categories ${}_R\mathrm{Mod}$ and Mod_R are defined using the same ring R, but they are in fact different abelian categories, so we should not expect, for instance, a relation between their projective global dimensions. In fact there exist rings R for which $\mathrm{rpdim}(R) = \mathrm{pdim}(\mathrm{Mod}_R)$ is different from $\mathrm{lpdim}(R) = \mathrm{pdim}({}_R\mathrm{Mod})$. Necessarily, such R must be non-commutative!
In the following we will focus, whenever a choice can be made, on left R-modules.

Example 14.16. For a left R-module M we have $\mathrm{pd}(M) = 0$ iff M is projective, and $\mathrm{id}(M) = 0$ iff M is injective, almost by definition. It is not difficult to find examples of R and M such that M is projective but not injective, or viceversa. Hence, in general, for a ring R and a left R-module M we have $\mathrm{pd}(M)\neq\mathrm{id}(M)$.

In spite of the previous example, we will prove the following theorem.

Theorem 14.17. *Let R be an associative ring. Then* $\mathrm{lpdim}(R) = \mathrm{lidim}(R)$ *and* $\mathrm{rpdim}(R) = \mathrm{ridim}(R)$.[49].

Note that the theorem also predicts that if either $\mathrm{lpdim}(R)$ or $\mathrm{rpdim}(R)$ is infinite, then so is the other. In order to prove the theorem, we will in fact prove the following proposition.

[49] In fact, if $\mathcal{C}$ is an abelian category with enough injectives and projectives, one can prove that $\mathrm{pdim}(\mathcal{C}) = \mathrm{idim}(\mathcal{C})$.

Proposition 14.18. *Let $n \geq 0$ be an integer and R a ring. Then the following are equivalent:*

(1) $\mathrm{lpdim}(R) \leq n$*;*
(2) $\mathrm{Ext}_R^{n+1}(M, N) = 0$ *for all left R-modules M, N;*
(3) $\mathrm{lidim}(R) \leq n$*.*

Thanks to Theorem 11.18, the condition (2) is more symmetric in the words "injective" and "projective" than the other two conditions (1) and (3), and we expect that any proof of the equivalence of (1) and (2) can be adapted to a proof of the equivalence of (2) and (3). In fact we will focus on the proof of the equivalence of (1) and (2), so we will focus on lengths of projective resolutions.

Proof that (1) implies (2) in Proposition 14.18. Let n and R satisfy $\mathrm{lpdim}(R) \leq n$, and let M, N be any left R-modules. Then M admits a projective resolution $P_\bullet$ of length at most n, and we can compute $\mathrm{Ext}_R^{n+1}(M, N)$ as the $n + 1^{\mathrm{th}}$ cohomology group of the cochain complex $\mathrm{Hom}_R(P_\bullet, N)$.
This cohomology group is supposed to be a subquotient of the abelian group $\mathrm{Hom}_R(P_{n+1}, N)$, which is assumed to be zero. □

Exercise 14.19. Adapt the previous argument to prove that (3) implies (2), by using an injective resolution of N instead of a projective resolution of M.

The next ingredient we will need towards a proof of Proposition 14.18is Schanuel's Lemma.

Lemma 14.20 (Schanuel's Lemma)**.** *Let M be a left R-module, suppose that we are given two SES of left R-modules*

$$K \xrightarrow{i} P \xrightarrow{p} M$$

$$K' \xrightarrow{i'} P' \xrightarrow{p'} M$$

with P and P' projective. Then there is an isomorphism of left R-modules $K \oplus P' \cong K' \oplus P$. In particular, K is projective if and only if K' is projective.

Proof. Consider the R-linear map $p \oplus p' \colon P \oplus P' \to M$: we will focuse on proving that $K' \oplus P$ is isomorphic to $\ker(p \oplus p')$; the proof that $K \oplus P'$ is also isomorphic to $\ker(p \oplus p')$ is completely analogous, and the two isomorphisms yield the first statement. The second statement follows from the fact that a direct summand of a projective module is projective.
Use that P is projective to construct a map ϕ making the following diagram commute

$$\begin{array}{ccccc} K & \xrightarrow{i} & P & \xrightarrow{p} & M \\ & & \downarrow{\scriptstyle\phi} & & \| \\ K' & \xrightarrow{i'} & P' & \xrightarrow{p'} & M \end{array}$$

Define now a map $\psi \colon P \oplus K' \to \ker(p \oplus p')$ by declaring $\psi|_{K'} \colon x \mapsto (0, x)$, and by declaring $\psi|_P \colon y \mapsto (y, -(y)\phi)$. The map ψ is surjective: every element $(z, w) \in \ker(p \oplus p')$ can be written as $(z, -(z)\phi) + (0, (z)\phi - w)$, where $(z)\phi - w \in P'$ is in the kernel of p', hence in the image of i'. The map ψ is also injective: if $(y, x)\psi = (y, -(y)\psi + x) = 0$, then $y = 0$, and then also $x = 0$. □

Exercise 14.21. Prove the dual Schanuel's Lemma: if $M \to I \to K$ and $M \to I' \to K'$ are SES's with I and I' injective modules, then $I \oplus K' \cong I' \oplus K$, and in particular K is injective if and only if K' is injective.

Schanuel's Lemma has the following striking consequence. Suppose that we want to compute $\mathrm{pd}(M)$ for some left R-module M; then the following algorithm returns $\mathrm{pd}(M)$, no matter what choices are made during it:

- if M is projective, return 0 and stop the algorithm;
- if M is not projective, choose any surjective map $\epsilon\colon P_0 \to M$ from any projective module P_0, and consider $\ker(\epsilon)$; if $\ker(\epsilon)$ is projective, return 1 and stop the algorithm;
- if also $\ker(\epsilon)$ is not projective, choose any surjective map $d_1^P\colon P_1 \to \ker(\epsilon)$ from any projective module P_1, and consider $\ker(d_1^P)$; if $\ker(d_1^P)$ is projective, return 2 and stop the algorithm;
- continue in this fashion, until for the first time you observe that $\ker(d_n^P)\colon P_n \to P_{n-1}$ is projective; then return $n+1$;
- if the algorithm never ends, return ∞.

In other words, there is no particularly clever way to shorten the length of a projective resolution of a module: if $P_\bullet$ is a projective resolution of M, Schanuel's Lemma, strictly speaking, ensures that if a run of the algorithm returns 1, then any run of the algorithm returns 1.

Definition 14.22. Two left R-modules M and M' are *projectively equivalent* if there are projective modules P and P' with $M \oplus P' \cong M' \oplus P$.

Note that if $M \oplus P' \cong M' \oplus P$ and $M' \oplus Q'' \cong M'' \oplus Q'$, then $M \oplus P' \oplus Q'' \cong M' \oplus P \oplus Q'' \cong M'' \oplus Q' \oplus P$, so Definition 14.22 gives in fact an equivalence relation on left R-modules.
Schanuel's Lemma can be generalised to the following statement: if M and M' are projectively equivalent and if $K \to P \to M$ and $K' \to P' \to M'$ are SES with P and P' projective, then K and K' are also projectively equivalent: indeed we may first find an isomorphism $M \oplus Q' \cong M' \oplus Q$, then consider the SESs $K \to P \oplus Q' \to M \oplus Q'$ and $K' \to P' \oplus Q \to M' \oplus Q$, then conclude by Schanuel's Lemma that $K \oplus P' \oplus Q \cong K' \oplus P \oplus Q'$.
After these considerations, consider two parallel runs of the above algorithm starting from the same left R-module M: at each step we obtain projectively equivalent modules, and both runs stop, at the same time, the first time that two kernels of surjective maps are (both) in the projective equivalence class of projective modules.

Exercise 14.23. Describe a dual algorithm returning the injective dimension of a left R-module N, by iteratively injecting into an injective R-module, taking the cokernel, and asking whether the latter is an injective module or not.

We will finish the proof of Proposition 14.18, and hence of Theorem 14.17, next time.

15. Global dimension, weak dimension, Hilbert's syzygy theorem

15.1. **End of proof of Proposition 14.18.** We start by completing the proof of Proposition 14.18. We focus on the proof that (1) implies (2) and on left R-modules. The first ingredient we need is the following characterisation of projective modules.

Lemma 15.1. *Let M be a left R-module. Then M is projective if and only if $\mathrm{Ext}^1_R(M,N) = 0$ for all $N \in {}_R\mathrm{Mod}$.*

Proof. Suppose first M projective, i.e. $\mathrm{Hom}_R(M,-)\colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is an exact (covariant) functor. Then $\mathrm{Ext}^1_R(M,-) = \mathbb{R}_{-1}(\mathrm{Hom}_R(M,-))$ is the zero functor, by an adaptation of Example 10.21 to right derived functors. In fact all functors $\mathrm{Ext}^n_R(M,-)$ for $n \neq 0$ are the zero functor.
Conversely, suppose that M is a left R-module with $\mathrm{Ext}^1_R(M,-)$ being the zero functor. In order to prove that $\mathrm{Hom}_R(M,-)$ is an exact functor (and hence M is projective) it suffices to show that $\mathrm{Hom}_R(M,-)$ maps SESs in ${}_R\mathrm{Mod}$ to SESs in ${}_{\mathbb{Z}}\mathrm{Mod}$ (see Proposition 4.25. Let therefore $N' \xrightarrow{i} N \xrightarrow{p} N''$ be a SES in ${}_R\mathrm{Mod}$. By Theorem 11.8, and recalling that $\mathrm{Ext}^0_R(-,-)$ is naturally isomorphic to $\mathrm{Hom}_R(-,-)$, we have a long exact sequence of cohomology groups featuring the following segment

$$\dots 0 \to \mathrm{Hom}_R(M,N') \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}(M,N'') \to \mathrm{Ext}^1_R(M,N') \to \dots$$

By hypothesis on M, the group $\mathrm{Ext}^1_R(M,N')$ vanishes; it follows that $\mathrm{Hom}_R(M,N') \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}(M,N'')$ is a SES in ${}_{\mathbb{Z}}\mathrm{Mod}$, which is what we wanted to show. □

Exercise 15.2. Prove the dual lemma: M is injective if and only if $\mathrm{Ext}^1_R(N,M) = 0$ for all $N \in {}_R\mathrm{Mod}$.

We are now ready to prove another implication in Proposition 14.18.

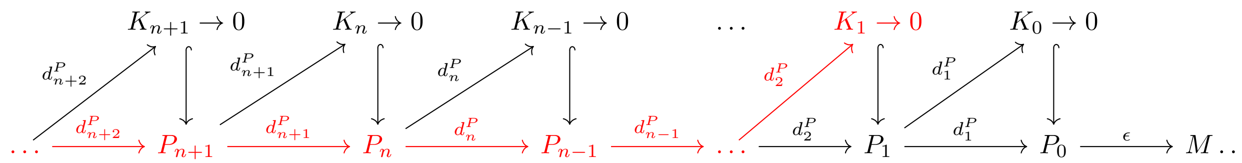
Proof that (2) implies (1) in Proposition 14.18. Let R be a ring, $n \geq 0$ and assume that Ext^{n+1}_R is the zero bifunctor ${}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$. We want to prove $\mathrm{lpdim}(R) \leq n$. By Lemma 15.1, in the case $n = 0$ we have that the condition "Ext^{n+1}_R is the zero bifunctor" implies that every left R-module is projective
Fix $M \in {}_R\mathrm{Mod}$. We are going to prove first a version of "(2) implies (1)" that only focuses on M: we are going to prove that if $\mathrm{Ext}_R(M,-)$ is the zero functor ${}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, then $pd(M) \leq n$. Once this is done, letting M vary we immediately obtain "(2) implies (1)".
Let $(P_\bullet, \epsilon)$ be a projective resolution of M, and for all $i \geq 1$ define[50] K_i to be $\ker(d^P_i) \subseteq P_i$; define also $K_0 = \ker(\epsilon)$; the situation is represented in the following diagram, where the row is exact:

$$\begin{array}{ccccccccccccc}
 & & K_{n+1} & & K_n & & K_{n-1} & & \dots & & K_1 & & K_0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{d^P_{n+2}} & P_{n+1} & \xrightarrow{d^P_{n+1}} & P_n & \xrightarrow{d^P_n} & P_{n-1} & \xrightarrow{d^P_{n-1}} & \dots & \xrightarrow{d^P_2} & P_1 & \xrightarrow{d^P_1} & P_0 & \xrightarrow{\epsilon} & M \dots
\end{array}$$

Our aim is to show that M admits a projective resolution of length at most n; to this purpose, it suffices to show that K_{n-1} is a projective module: then we can consider $\dots 0 \to K_{n-1} \hookrightarrow P_{n-1} \xrightarrow{d^P_{n-1}} \dots \xrightarrow{d^P_1} P_0 \to 0 \dots$ as a length-n projective resolution of M. In fact, the argument from the last lecture involving Schanuel's Lemma and the algorithm implies that this is our only hope: if K_{n-1} fails being projective, then it would just be false that $\mathrm{pd}(M) \leq n$.

[50] The word "syzygy" is often used in the literature to refer to the modules K_i; the word was already used in astronomy with the meaning of "conjunction" (when three or more bodies are aligned), when Hilbert imported it into mathematics as an alternative to the word "relation".

Remember that K_i is also the image of the map d^P_{i+1}: we obtain the following nice fact: for each $i \geq 0$ the chain complex $\dots \xrightarrow{d^P_{i+3}} P_{i+2} \xrightarrow{d^P_{i+2}} P_{i+1} \to 0 \to 0 \to 0 \dots$, with P_{i+1} put in degree 0, is a projective resolution of K_i, with augmentation $d^P_{i+1} \colon P_{i+1} \to K_i$. The situation is represented in the following diagram, in which sequences as the red one are exact



In particular the following row is exact, and represents a projective resolution of K_{n-1} together with its augmentation:

$$\dots \xrightarrow{d^P_{n+3}} P_{n+2} \xrightarrow{d^P_{n+2}} P_{n+1} \xrightarrow{d^P_{n+1}} P_n \xrightarrow{d^P_n} K_{n-1} \longrightarrow 0 \dots$$

To prove that K_{n-1} is projective, we use Lemma 15.1 and check that $\mathrm{Ext}^1_R(K_{n-1}, N) = 0$ for any left R-module N. We have a projective resolution of K_{n-1} ready, so let us just apply the functor $\mathrm{Hom}_R(-, N)$ to it: we obtain the following chain complex, where $\mathrm{Hom}_R(P_n, N)$ lies in degree 0, and (homological) degrees decrease from left to right

$$\dots 0 \longrightarrow \mathrm{Hom}_R(P_n, N) \xrightarrow{\mathrm{Hom}_R(d^P_{n+1}, N)} \mathrm{Hom}_R(P_{n+1}, N) \xrightarrow{\mathrm{Hom}_R(d^P_{n+2}, N)} \mathrm{Hom}_R(P_{n+2}, N) \xrightarrow{\mathrm{Hom}_R(d^P_{n+3}, N)} \dots$$

The group $\mathrm{Ext}^1_R(K_{n-1}, N)$ can be computed as $\ker(\mathrm{Hom}_R(d^P_{n+2}))/\mathrm{Im}(\mathrm{Hom}_R(d^P_{n+1}))$. Notice that also $\mathrm{Ext}^{n+1}_R(M, N)$ can be computed as the very same quotient: the reason why this happens is that the triple "$P_{n+2} \to P_{n+1} \to P_n$" occurs both in the original projective resolution of M, and in the obtained projective resolution of K_{n-1}. We assumed that $\mathrm{Ext}^{n+1}_R(M, -)$ is the zero functor, so we deduce that $\mathrm{Ext}^1_R(K_{n-1}, N)$ is zero for all left R-modules N, which implies by Lemma 15.1 that K_{n-1} is projective, which in turn implies $\mathrm{pd}(M) \leq n$. □

Exercise 15.3. Let M be a left R-module and let $(I^\bullet, \eta)$ be an injective resolution of M. Define $K^i = \mathrm{coker}(d^{i-1}_I \colon I^{i-1} \to I^i)$ for all $i \geq 1$, and define $K^0 = \mathrm{coker}(\eta \colon M \to I^0)$. For $n \geq 0$ prove that the functor $\mathrm{Ext}^{n+1}_R(-, M)$ is the zero functor if and only if K^{n-1} is an injective module. Deduce that (2) implies (3) in Proposition 14.18.

With the discussion of this subsection the proof of Theorem 14.17 is complete: we will henceforth call "left global dimension of R" the number $\mathrm{lpdim}(R) = \mathrm{lidim}(R)$, which may be infinite, and denote it $\mathrm{ldim}(R)$ for simplicity. Similarly we denote by $\mathrm{rdim}(R) = \mathrm{rpdim}(R) = \mathrm{ridim}(R)$ the right global dimension of R.

15.2. **An application of Baer's criterion.** Recall Proposition 5.14: we can use it to give a formula computing the left global dimension.

Theorem 15.4 (Auslander)**.** *Let R be a ring. Then* $\mathrm{ldim}(R)$ *can be computed as the supremum of the set* $\{\mathrm{pd}(R/J)\}$*, where J ranges among left ideals of R. In other words, 1-generated/cyclic left R-modules suffice to detect the left global dimension of R (using the definition as left projective dimension of R).*

Proof. It is clear that $\sup\{\mathrm{pd}(R/J)\,|\,J \subseteq R \text{ left ideal}\} \le \sup\{\mathrm{pd}(M)\,|\,M \in {}_R\mathrm{Mod}\}$, and in particular if the first is infinite, then so is the second and hence the two agree. Assume now $n := \sup\{\mathrm{pd}(R/J)\,|\,J \subseteq R \text{ left ideal}\}$ is finite. Let M be a left R-module; for all $J \subset R$ we have $\mathrm{Ext}_R^{n+1}(R/J, M) = 0$, as can be checked by choosing a projective resolution of R/J of length at most n (which exists by definition of n). Let now $(I^\bullet, \eta)$ be instead an injective resolution of M, and let K^{n-1} be as in Exercise 15.3: we can identify the groups $\mathrm{Ext}_R^{n+1}(R/J, M)$ and $\mathrm{Ext}_R^1(R/J, K^{n-1})$ by considering the short piece of chain complex $\mathrm{Hom}_R(R/J, I^n) \to \mathrm{Hom}_R(R/J, I^{n+1}) \to \mathrm{Hom}_R(R/J, I^{n+2})$ and computing the homology group in the middle degree. We conclude that $\mathrm{Ext}_R^1(R/J, K^{n-1}) = 0$ for all left ideals $J \subseteq R$.
We then write the long exact sequence of $\mathrm{Ext}_R(-, K^{n-1})$-groups associated with the SES of left R-modules $J \hookrightarrow R \to R/J$, for all left ideals $J \subseteq R$: it begins with

$$\dots 0 \to \mathrm{Hom}_R(R/J, K^{n-1}) \to \mathrm{Hom}_R(R, K^{n-1}) \to \mathrm{Hom}_R(J, K^{n-1}) \to \mathrm{Ext}_R^1(R/J, K^{n-1}) \to \dots$$

Since $\mathrm{Ext}_R^1(R/J, K^{n-1}) = 0$ for all J, we deduce that the map $\mathrm{Hom}_R(R, K^{n-1}) \to \mathrm{Hom}_R(J, K^{n-1})$ (induced by restriction of functions from R to J) is surjective; in other words, every R-linear map $J \to K^{n-1}$ can be extended to an R-linear map $R \to K^{n-1}$, for all left ideals $J \subseteq R$. This is precisely Baer's criterion to check that K^{n-1} is an injective left R-module: we conclude that K^{n-1} is an injective left R-module.
Since K^{n-1} is injective, we obtain an injective resolution of M of length n:

$$\dots 0 \longrightarrow I^0 \xrightarrow{d_I^0} \dots \xrightarrow{d_I^{n-1}} I^{n-1} \longrightarrow K^{n-1} \longrightarrow 0 \longrightarrow \dots$$

It follows that $\mathrm{id}(M) \le n$; since the arguments works for all $M \in {}_R\mathrm{Mod}$, we conclude that $\mathrm{lidim}(R) \le n$, and now we use that $\mathrm{lidim}(R)$ is one of the two ways to compute $\mathrm{ldim}(R)$. The other way is $\sup\{\mathrm{pd}(M)\,|\,M \in {}_R\mathrm{Mod}\}$, and so (recalling how n was defined) we obtain the inequality $\sup\{\mathrm{pd}(M)\,|\,M \in {}_R\mathrm{Mod}\} \le \sup\{\mathrm{pd}(R/J)\,|\,J \subseteq R \text{ left ideal}\}$, which together with the inequality from the beginning of the proof concludes the proof.

□

15.3. **A glimpse into the flat dimension.** Proposition 14.18 uses the vanishing of Ext_R^{n+1}, as a bifunctor ${}_R\mathrm{Mod}^{op} \boxtimes {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, as a criterion for the upper bounds $\mathrm{lpdim}(R) \le n$ and $\mathrm{lidim}(R) \le n$. Can we write a similar proposition involving the vanishing of Tor_{n+1}^R instead?

Definition 15.5. Let M be a left R-module. The flat dimension of M, denoted $\mathrm{fd}(M)$, is the infimum of lengths of flat resolutions of M. The left flat dimension[51] of R, denoted $\mathrm{lfdim}(R)$, is the supremum of $\mathrm{fd}(M)$ for $M \in {}_R\mathrm{Mod}$.
Similarly, define the flat dimension of right R-modules, and the right flat dimension $\mathrm{rfdim}(R)$.

We have the following lemma, which parallels Lemma 15.1, and whose proof is left as exercise.

Lemma 15.6. *Let N be a left R-module. Then N is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for all $M \in \mathrm{Mod}_R$. Similarly, let M be a right R-module. Then M is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for all $M \in \mathrm{Mod}_R$.*

[51]This is called "weak" dimension in Rotman and in the literature in general, but let me use a terminology which is parallel to what we already did.

We can now mimick the arguments of the previous subsection/last lecture. In the following it is important to notice that we do not rely on an analogue of Schanuel's Lemma for flat modules, especially because such analogue is not available.

Lemma 15.7. *Let M be a right R-module and $n \geq 0$; the following are equivalent*

(i) $\mathrm{fd}(M) \leq n$*;*

(ii) $\mathrm{Tor}_{n+1}^R(M, N) = 0$ *for all left R-modules N.*

A similar statement holds with left and right modules swapped.

Proof. First, assume (i). Recall that for a left R-module N we can compute the abelian group $\mathrm{Tor}_{n+1}^R(M, N)$, up to isomorphism, by choosing a flat resolution of M and tensoring it with N. By (i), there is a flat resolution $F_\bullet$ of M of length $\leq n$; hence the chain complex $F_\bullet \otimes_R N$ vanishes in degree $n + 1$, and this forces the $n + 1^{\text{th}}$ homology group, i.e. $\mathrm{Tor}_{n+1}^R(M, N)$, to vanish.
Now assume (ii). If $n = 0$, (ii) and Lemma 15.6 immediately imply that M is flat, and hence M admits $\theta_0(M)_\bullet$ as flat resolution. Suppose now (ii) holds and $n \geq 1$. Let $(F_\bullet, \epsilon)$ be a flat resolution of M, and similarly as in the previous subsection, define $K_{n-1} = \ker(d_{n-1}^F\colon F_{n-1} \to F_{n-2})$ (in the case $n = 1$, define $K_0 = \ker(\epsilon\colon F_0 \to M)$). Note that $\dots \overset{d_{n+2}^F}{\to} F_{n+1} \overset{d_{n+1}^F}{\to} F_n \to 0 \dots$ is a flat resolution of K_{n-1}; if N is any left R-module, the homology group computed in the middle of the piece of chain complex $F_{n+2} \otimes_R N \to F_{n+1} \otimes_R N \to F_n \otimes_R N$ can be identified with both $\mathrm{Tor}_1^R(K_{n-1}, N)$ and $\mathrm{Tor}_{n+1}^R(M, N)$; by (ii) the latter vanishes, hence the former vanishes, for all $N \in {}_R\mathrm{Mod}$. We can now apply Lemma 15.6 and conclude that K_{n-1} is flat; as a consequence, the following is a flat resolution of M of length n, witnessing that $\mathrm{fd}(M) \leq n$:

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow K_{n-1} \hookrightarrow F_{n-1} \xrightarrow{d_{n-1}^F} \dots \xrightarrow{d_1^F} F_0 \longrightarrow 0 \dots$$

□

Proposition 15.8. *Let $n \geq 0$ be an integer and R a ring. Then the following are equivalent:*

(1) $\mathrm{lfdim}(R) \leq n$*;*

(2) $\mathrm{Tor}_{n+1}^R(M, N) = 0$ *for all right R-modules M and left R-modules N;*

(3) $\mathrm{rfdim}(R) \leq n$*.*

Proof. The equivalence of (1) and (2) is a straightforward consequence of Lemma 15.7, by letting M vary among right R-modules. The equivalence of (2) and (3) is proved by a symmetric argument, swapping the roles of left and right modules. □

We can now derive from Proposition 15.8 the following theorem, in a way that is completely analgous to how we derived Theorem 14.17 from Proposition 14.18.

Theorem 15.9. *Let R be an associative ring. Then* $\mathrm{lfdim}(R) = \mathrm{rfdim}(R)$*.*

At first glance it might seem strange that in the case of flat dimensions we manage to prove that the left and the right versions are equal, whereas in the case of projective dimensions we couldn't. This is essentially a consequence of the fact that $- \otimes_R -$ combines left and right R-modules, whereas $\mathrm{Hom}_R(-, -)$ does not. From now on we will denote by $fdim(R)$ the common value $\mathrm{lfdim}(R) = \mathrm{rfdim}(R)$.
We can also compare Theorem 15.4 with the following theorem, which is [Rot, Theorem 8.25], and which we leave without proof.

Theorem 15.10. *Let R be a ring; then we have*

$$\mathrm{fdim}(R) = \sup\{\mathrm{fd}(R/J) \mid J \subseteq R \textit{ left ideal}\} = \sup\{\mathrm{fd}(R/J) \mid J \subseteq R \textit{ right ideal}\},$$

where in the first sup *we consider flat dimensions of left R-modules, and in the second we consider flat dimensions of right R-modules.*

15.4. **Hilbert's syzygy theorem.** Our next goal, the last one in the course, is to give a proof of the following theorem.

Theorem 15.11 (Hilbert's syzygy theorem)**.** *Let $\mathbb{F}$ be a field, and consider the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ in n variables. Then*

$$\mathrm{rdim}(\mathbb{F}[x_1, \ldots, x_n]) = \mathrm{ldim}(\mathbb{F}[x_1, \ldots, x_n]) = n.$$

We note that, since $\mathbb{F}[x_1, \ldots, x_n]$ is commutative, the categories ${}_{\mathbb{F}[x_1,\ldots,x_n]}\mathrm{Mod}$ and $\mathrm{Mod}_{\mathbb{F}[x_1,\ldots,x_n]}$ of left and right modules are isomorphic, and thus $\mathrm{rdim}(\mathbb{F}[x_1, \ldots, x_n]) = \mathrm{ldim}(\mathbb{F}[x_1, \ldots, x_n])$. Hilbert proved in 1890 a statement that can be reformulated as follows: every ideal J in the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ admits a free resolution of length at most $n-1$. Hilbert's statement implies immediately that $\mathrm{pd}(\mathbb{F}[x_1, \ldots, x_n]/J) \leq n$ for all ideals $J \subseteq \mathbb{F}[x_1, \ldots, x_n]$, and thus by Theorem 15.4 we obtain the inequality $\mathrm{rdim}(\mathbb{F}[x_1, \ldots, x_n]) = \mathrm{ldim}(\mathbb{F}[x_1, \ldots, x_n]) \leq n$; the fact that equality holds requires another additional argument, and you are more than invited, in order to satisfy your desire of knowledge about history of mathematics, to read Hilbert's original article "Über die Theorie der algebraischen Formen" and determine whether Hilbert also proved a statement which, reformulated in modern terms, gives the equality $\mathrm{rdim}(\mathbb{F}[x_1, \ldots, x_n]) = \mathrm{ldim}(\mathbb{F}[x_1, \ldots, x_n]) = n$.
We will derive Theorem 15.11 from a more general statement; before stating a proposition, we need a definition.

Definition 15.12. Let R be an associative ring. We denote by $R[x]$ the ring of polynomials in one variable x. An element of $R[x]$ has the form $f(x) = r_k x^k + \cdots + r_1 x + r_0$, with $r_0, \ldots, r_k \in R$; as an additive group, $R[x]$ is isomorphic to $\bigoplus_{i \geq 0} R$. The product is defined by setting on monomials by the rule $(r_k x^k) \cdot (r_h x^h) = (r_k r_h) x^{k+h}$ (note that in the last equality we have formally swapped x^k with r_h), and is extended bi-additively to all polynomials.

Exercise 15.13. We can also define $R[x]$ by a universal property: it is a ring $\bar{S}$ with a specified map $\bar{i} \colon R \to \bar{S}$ and with a specified element x, such that $\bar{i}(r)x = x\bar{i}(r)$ for all $r \in R$, and such that the triple $(\bar{S}, x, \bar{i})$ is *universal* among triples (S, y, i) with $i \colon R \to S$ a map of rings and y an element in S satisfying $i(r)y = yi(r)$ for all $r \in R$. Think about the details of the approach to define $R[x]$ by universal property, especially formulate explicitly the universal property (and check that it holds for the usual $R[x]$).

We can now state the main proposition

Proposition 15.14. *Let R be a ring. Then* $\mathrm{ldim}(R[x]) = \mathrm{ldim}(R) + 1$*; in particular, either side of the equality is infinite if and only if also the other side is infinite.*

We stated Proposition 15.14 in the context of possibly non-commutative rings because this is the level of generality in which (without much more effort) it can be proved; nevertheless, in order to show Theorem 15.11, we only need the "commutative" part of Proposition 15.14.

Proof of Theorem 15.11 assuming Proposition 15.14. We prove the theorem by induction on n. For $n = 0$, we note that a field $\mathbb{F}$ satisfies $\mathrm{ldim}(F) = 0$, as every (left) $\mathbb{F}$-module is free. Assuming the statement for $n = k-1 \geq 0$, we then note that if we set $R := \mathbb{F}[x_1, \ldots, x_{k-1}]$, then there is an isomorphism of rings $R[x] \cong \mathbb{F}[x_1, \ldots, x_k]$, by identifying x with the variable x_k.
We have a sequence of equalities, the third of which follows from Proposition 15.14:

$$\mathrm{ldim}(\mathbb{F}[x_1, \ldots, x_k]) = \mathrm{ldim}(R[x]) = \mathrm{ldim}(R) + 1 = (k-1) + 1 = k,$$

i.e. the statement also holds for $n = k$. □

The proof of Proposition 15.14 will be the object of the rest of this lecture and the next lecture. The analogue statement that $\mathrm{rdim}(R[x]) = \mathrm{rdim}(R)+1$ can be proved by similar arguments, but we will focus on left modules. Whenever we consider a projective dimension of a module, we will use the ring as an index in order to stress over which ring we are considering the module (this was unnecessary until now, as we were basically always working with a single ring).

Definition 15.15. Let M be a left R-module. We denote by $M[x]$ the left $R[x]$-module $R[x] \otimes_R M$, obtained by tensoring M with the $R[x]-R$-bimodule $R[x]$; here we consider $R[x]$ as a right R-module by using the inclusion of rings $R \hookrightarrow R[x]$.

Lemma 15.16. *Let M be a left R-module; if M is projective over R, then $M[x]$ is projective over $R[x]$. Let N be a left $R[x]$-module; if N is projective over $R[x]$, then N, considered as a left R-module by restriction of scalars, is projective over R.*

Proof. Assume first that M is projective over R, and hence that there is an isomorphism of left R-modules $M \oplus P = \bigoplus_{i \in I} R$; tensoring on left with $R[x]$, and remembering that tensor products are distributive with respect to direct sums, we obtain an isomorphism of left $R[x]$-modules

$$M[x] \oplus P[x] \cong R[x] \otimes_R M \oplus R[x] \otimes_R P \cong R[x] \otimes_R \left(\bigoplus_{i \in I} R \right) \cong \bigoplus_{i \in I} R[x]$$

and hence $M[x]$ is also projective over $R[x]$.
Assume now that N is projective over $R[x]$, and hence there is an isomorphism of left $R[x]$-modules $N \oplus Q \cong \bigoplus_{i \in I} R[x]$. Recall that the map of rings $R \hookrightarrow R[x]$ induces a functor ${}_{R[x]}\mathrm{Mod} \to {}_R\mathrm{Mod}$ by restriction of scalars. In particular, we can consider $N \oplus Q \cong \bigoplus_{i \in I} R[x]$ also as an isomorphism of left R-modules. We then note the following isomorphism of R-modules: $R[x] \cong \bigoplus_{j=0}^{\infty} R$.
Putting everything together, we have an isomorphism of left R-modules

$$N \oplus Q \cong \bigoplus_{i \in I} \bigoplus_{j=0}^{\infty} R$$

and hence N is projective over R. □

Lemma 15.17. *Let M be a left R-module; then $\mathrm{pd}_R(M) = \mathrm{pd}_{R[x]}(M[x])$.*

Proof. We prove that, for an integer $n \geq 0$, the inequality $\mathrm{pd}_R(M) \leq n$ holds if and only if the inequality $\mathrm{pd}_{R[x]}(M[x]) \leq n$ holds; once this is proved, the statement follows immediately.
Suppose first $\mathrm{pd}_R(M) \leq n$; then we can find a projective resolution

$$\cdots \to 0 \to P_n \to \cdots \to P_1 \to P_0 \to 0 \ldots$$

of M over R of length n (we may have $P_i = 0$ for some values of i towards the left end of the resolution). We can adjoin M to the previous and obtain an exact sequence of left R-modules

$$\cdots \to 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0 \ldots$$

Tensoring with $R[x]$ over R, and recalling that $R[x]$ is free as a right R-module (hence flat), we obtain an exact sequence

$$\cdots \to 0 \to P_n[x] \to \cdots \to P_1[x] \to P_0[x] \to M[x] \to 0 \ldots.$$

By Lemma 15.16 the modules $P_i[x]$ are projective over $R[x]$, and hence the previous, after removing $M[x]$, can be regarded as a projective resolution of $M[x]$ over $R[x]$ of length n, witnessing $\mathrm{pd}_{R[x]}(M[x]) \leq n$.
Viceversa, suppose that $\mathrm{pd}_{R[x]}(M[x]) \leq n$, and fix now a projective resolution

$$\cdots \to 0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to 0 \ldots$$

of $M[x]$ over $R[x]$ of length n. We can adjoin $M[x]$ to the previous and obtain an exact sequence of left $R[x]$-modules

$$\cdots \to 0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to M[x] \to 0 \ldots;$$

applying the restriction of scalar functor ${}_{R[x]}\mathrm{Mod} \to {}_R\mathrm{Mod}$ we can consider the previous also as an exact sequence of left R-modules; by Lemma 15.16 each Q_i is projective over R, and hence we conclude that $\mathrm{pd}_R(M[x]) \leq n$.
We then notice that $M[x]$ is isomorphic to $\bigoplus_{j=0}^{\infty} M$ as a left R-module; it follows that there is an isomorphism of functors

$$\mathrm{Ext}_R^{n+1}(M[x], -) \cong \bigoplus_{j=0}^{\infty} \mathrm{Ext}_R^{n+1}(M, -) \colon {}_R\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod};$$

the inequality $\mathrm{pd}_R(M[x]) \leq n$ implies[52] that $\mathrm{Ext}_R^{n+1}(M[x], -)$ is the zero functor; hence all direct summands of $\bigoplus_{j=0}^{\infty} \mathrm{Ext}_R^{n+1}(M, -)$ are the zero functor, and this implies in turn, by the argument of the proof of Proposition 14.18, that $\mathrm{pd}_R(M) \leq n$. □

A mild, direct consequence of Lemma 15.17 is that $\mathrm{ldim}(R[x]) \geq \mathrm{ldim}(R)$; in particular, if $\mathrm{ldim}(R) = \infty$, then also $\mathrm{ldim}(R[x]) = \infty$, and the statement of Proposition 15.14 holds; hence it doesn't harm, from now on, to assume that $\mathrm{ldim}(R)$ is finite.

Lemma 15.18. *Let N be a left $R[x]$-module, and consider N also as a left R-module by restriction of scalars. Then there exists a SES of left $R[x]$-modules of the form*

$$N[x] \xrightarrow{\iota} N[x] \xrightarrow{\pi} N,$$

featuring the left $R[x]$-modules $N[x] = R[x] \otimes_R N$.

Proof. We define $\pi \colon N[x] \to N$ to be the map sending $f(x) \otimes m \mapsto f(x) \cdot m$, for all $m \in N$ and all polynomials $f(x) \in R[x]$[53]. Since $1 \otimes m \mapsto m$, the map π is surjective.
An element in $N[x] = R[x] \otimes_R N$ can be written in a unique way as a finite sum $\sum_{i=0}^{\infty} x^i \otimes m_i$, with $m_i \in N$: this uses the direct sum decomposition $N[x] \cong$

[52] Is in fact equivalent.

[53] In fact, we should first define an R-bilinear map $R[x] \times N \to N$ by $(f(x), m) \mapsto f(x) \cdot m$, then use the universal property of the tensor product

$\bigoplus_{i=0}^{\infty} N$, which is an isomorphism of left R-modules. We define $\iota\colon N[x] \to N[x]$ by the following formula, where all tensors are over R:

$$\sum_{i=0}^{k} x^i \otimes m_i \mapsto \sum_{i=0}^{k} x^i \cdot (1 \otimes (x \cdot m_i) - x \otimes m_i) = \sum_{i=0}^{k} \left(x^i \otimes (x \cdot m_i) - x^{i+1} \otimes m_i\right)$$

$$= 1 \otimes (x \cdot m_0) + \sum_{i=1}^{k} \left(x^i \otimes (x \cdot m_i - m_{i-1})\right) - x^{k+1} \otimes m_k.$$

Note that the last expression gives the value $\left(\sum_{i=0}^{k} x^i \otimes m_i\right) \iota$ again in the normal form corresponding to the isomorphism of left R-modules $N[x] \cong \bigoplus_{i=0}^{\infty} N$.
The map ι is injective: suppose that the element $\sum_{i=0}^{k} x^i \otimes m_i$ is sent to zero by ι, then the component $-x^{k+1} \otimes m_k$ in the last expression for $\left(\sum_{i=0}^{k} x^i \otimes m_i\right) \iota$ must vanish: it follows that $m_k = 0$; we then argue that also $x^k \otimes (x \cdot m_k - m_{k-1})$ must vanish, as it is one component of $\left(\sum_{i=0}^{k} x^i \otimes m_i\right) \iota$; we already know that $m_k = 0$, hence we conclude that also $m_{k-1} = 0$. We proceed inductively until we obtain that $m_0 = 0$; the fact that $1 \otimes (x \cdot m_0) = 0$ is then automatic, and we conclude that our source element $\sum_{i=0}^{k} x^i \otimes m_i$ was zero in $N[x]$.
The composition $\iota \circ \pi$ is the zero map, as it is straightforward to check from the formulas. To prove that $\ker(\pi) = \mathrm{Im}(\iota)$, let $\sum_{i=0}^{k} x^i \otimes m_i \in N[x]$ and note that up to adding and subtracting elements in $\mathrm{Im}(\iota)$ we can transform our element into one of the form $1 \otimes m_0'$: we first eliminate the summand $x^k \otimes m_k$ by adding the element $x^{k-1} \otimes (x \cdot m_k) - x^k \otimes m_k = \left(x^{k-1} \otimes m_k\right) \iota$: this results in changing the $k-1^{\text{st}}$ component of the sum from $x^{k-1} \otimes m_{k-1}$ to $x^{k-1} \otimes (m_{k-1} + x \cdot m_k)$; the new element can be expressed as a sum of smaller length, and repeating this procedure k times we obtain an element of the form $1 \otimes m_0'$. If our original element $\sum_{i=0}^{k} x^i \otimes m_i \in N[x]$ is sent to zero by π, then we also have $(1 \otimes m_0')\pi = m_0' = 0$, and therefore our original element was in fact in the image of ι. □

As a consequence of the previous Lemma, we can prove the inequality $\mathrm{ldim}(R[x]) \leq \mathrm{ldim}(R) + 1$, which is half of Proposition 15.14. Assume $n := \mathrm{ldim}(R)$ is finite, and let N be a left $R[x]$-module. We want to show that $\mathrm{pd}_{R[x]}(N) \leq n + 1$. By the argument of the proof of Proposition 14.18, this is equivalent to showing that the functor $\mathrm{Ext}^{n+2}_{R[x]}(N, -)\colon {}_{R[x]}\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is the zero functor. Let therefore $N' \in {}_{R[x]}\mathrm{Mod}$ be another module, and let us try to prove that $\mathrm{Ext}^{n+2}_{R[x]}(N, N') = 0$. We can now regard the latter group as the image of N along the functor $\mathrm{Ext}^{n+2}_{R[x]}(-, N')$. The SES of left $R[x]$-modules $N[x] \to N[x] \to N$ from Lemma 15.18 gives rise to a long exact sequence of $\mathrm{Ext}_R(-, N')$-groups, containing in particular the triple of groups $\mathrm{Ext}^{n+1}_{R[x]}(N[x], N') \to \mathrm{Ext}^{n+2}_{R[x]}(N, N') \to \mathrm{Ext}^{n+2}_{R[x]}(N[x], N')$. Since $\mathrm{pd}_{R[x]}(N[x]) = \mathrm{pd}_R(N) \leq n$ by Lemma 15.17, the two external groups vanish; it follows that also the middle group vanishes.

16. The other half of Proposition 15.14, an example, Serre's theorem

For a ring R we have proved the inequality $\mathrm{ldim}(R[x]) \leq \mathrm{ldim}(R)+1$; we want now to prove the opposite inequality, and it suffices to prove $\mathrm{ldim}(R[x]) \geq \mathrm{ldim}(R)+1$ in the case in which both terms are finite (if they are both infinite, then our convention is

that $\infty \geq \infty$). We set therefore $n := \mathrm{ldim}(R) \geq 0$. In order to prove $\mathrm{ldim}(R) \geq n+1$ it suffices to prove that there exists a left $R[x]$-module with projective dimension $n+1$.
Following [Rot 8.2], we try to prove a statement (Proposition 16.7) which is a bit more general and applies to more situations than just the couple of rings R and $R[x]$.

Example 16.1. Let S be a ring and let $x \in S$ be a central element, i.e. $xs = sx$ for all $s \in S$. Then the left ideal $(x) \subseteq S$ coincides with the right ideal $(x) \subseteq S$, and also with the bilateral ideal $(x) \subseteq S$. The quotient abelian group $S/(x)$ inherits a ring structure from S, such that the projection to the quotient $\mathfrak{p}\colon S \to S/(x)$ is a surjective homomorphism of rings.

Example 16.2. Suppose that $x \in S^\times$ is a unit; then (x) coincides with the entire S, and thus $S/(x)$ is the zero ring.

Example 16.3. The element $x \in R[x]$ is central. Moreover $R[x]/(x)$ is isomorphic to the ring R, and by a slight abuse of notation we also call $\mathfrak{p}$ the surjective ring homomorphism $R[x] \to R$ given by evaluating polynomials at $x = 0$.

Notation 16.4. Let S and x be a in Example 16.1; then the map of rings $\mathfrak{p}\colon S \to S/(x)$ gives rise to an exact functor $\mathfrak{p}^*\colon {}_{S/(x)}\mathrm{Mod} \to {}_S\mathrm{Mod}$ by restriction of scalars.

The fact that $\mathfrak{p}$ is a *surjective* homomorphism of rings has a consequence: the functor $\mathfrak{p}^*$ is *fully faithful*: for any two left $S/(x)$-modules M, M' we have that a map of sets $f\colon M \to M'$ is $S/(x)$-linear if and only if the same map, considered as a map between the S-modules $\mathfrak{p}^*M \to \mathfrak{p}^*M'$, is S-linear.

Notation 16.5. In the other direction, we have a functor $\mathfrak{p}_*\colon {}_S\mathrm{Mod} \to {}_{S/(x)}\mathrm{Mod}$: this is given by considering $S/(x)$ as a $S/(x) - S$-bimodule (the right S-module structure being given by the homomorphism of rings $\mathfrak{p}$), and by considering the functor $S/(x) \otimes_S -$.

Concretely, if N is a left S-module, then $xN \subset N$ (i.e. the subset of multiples of x in N) is a left sub-S-module of N (here we use that x is central in S), and the quotient S-module N/xN is isomorphic, as a left S-module, to $\mathfrak{p}_*(N) \in {}_{S/(x)}\mathrm{Mod}$.

Exercise 16.6. Prove that the functor $\mathfrak{p}_*$ is left adjoint to the functor $\mathfrak{p}^*$: i.e. we have an adjunction

$$\mathfrak{p}_*\colon {}_S\mathrm{Mod} \rightleftarrows {}_{S/(x)}\mathrm{Mod}\colon \mathfrak{p}^*$$

The following is [Rot, Proposition 8.39].

Proposition 16.7 (Kaplansky)**.** *Let S be a ring, let $x \in S$ be a central element, and suppose that x is neither a unit nor a zero-divisor*[54]*. Let M be a (non-zero) left $S/(x)$-module such that* $\mathrm{pd}_{S/(x)}(M) = n$ *is finite. Then* $\mathrm{pd}_S(\mathfrak{p}^*M) = n+1$.

Before proving Proposition 16.7, we note that it immediately helps us in our purposes: for if M is a left R-module with $\mathrm{pd}_R(M) = n$ (such a module exists because we assumed $n = \mathrm{ldim}(R)$), then $\mathfrak{p}^*(M)$ is an example of a left $R[x]$-module with $\mathrm{pd}_{R[x]}(M) = n+1$, implying that $\mathrm{ldim}(R[x]) \geq n+1$, which is exactly what we

[54]In general, for a non-commutative ring, one has to distinguish a notion of *left zero divisor* and one of *right zero divisor*; since x is central, however, it is a left zero divisor if and only if it is a right zero divisor.

wanted to show. The hypothesis that x is not a unit in S is only used to exclude that $S/(x)$ is the zero ring, and thus that every $S/(x)$-module is the zero module.

Proof of Proposition 16.7. We prove the proposition by induction on $n \geq 0$.

- Suppose first $n = 0$, i.e. every $S/(x)$-module is projective, in particular M. Surely $\mathfrak{p}^*M$ cannot be projective over S: for the multiplication map $x \cdot -$ is injective on every free left S-module, and by restriction it is injective also on each submodule of a free left S-module, in particular on a projective S-module which can be exhibited as a direct summand of a free S-module. Now remember that $\mathfrak{p}^*M$ is a non-zero module, but $x \cdot -$ is the zero map $\mathfrak{p}^*M \to \mathfrak{p}^*M$: hence $\mathfrak{p}^*M$ is not projective over S.

  By the hypothesis that x is not a zero-divisor we also get a SES of left S-modules
  $$S \xrightarrow{\cdot x} S \xrightarrow{\mathfrak{p}} S/(x),$$
  and this can be considered as a free resolution of $S/(x)$ over S of length 1, hence $\mathrm{pd}_S(S/(x)) \leq 1$. Similarly, for every free left $S/(x)$-module F, we obtain a free resolution of $\mathfrak{p}^*F$ over S of length 1, by taking a suitable direct sum of copies of the above resolution of $S/(x)$: it follows that $\mathrm{pd}_S(\mathfrak{p}^*F) \leq 1$. Now we assumed that M is projective over $S/(x)$, hence we may find a free $S/(x)$-module F and a decomposition $F \cong M \oplus M'$ of left $S/(x)$-modules; applying $\mathfrak{p}^*$ we obtain a decomposition $\mathfrak{p}^*F \cong \mathfrak{p}^*M \oplus \mathfrak{p}^*M'$ of left S-modules. By the proof of Proposition 14.18 the inequality $\mathrm{pd}_S(\mathfrak{p}^*F) \leq 1$ is equivalent to the vanishing of the functor $\mathrm{Ext}^2_S(\mathfrak{p}^*F, -)$; this functor can be written as a direct sum of functors
  $$\mathrm{Ext}^2_S(\mathfrak{p}^*F, -) \cong \mathrm{Ext}^2_S(\mathfrak{p}^*M, -) \oplus \mathrm{Ext}^2_S(\mathfrak{p}^*M', -);$$
  it follows that also the functor $\mathrm{Ext}^2_S(\mathfrak{p}^*M, -)$ vanishes, and hence, again by the proof of Proposition 14.18, we obtain $\mathrm{pd}_S(\mathfrak{p}^*M) \leq 1$ as desired.
- Suppose now $n = 1$. Fix a SES of left $S/(x)$-modules $K \to F \to M$ with F free; note that the hypothesis $\mathrm{pd}_{S/(x)}(M) = 1$, together with the algorithm relying on Schanuel's lemma, imply that K is projective over $S/(x)$. The previous step of the induction gives $\mathrm{pd}_S(\mathfrak{p}^*K) = 1$. Exactly as in the previous case, we also show that $\mathrm{pd}_S(\mathfrak{p}^*(F)) \leq 1$, by exhibiting a free resolution of F over S of length 1. We first claim that $\mathrm{pd}_S(\mathfrak{p}^*M) \leq 2$; this is equivalent by the proof of Proposition 14.18 to the vanishing of $\mathrm{Ext}^3_S(\mathfrak{p}^*M, N)$ for all $N \in {}_S\mathrm{Mod}$; this can in turn be checked by the long exact sequence associated with the SES $\mathfrak{p}^*K \to \mathfrak{p}^F \to \mathfrak{p}^*M$ and the functors $\mathrm{Ext}_S(-, N)$: we have one piece reading
  $$\ldots \mathrm{Ext}^2_S(\mathfrak{p}^*K, N) \longrightarrow \mathrm{Ext}^3_S(\mathfrak{p}^*M, N) \longrightarrow \mathrm{Ext}^3(\mathfrak{p}^*F, N) \ldots$$
  and the vanishing of the outer groups implies the vanishing of the middle group. Great! Our purpose is to show $\mathrm{pd}_S(\mathfrak{p}^*M) = 2$ and we just checked $\mathrm{pd}_S(\mathfrak{p}^*M) \leq 2$! Now fix a SES of left S-modules $L \xrightarrow{i} F' \xrightarrow{p} \mathfrak{p}^*M$, with F' free over S; without loss of generality, assume that $L \subset F'$ is a sub-S-module. Consider also the sub-S-module $xF' \subset F'$: by S-linearity of p we have, for all $m \in F'$, $(xm)p = x((m)p) = 0$, because $x \cdot -$ is the zero map on $\mathfrak{p}^*M$. It follows that $xF' \subseteq \ker(p) = L$. We have thus inclusions of left

S-modules $xL \subseteq xF' \subseteq L \subseteq F'$. Our first aim is to show that $L/xL = \mathfrak{p}_* L$ is *not* projective over $S/(x)$: this will take a little time.

The following can be regarded both as a SES of left S-modules

$$L/xF' \longrightarrow F'/xF' \longrightarrow \mathfrak{p}^* M$$

or can be rewritten in a more fancy way to give a SES of left $S/(x)$-modules

$$\Lambda \longrightarrow \mathfrak{p}_* F' \longrightarrow M,$$

by defining $\Lambda = L/xF'$, with the induced $S/(x)$-module structure (the quotient L/xF' is in fact a left S-module on which $x \cdot -$ is the zero map). We assumed that $\mathrm{pd}_{S/(x)}(M) = 1$, and since $\mathfrak{p}_* F' = S/(x) \otimes_S F'$ is a free left $S/(x)$-module, we have that Λ must be projective (think of the algorithm to compute $\mathrm{pd}_{S/(x)}(M)$). Consider now the following SES of left $S/(x)$-modules, where xF/xL is given the natural structure of $S/(x)$-module (it is indeed an S-module on which $x \cdot -$ is the zero map):

$$xF'/xL \longrightarrow L/xL = \mathfrak{p}_* L \longrightarrow \Lambda = L/xF'.$$

Since Λ is projective over $S/(x)$, we obtain an isomorphism of left $S/(x)$-modules $\mathfrak{p}_* L \cong xF'/xL \oplus \Lambda$. Now use that x is a non-zero divisor, hence $x \cdot -$ is injective on F' and hence, by restriction, on $L \subset F'$: thus $xF' \cong F'$ (as S-modules), $xL \cong L$ and $xF'/xL \cong F'/L \cong \mathfrak{p}^* M$ as S-modules; but for two left $S/(x)$-modules it is equivalent to be isomorphic as $S/(x)$-modules, or as S-modules after applying $\mathfrak{p}^*$. We conclude that $\mathfrak{p}_* L = L/xL$ admits a direct sum decomposition (over $S/(x)$) one of whose summands is M. This implies directly that $\mathrm{pd}_{S/(x)}(\mathfrak{p}_* L) \geq \mathrm{pd}_{S/(x)}(M) = 1$, and in particular L/xL is *not* projective over $S/(x)$.

Now suppose by absurd that L is projective over S; then there is an isomorphism $L \oplus Q \cong F''$ for some free S-module F''; applying $\mathfrak{p}_*$ we would obtain $\mathfrak{p}_* L \oplus \mathfrak{p}_* Q \cong \mathfrak{p}_* F''$ with $\mathfrak{p}_* F'' = F''/xF''$ being a free $S/(x)$-module: it would follow that $\mathfrak{p}_* L$ is projective, which is exactly what we just proved is not the case! Hence L is not projective. Now the algorithm looks at our SES of S-modules $L \to F' \to \mathfrak{p}^* M$ and instead of returning $\mathrm{pd}_S(\mathfrak{p}^* M) = 1$ (as it would do if L were projective), it continues at least another step: thus $\mathrm{pd}_S(\mathfrak{p}^* M) \geq 2$, which together with the previous part of the argument finally shows that $\mathrm{pd}_S(\mathfrak{p}^* M) = 2$.

- Don't worry, the difficult part is already over! Assume now $n \geq 2$, and fix again a SES of $S/(x)$-modules $K \to F \to M$ with F free. The algorithm tells us that $\mathrm{pd}_{S/(x)}(K) = n-1$, and the inductive hypothesis applied to K tells us that $\mathrm{pd}_S(\mathfrak{p}^* K) = n-1+1 = n$. Recall also that $\mathrm{pd}_S(\mathfrak{p}^* F) \leq 1$, once again by exhibiting a concrete free resolution of $\mathfrak{p}^* F$ over S of length 1. In particular the functors $\mathrm{Ext}^i_S(\mathfrak{p}^* F, -)$ vanish for $i \geq n$ (here it is crucial that $n \geq 2$, and that's also why we had to suffer so much on the case $n = 1$!).

  Let now N be a left S-module, and consider the long exact sequence of $\mathrm{Ext}_S(-, N)$ groups associated with the SES of left S-modules $\mathfrak{p}^* K \to \mathfrak{p}^* F \to M$: inside it we find, for all $i \geq n$, short pieces of the form

$$\ldots \mathrm{Ext}^i_S(\mathfrak{p}^* F, N) \longrightarrow \mathrm{Ext}^i_S(\mathfrak{p}^* K, N) \longrightarrow \mathrm{Ext}^{i+1}_S(\mathfrak{p}^* M, N) \longrightarrow \mathrm{Ext}^{i+1}_S(\mathfrak{p}^* F, N) \ldots$$

For $i \geq n+1$ all of the above terms, except possibly $\mathrm{Ext}_S^{i+1}(\mathfrak{p}^*M, N)$, are known to vanish, again because $\mathrm{pd}_S(\mathfrak{p}^*F) \leq 1$ and $\mathrm{pd}_S(\mathfrak{p}^*K) = n$: it follows that also $\mathrm{Ext}_S^{i+1}(\mathfrak{p}^*M, N) = 0$, hence $\mathrm{Ext}_S^{i+1}(\mathfrak{p}^*M, -)$ is the zero functor for all $i \geq n+1$, which implies immediately that $\mathrm{pd}_S(\mathfrak{p}^*M) \leq n+1$. For $i = n$ instead, only the terms $\mathrm{Ext}_S^n(\mathfrak{p}^*F, N)$ and $\mathrm{Ext}_S^{n+1}(\mathfrak{p}^*F, N)$ are known to vanish; in fact, since $\mathrm{pd}_S(\mathfrak{p}^*K) = n$, the functor $\mathrm{Ext}_S^n(\mathfrak{p}^*K, -)$ is not the zero functor, hence we can assume to have chosen N such that $\mathrm{Ext}_S^n(\mathfrak{p}^*K, N) \neq 0$; exactness gives us in this case an isomorphism $\mathrm{Ext}_S^n(\mathfrak{p}^*K, N) \cong \mathrm{Ext}_S^{n+1}(\mathfrak{p}^*M, N)$, but then the same module N witnesses that $\mathrm{Ext}^{n+1}(\mathfrak{p}^*M, -)$ is not the zero functor, and hence $\mathrm{pd}_S(\mathfrak{p}^*M) \geq n+1$. □

We have thus completed the proof of Proposition 15.14, which in turn implies Theorem 15.11.

Example 16.8. Let $\mathbb{F}$ be a field, and consider $R = \mathbb{F}[x, y]$ as a ring, and $M = \mathbb{F}[x, y]/(x, y)$ as an R-module. Set $P_0 = R$: we have a surjective quotient map $R \to M$ with $K_0 = (x, y)$ as kernel; in particular the following is exact:

$$\ldots 0 \longrightarrow K_0 = (x, y) \longrightarrow P_0 = R \longrightarrow M \longrightarrow 0$$

Set $P_1 = R \oplus R$, call X and Y the generators, and define a surjective map $d_1^P : P_1 \to K_0$ by sending $X \mapsto x \in P_0$ and $Y \mapsto y \in P_0$. The kernel K_1 of d_1^P is generated by the element $yX - xY \in P_1$, and is a free R-module on one generator. In particular, setting $P_2 = K_1$, the following is a free resolution of M of length 2:

$$\ldots \longrightarrow 0 \longrightarrow P_2 = R\langle yX - xY\rangle \xhookrightarrow{d_2^P} P_1 = R\langle X\rangle \oplus R\langle Y\rangle \xrightarrow{d_1^P} P_0 = R \longrightarrow 0 \ldots$$

Exercise 16.9. Can you generalise the previous example to $R = \mathbb{F}[x_1, \ldots, x_n]$ and $M = \mathbb{F}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$? Here are some hints.

- For all $1 \leq i \leq n$ there is a resolution of $M_i := \mathbb{F}[x_i]/(x_i)$ over $\mathbb{F}[x_i]$ of length 1 given by $\ldots 0 \to \mathbb{F}[x_i] = P_1^{(i)} \xrightarrow{\cdot x_i} \mathbb{F}[x_i] = P_0^{(i)} \to 0 \ldots$.
- The tensor product over $\mathbb{F}$ is our best friend: every $\mathbb{F}$-module is flat, so we imagine not to lose exactness, where needed, by tensoring over $\mathbb{F}$.
- In what sense $M = M_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} M_k$? First, make sense of this long tensor product; second, make sense of the fact that the single tensor factors are just modules over the small rings $\mathbb{F}[x_i]$, but the result should be a module over the big ring R.
- Make sense of the expression "$\mathrm{Tot}(P_\bullet^{(1)} \otimes_{\mathbb{F}} \ldots P_\bullet^{(n)})$" as a chain complex of R-modules of length n; prove that it is a projective resolution, over R, of M.

16.1. **Serre's theorem on global dimension of local Noetherian rings.** We conclude the course by understanding the following statement, whose proof we omit.

Theorem 16.10 (Serre)**.** *Let R be a commutative* Noetherian local *ring. Then there are two possibilities:*

- *either R is not* regular*, in which case* $\mathrm{ldim}(R) = \infty$[55]*;*

[55] We use ldim, but in the commutative context rdim would be the same...

- *or* R *is* regular*, in which case* $\mathrm{ldim}(R)$ *is finite and is equal to the* Krull dimension *of* R*.*

I emphasized the words that we have not encountered yet in the course: these words have in fact not much to do directly with homological algebra, and for this reason the previous theorem should be surprising.

Definition 16.11. Let R be a (non-zero) commutative ring. An ideal in R is a subset $I \subset R$ which is also a (proper) sub-R-module, i.e. $I \neq R$. We say that R is *Noetherian* if every ideal is a finitely generated R-module.

Definition 16.12. An ideal $I \subset R$ is a *prime* ideal if the following hold: for every $x, y \in R \setminus I$, we have $xy \in R \setminus I$.
The *Krull dimension* of a commutative ring R is the supremum of all $n \geq 0$ for which one can find a sequence of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n \subset R$, with all containments being inequalities.

Definition 16.13. An ideal $I \subset R$ is a *maximal* ideal if there exist no ideal $J \subset R$ with $I \subset J$ (and $I \neq J$). We say that R is *local* if it has a unique maximal ideal[56].

In general, if R is a commutative Noetherian local ring with maximal ideal $M \subset R$, then one can prove that M cannot be generated by less than n elements, where n is the Krull dimension of R. This should be surprising, because at first glance there is no connection between the problems of finding long chains of prime ideals in R, and finding small sets of generators for M.

Definition 16.14. We say that a Noetherian local ring R is *regular* if the maximal ideal M can be generated by exactly n of its elements, where n is the Krull dimension of R.

Example 16.15. Consider the ring $R = \mathbb{Z}/4$; it has two ideals: one is $([0]_4)$, the other is the ideal $M = ([2]_4)$, which is therefore the unique maximal ideal: R is local. Since R is finite as a set, it is clearly Noetherian. The ideal $([0]_4)$ is not a prime ideal, as the product $[2]_4[2]_4 = [0]_4$ witnesses. It follows that the Krull dimension of R is 0, exhibited by setting $P_0 = M$ and acknowledging that a longer chain of prime ideals just doesn not exist.
If R were regular, we could generate M by... zero elements! That is, M should be the zero ideal; this is not the case, hence we conclude that R is not regular.
Serre's theorem implies that $\mathrm{ldim}(\mathbb{Z}_4) = \infty$. Does it remind you of anything? Yes! We already proved that $\mathrm{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \neq 0$ for all $n \geq 0$, and have thus already checked that $\mathrm{ldim}(\mathbb{Z}_4) = \infty$. Thus Serre's theorem (first part) generalises this example we already saw.

For a commutative ring R and a prime ideal $P \subset R$, note that $\mathcal{S} = R \setminus P$ is a multiplicative subset of R, and thus we can form the localisation $\mathcal{S}^{-1}R$; it turns out that the localisation $\mathcal{S}^{-1}R$... is a local ring! If R was Noetherian, also the localisation is Noetherian.
For a generic commutative Noetherian ring R one has the following formula:

$$\mathrm{ldim}(R) = \sup \left\{ \mathrm{ldim}(\mathcal{S}^{-1}R) \,|\, \mathcal{S} = R \setminus P \text{ for a prime ideal } P \subset R \right\}.$$

[56] One can use Zorn Lemma and prove that every commutative ring R has at least one maximal ideal.

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